7 RIGID BODY MOTION

7.1 Introduction

In the last chapter we analyzed the motion of rigid bodies undergoing fixed axis rotation. In this chapter we shall attack the more general problem of analyzing the motion of rigid bodies which can rotate about any axis. Rather than emphasize the formal mathematical details, we will try to gain insight into the basic principles. We will discuss the important features of the motion of gyroscopes and other devices which have large spin angular momentum, and we will also look at a variety of other systems. Our analysis is based on a very simple idea—that angular momentum is a vector. Although this is obvious from the definition, somehow its significance is often lost when one first encounters rigid body motion. Understanding the vector nature of angular momentum leads to a very simple and natural explanation for such a mysterious effect as the precession of a gyroscope.

A second topic which we shall treat in this chapter is the conservation of angular momentum. We touched on this in the last chapter but postponed any incisive discussion. Here the problem is physical subtlety rather than mathematical complexity.

7.2 The Vector Nature of Angular Velocity and Angular Momentum

In order to describe the rotational motion of a body we would like to introduce suitable coordinates. Recall that in the case of translational motion, our procedure was to choose some convenient coordinate system and to denote the position of the body by a vector **r**. The velocity and acceleration were then found by successively differentiating **r** with respect to time.

Suppose that we try to introduce angular coordinates θ_x, θ_y, and θ_z about the x, y, and z axes, respectively. Can we specify the angular orientation of the body by a vector?

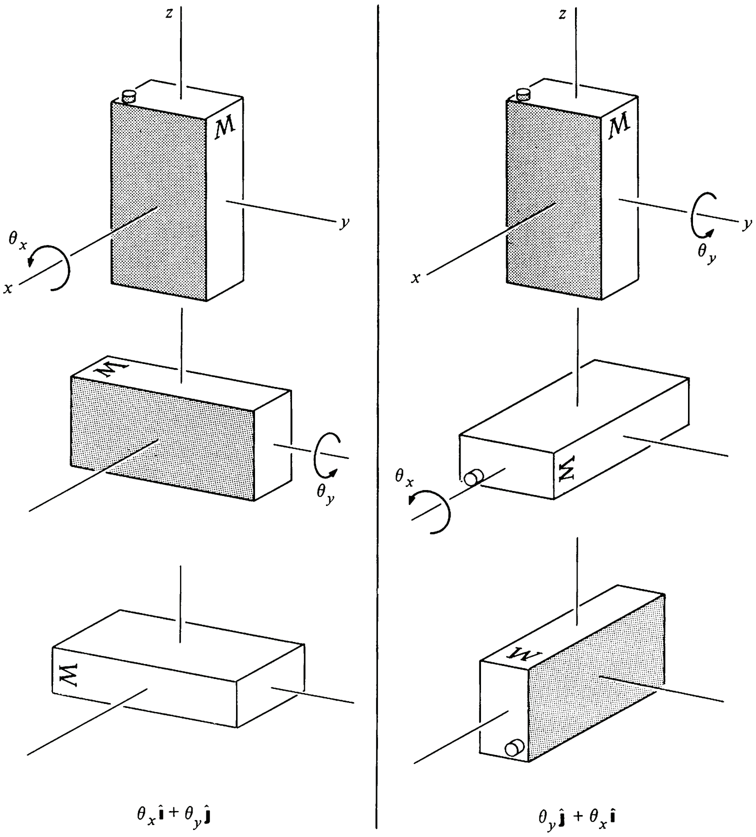
$$\boldsymbol{\theta} \stackrel{?}{=} (\theta_x\hat{\mathbf{i}} + \theta_y\hat{\mathbf{j}} + \theta_z\hat{\mathbf{k}})$$

Unfortunately, this procedure can *not* be made to work; there is no way to construct a vector to represent an angular orientation.

The reason that $\theta_x\hat{\mathbf{i}}$ and $\theta_y\hat{\mathbf{j}}$ cannot be vectors is that the order in which we add them affects the final result: $\theta_x\hat{\mathbf{i}} + \theta_y\hat{\mathbf{j}} \neq \theta_y\hat{\mathbf{j}} + \theta_x\hat{\mathbf{i}}$, as we show explicitly in Example 7.1. For honest-to-goodness vectors like $x\hat{\mathbf{i}}$ and $y\hat{\mathbf{j}}$, $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = y\hat{\mathbf{j}} + x\hat{\mathbf{i}}$. Vector addition is commutative.

Example 7.1 Rotations through Finite Angles

Consider a can of maple syrup oriented as shown, and let us investigate what happens when we rotate it by an angle of $\pi/2$ around the x axis, and then by $\pi/2$ around the y axis, and compare the result with executing the same rotations but in reverse order.

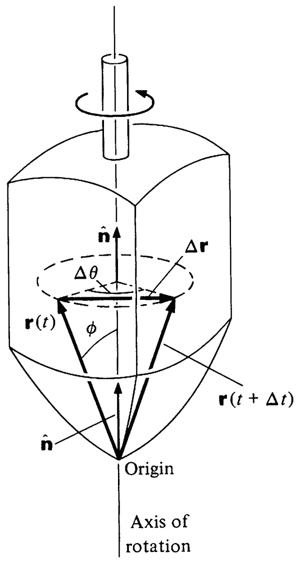


The diagram speaks for itself:

$\theta_x\mathbf{\hat{i}} + \theta_y\mathbf{\hat{j}} \neq \theta_y\mathbf{\hat{j}} + \theta_x\mathbf{\hat{i}}.$

Fortunately, all is not lost; although angular position cannot be represented by a vector, it turns out that angular velocity, the rate of change of angular position, is a perfectly good vector. We can define angular velocity by

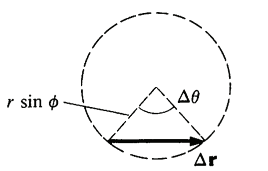
$$\begin{aligned}\boldsymbol{\omega} &= \frac{d\theta_x}{dt}\mathbf{\hat{i}} + \frac{d\theta_y}{dt}\mathbf{\hat{j}} + \frac{d\theta_z}{dt}\mathbf{\hat{k}} \\ &= \omega_x\mathbf{\hat{i}} + \omega_y\mathbf{\hat{j}} + \omega_z\mathbf{\hat{k}}.\end{aligned}$$



The important point is that although rotations through finite angles do not commute, infinitesimal rotations do commute, so that $\boldsymbol{\omega} = \lim_{\Delta t \to 0} (\Delta\boldsymbol{\theta}/\Delta t)$ represents a true vector. The reason for this is discussed in Note 7.1 at the end of the chapter. Assuming that angular velocity is indeed a vector, let us find how the velocity of any particle in a rotating rigid body is related to the angular velocity of the body.

Consider a rigid body rotating about some axis. We designate the instantaneous direction of the axis by $\hat{\mathbf{n}}$ and choose a coordinate system with its origin on the axis. The coordinate system is fixed in space and is inertial. As the body rotates, each of its particles describes a circle about the axis of rotation. A vector $\mathbf{r}$ from the origin to any particle tends to sweep out a cone. The drawing shows the result of rotation through angle $\Delta\theta$ about the axis along $\hat{\mathbf{n}}$. The angle ϕ between $\hat{\mathbf{n}}$ and $\mathbf{r}$ is constant, and the tip of $\mathbf{r}$ moves on a circle of radius $r \sin \phi$.

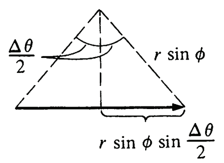
The magnitude of the displacement $|\Delta\mathbf{r}|$ is



$$|\Delta\mathbf{r}| = 2r \sin \phi \sin \frac{\Delta\theta}{2}.$$

For $\Delta\theta$ very small, we have

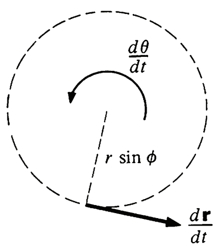
$$\sin \frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2} \qquad \text{and} \qquad |\Delta\mathbf{r}| \approx r \sin \phi \, \Delta\theta.$$

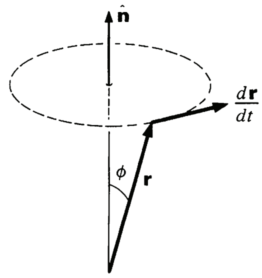


If $\Delta\theta$ occurs in time Δt, we have $|\Delta\mathbf{r}|/\Delta t \approx r \sin \phi \, (\Delta\theta/\Delta t)$. In the limit $\Delta t \to 0$,

$$\left|\frac{d\mathbf{r}}{dt}\right| = r \sin \phi \frac{d\theta}{dt}.$$

In the limit, $d\mathbf{r}/dt$ is tangential to the circle, as shown below. Recalling the definition of vector cross product (Sec. 1.2*e*), we see that the magnitude of $d\mathbf{r}/dt$, $|d\mathbf{r}/dt| = r \sin \phi \, d\theta/dt$, and its direction, perpendicular to the plane of $\mathbf{r}$ and $\hat{\mathbf{n}}$, are given cor-



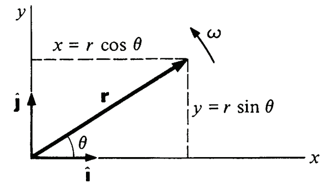


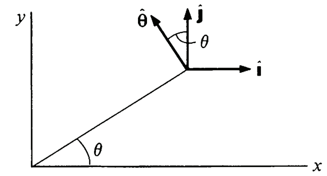
rectly by $d\mathbf{r}/dt = \hat{\mathbf{n}} \times \mathbf{r}\, d\theta/dt$. Since $d\mathbf{r}/dt = \mathbf{v}$ and $\hat{\mathbf{n}}\, d\theta/dt = \boldsymbol{\omega}$, we have

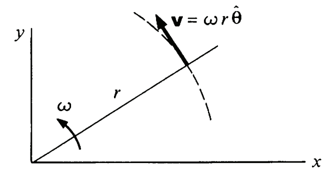
$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \qquad 7.1$$

Example 7.2

Rotation in the xy Plane







To connect Eq. (7.1) with a more familiar case—rotation in the xy plane—suppose that we evaluate $\mathbf{v}$ for the rotation of a particle about the z axis. We have $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. Hence,

$$\begin{aligned}\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r}\\ &= \omega\hat{\mathbf{k}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\\ &= \omega(x\hat{\mathbf{j}} - y\hat{\mathbf{i}}).\end{aligned}$$

In plane polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and therefore

$$\mathbf{v} = \omega r(\hat{\mathbf{j}}\cos\theta - \hat{\mathbf{i}}\sin\theta).$$

But $\hat{\mathbf{j}}\cos\theta - \hat{\mathbf{i}}\sin\theta$ is a unit vector in the tangential direction $\hat{\boldsymbol{\theta}}$. Therefore,

$$\mathbf{v} = \omega r\hat{\boldsymbol{\theta}}.$$

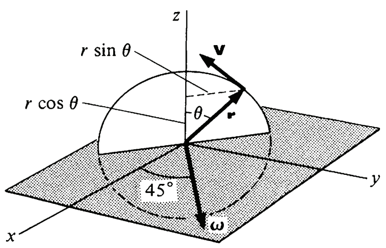
This is the velocity of a particle moving in a circle of radius r at angular velocity ω.

It is sometimes difficult to appreciate at first the vector nature of angular velocity since we are used to visualizing rotation about a fixed axis, which involves only one component of angular velocity. We are generally much less familiar with simultaneous rotation about several axes.

We have seen that we can treat angular velocity as a vector in the relation $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. It is important to assure ourselves that this relation remains valid if we resolve $\boldsymbol{\omega}$ into components like any other vector. In other words, if we write $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$, is it true that $\mathbf{v} = (\boldsymbol{\omega}_1 \times \mathbf{r}) + (\boldsymbol{\omega}_2 \times \mathbf{r})$? As the following example shows, the answer is yes.

Example 7.3

Vector Nature of Angular Velocity



Consider a particle rotating in a vertical plane as shown in the sketch. The angular velocity $\boldsymbol{\omega}$ lies in the xy plane and makes an angle of 45° with the xy axes.

First we shall calculate $\mathbf{v}$ directly from the relation $\mathbf{v} = d\mathbf{r}/dt$. To find $\mathbf{r}$, note from the sketch at left that $z = r\cos\theta$, $x = -r\sin\theta/\sqrt{2}$ and $y = r\sin\theta/\sqrt{2}$. Hence,

$$\mathbf{r} = r\left(\frac{-1}{\sqrt{2}}\sin\theta\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right)$$

and differentiating, we have, since r = constant,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$
$$= r\left[\frac{-1}{\sqrt{2}}\cos\theta\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}\right]\frac{d\theta}{dt}$$
$$= \omega r\left[\frac{-1}{\sqrt{2}}\cos\theta\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}\right], \qquad 1$$

where we have used $d\theta/dt = \omega$.

Next we shall find the velocity from $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Assuming that $\boldsymbol{\omega}$ can be resolved into components,

$$\boldsymbol{\omega} = \frac{\omega}{\sqrt{2}}\hat{\mathbf{i}} + \frac{\omega}{\sqrt{2}}\hat{\mathbf{j}},$$

we have

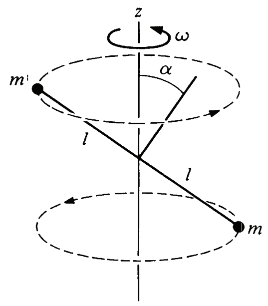
$$\boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\omega}{\sqrt{2}} & \frac{\omega}{\sqrt{2}} & 0 \\ \frac{-r\sin\theta}{\sqrt{2}} & \frac{r\sin\theta}{\sqrt{2}} & r\cos\theta \end{vmatrix}$$
$$= \omega r\left(\frac{-1}{\sqrt{2}}\cos\theta\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}\right)$$

in agreement with Eq. (1).

As we expect, there is no problem in treating $\boldsymbol{\omega}$ like any other vector.

In the following example we shall see that a problem can be greatly simplified by resolving $\boldsymbol{\omega}$ into components along convenient axes. The example also demonstrates that angular momentum is not necessarily parallel to angular velocity.

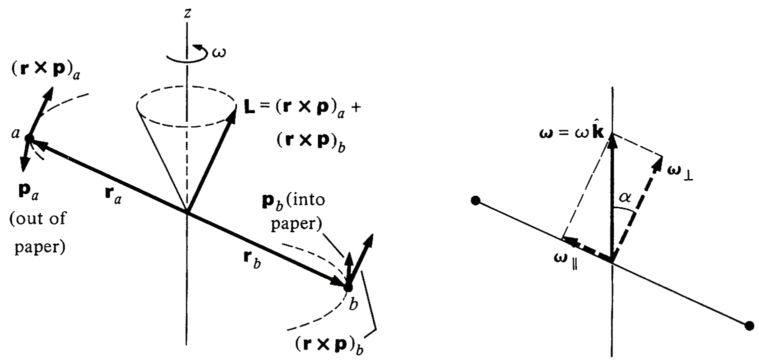
Example 7.4 **Angular Momentum of a Rotating Skew Rod**

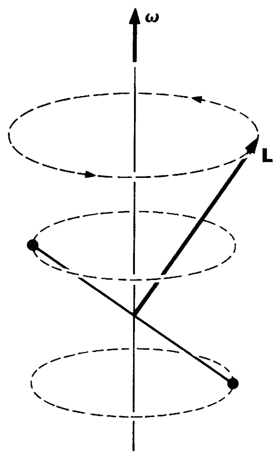


Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis which rotates at angular speed ω. The rod is skewed at angle α, as shown in the sketch. The problem is to find the angular momentum of the system.

The most direct method is to calculate the angular momentum from the definition $\mathbf{L} = \Sigma(\mathbf{r}_j \times \mathbf{p}_j)$. Each mass moves in a circle of radius $l\cos\alpha$ with angular speed ω. The momentum of each mass is $|\mathbf{p}| = m\omega l\cos\alpha$, tangential to the circular path. Taking the midpoint of the skew rod as origin, $|\mathbf{r}| = l$. $\mathbf{r}$ lies along the rod and is perpendicular to

p. Hence $|\mathbf{L}| = 2m\omega l^2 \cos\alpha$. **L** is perpendicular to the skew rod and lies in the plane of the rod and the z axis, as shown in the left hand drawing, below. **L** turns with the rod, and its tip traces out a circle about the z axis.





We now turn to a method for calculating **L** which emphasizes the vector nature of $\boldsymbol{\omega}$. First we resolve $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$ into components $\boldsymbol{\omega}_\perp$ and $\boldsymbol{\omega}_\parallel$, perpendicular and parallel to the skew rod. From the right hand drawing, above, we see that $\omega_\perp = \omega \cos\alpha$, and $\omega_\parallel = \omega \sin\alpha$.

Since the masses are point particles, $\boldsymbol{\omega}_\parallel$ produces no angular momentum. Hence, the angular momentum is due entirely to $\boldsymbol{\omega}_\perp$. The angular momentum is readily evaluated: the moment of inertia about the direction of $\boldsymbol{\omega}_\perp$ is $2ml^2$, and the magnitude of the angular momentum is

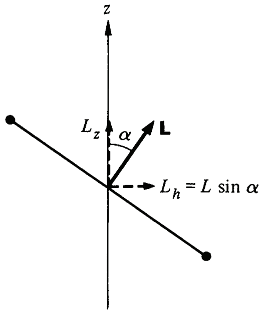
$$\begin{aligned} L &= I\omega_\perp \\ &= 2ml^2\omega_\perp \\ &= 2ml^2\omega \cos\alpha. \end{aligned}$$

L points along the direction of $\boldsymbol{\omega}_\perp$. Hence, **L** swings around with the rod; the tip of **L** traces out a circle about the z axis. (We encountered a similar situation in Example 6.2 with the conical pendulum.) Note that **L** is not parallel to $\boldsymbol{\omega}$. This is generally true for nonsymmetric bodies.

The dynamics of rigid body motion is governed by $\boldsymbol{\tau} = d\mathbf{L}/dt$. Before we attempt to apply this relation to complicated systems, let us gain some insight into its physical meaning by analyzing the torque on the rotating skew rod.

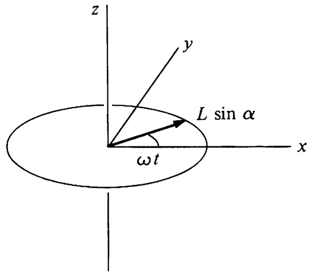
Example 7.5 **Torque on the Rotating Skew Rod**

In Example 7.4 we showed that the angular momentum of a uniformly rotating skew rod is constant in magnitude but changes in direction. **L** is fixed with respect to the rod and rotates in space with the rod.



The torque on the rod is given by $\boldsymbol{\tau} = d\mathbf{L}/dt$. We can find $d\mathbf{L}/dt$ quite easily by decomposing $\mathbf{L}$ as shown in the sketch. (We followed a similar procedure in Example 6.6 for the conical pendulum.) The component L_z parallel to the z axis, $L \cos \alpha$, is constant. Hence, there is no torque in the z direction. The horizontal component of $\mathbf{L}$, $L_h = L \sin \alpha$, swings with the rod. If we choose xy axes so that L_h coincides with the x axis at $t = 0$, then at time t we have

$$L_x = L_h \cos \omega t$$
$$= L \sin \alpha \cos \omega t$$
$$L_y = L_h \sin \omega t$$
$$= L \sin \alpha \sin \omega t.$$



Hence,

$$\mathbf{L} = L \sin \alpha(\hat{\mathbf{i}} \cos \omega t + \hat{\mathbf{j}} \sin \omega t) + L \cos \alpha \hat{\mathbf{k}}.$$

The torque is

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$
$$= L\omega \sin \alpha(-\,\omega\hat{\mathbf{i}} \sin \omega t + \hat{\mathbf{j}} \cos \omega t).$$

Using $L = 2ml^2\omega \cos \alpha$, we obtain

$$\tau_x = -2ml^2\omega^2 \sin \alpha \cos \alpha \sin \omega t$$
$$\tau_y = 2ml^2\omega^2 \sin \alpha \cos \alpha \cos \omega t.$$

Hence,

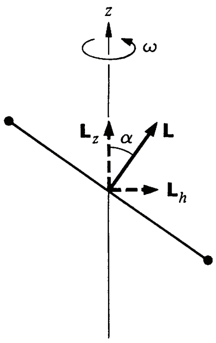
$$\tau = \sqrt{\tau_x{}^2 + \tau_y{}^2}$$
$$= 2ml^2\omega^2 \sin \alpha \cos \alpha$$
$$= \omega L \sin \alpha.$$

Note that $\tau = 0$ for $\alpha = 0$ or $\alpha = \pi/2$. Do you see why? Also, can you see why the torque should be proportional to ω^2?

This analysis may seem roundabout, since the torque can be calculated directly by finding the force on each mass and using $\boldsymbol{\tau} = \Sigma \mathbf{r}_j \times \mathbf{f}_j$. However, the procedure used above is just as quick. Furthermore, it illustrates that angular velocity and angular momentum are *real* vectors which can be resolved into components along any axes we choose.

Example 7.6 Torque on the Rotating Skew Rod (Geometric Method)

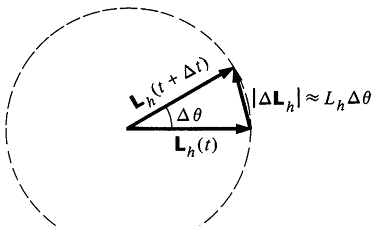
In Example 7.5 we calculated the torque on the rotating skew rod by resolving $\mathbf{L}$ into components and using $\boldsymbol{\tau} = d\mathbf{L}/dt$. We repeat the calculation in this example using a geometric argument which emphasizes



the connection between torque and the rate of change of **L**. This method illustrates a point of view that will be helpful in analyzing gyroscopic motion.

As in Example 7.5, we begin by resolving **L** into a vertical component $L_z = L \cos \alpha$ and a horizontal component $L_h = L \sin \alpha$ as shown in the sketch. Since $\mathbf{L}_z$ is constant, there is no torque about the z axis. $\mathbf{L}_h$ is constant in magnitude but is rotating with the rod. The time rate of change of **L** is due solely to this effect.

Once again we are dealing with a rotating vector. From Sec. 1.8 or Example 6.6, we know that $dL_h/dt = \omega L_h$. However, since it is so important to be able to visualize this result, we derive it once more. From the vector diagram we have



$$|\Delta \mathbf{L}_h| \approx |\mathbf{L}_h| \Delta\theta$$

$$\frac{dL_h}{dt} = L_h \frac{d\theta}{dt}$$

$$= L_h \omega.$$

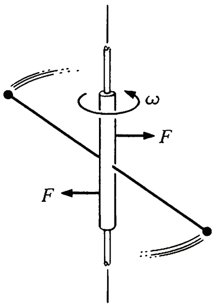
The torque is given by

$$\tau = \frac{dL_h}{dt}$$

$$= L_h \omega$$

$$= \omega L \sin \alpha,$$

which is identical to the result of the last example. The torque $\boldsymbol{\tau}$ is parallel to $\Delta\mathbf{L}$ in the limit. For the skew rod, $\boldsymbol{\tau}$ is in the tangential direction in the horizontal plane and rotates with the rod.

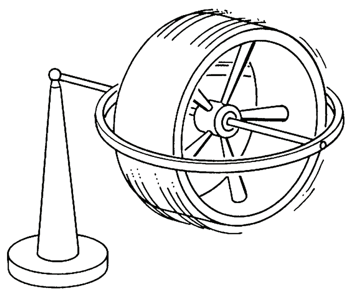


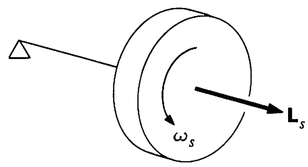
You may have thought that torque on a rotating system always causes the speed of rotation to change. In this problem the speed of rotation is constant, and the torque causes the direction of **L** to change. The torque is produced by the forces on the rotating bearing of the skew rod. For a real rod this would have to be an extended structure, something like a sleeve. The torque causes a time varying load on the sleeve which results in vibration and wear. Since there is no way for a uniform gravitational field to exert a torque on the skew rod, the rod is said to be *statically balanced*. However, there is a torque on the skew rod when it is rotating, which means that it is not *dynamically balanced*. Rotating machinery must be designed for dynamical balance if it is to run smoothly.

7.3 The Gyroscope

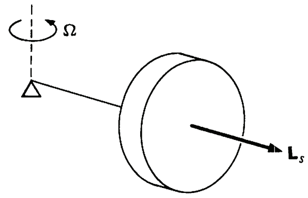
We now turn to some aspects of gyroscope motion which can be understood by using the basic concepts of angular momentum, torque, and the time derivative of a vector. We shall discuss each step carefully, since this is one area of physics where intuition may not be much help. Our treatment of the gyroscope in this section is by no means complete. Instead of finding the general motion of the gyroscope directly from the dynamical equations, we bypass this complicated mathematical problem and concentrate on uniform precession, a particularly simple and familiar type of gyroscope motion. Our aim is to show that uniform precession is consistent with $\boldsymbol{\tau} = d\mathbf{L}/dt$ and Newton's laws. While this approach cannot be completely satisfying, it does illuminate the physical principles involved.

The essentials of a gyroscope are a spinning flywheel and a suspension which allows the axle to assume any orientation. The familiar toy gyroscope shown in the drawing is quite adequate for our discussion. The end of the axle rests on a pylon, allowing the axis to take various orientations without constraint.

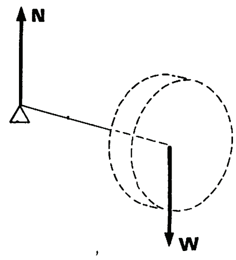




The right hand drawing above is a schematic representation of the gyroscope. The triangle represents the free pivot, and the flywheel spins in the direction shown.

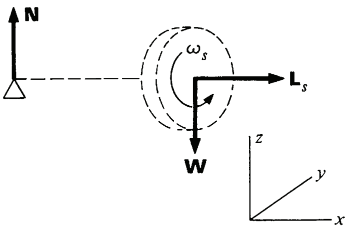


If the gyroscope is released horizontally with one end supported by the pivot, it wobbles off horizontally and then settles down to *uniform precession*, in which the axle slowly rotates about the vertical with constant angular velocity Ω. One's immediate impulse is to ask why the gyroscope does not fall. A possible answer is suggested by the force diagram. The total vertical force is $N - W$, where N is the vertical force exerted by the pivot and W is the weight. If $N = W$, the center of mass cannot fall.



This explanation, which is quite correct, is not satisfactory. We have asked the wrong question. Instead of wondering why the gyroscope does not fall, we should ask why it does not swing about the pivot like a pendulum.

As a matter of fact, if the gyroscope is released with its flywheel stationary, it behaves exactly like a pendulum; instead of precessing horizontally, it swings vertically. The gyroscope precesses



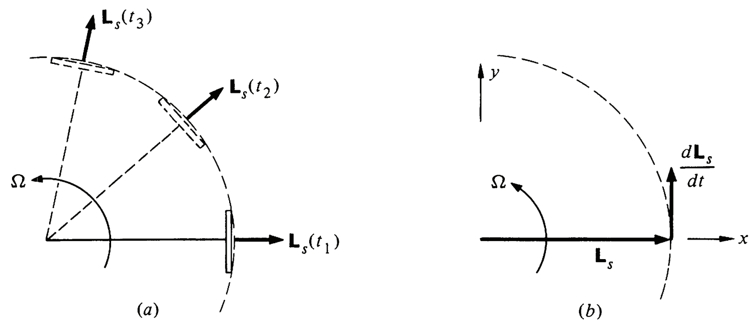
only if the flywheel is spinning rapidly. In this case, the large spin angular momentum of the flywheel dominates the dynamics of the system.

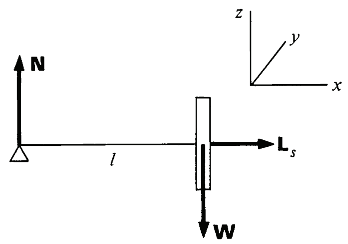
Nearly all of the gyroscope's angular momentum lies in $\mathbf{L}_s$, the spin angular momentum. $\mathbf{L}_s$ is directed along the axle and has magnitude $L_s = I_0\omega_s$, where I_0 is the moment of inertia of the flywheel about its axle. When the gyroscope precesses about the z axis, it has a small orbital angular momentum in the z direction. However, for uniform precession the orbital angular momentum is constant in magnitude and direction and plays no dynamical role. Consequently, we shall ignore it here.

$\mathbf{L}_s$ always points along the axle. As the gyroscope precesses, $\mathbf{L}_s$ rotates with it. (See figure a below.) We have encountered rotating vectors many times, most recently in Example 7.6. If the angular velocity of precession is Ω, the rate of change of $\mathbf{L}_s$ is given by

$$\left|\frac{d\mathbf{L}_s}{dt}\right| = \Omega L_s.$$

The direction of $d\mathbf{L}_s/dt$ is tangential to the horizontal circle swept out by $\mathbf{L}_s$. At the instant shown in figure b, $\mathbf{L}_s$ is in the x direction and $d\mathbf{L}_s/dt$ is in the y direction.





There must be a torque on the gyroscope to account for the change in $\mathbf{L}_s$. The source of the torque is apparent from the force diagram at left. If we take the pivot as the origin, the torque is due to the weight of the flywheel acting at the end of the axle. The magnitude of the torque is

$$\tau = lW.$$

$\boldsymbol{\tau}$ is in the y direction, parallel to $d\mathbf{L}_s/dt$, as we expect.

We can find the rate of precession Ω from the relation

$$\left|\frac{d\mathbf{L}_s}{dt}\right| = \tau.$$

Since $|d\mathbf{L}_s/dt| = \Omega L_s$ and $\tau = lW$, we have

$$\Omega L_s = lW.$$

or

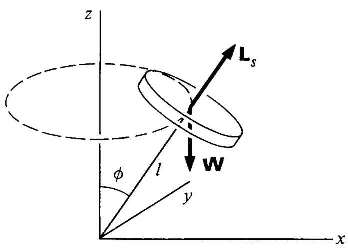
$$\Omega = \frac{lW}{I_0\omega_s}. \qquad 7.2$$

Alternatively, we could have analyzed the motion about the center of mass. In this case the torque is $\tau_0 = Nl = Wl$ as before, since $N = W$.

Equation (7.2) indicates that Ω increases as the flywheel slows. This effect is easy to see with a toy gyroscope. Obviously Ω cannot increase indefinitely; eventually uniform precession gives way to a violent and erratic motion. This occurs when Ω becomes so large that we cannot neglect small changes in the angular momentum about the vertical axis due to frictional torque. However, as is shown in Note 7.2, uniform precession represents an exact solution to the dynamical equations governing the gyroscope.

Although we have assumed that the axle of the gyroscope is horizontal, the rate of uniform precession is independent of the angle of elevation, as the following example shows.

Example 7.7 **Gyroscope Precession**



Consider a gyroscope in uniform precession with its axle at angle ϕ with the vertical. The component of $\mathbf{L}_s$ in the xy plane varies as the gyroscope precesses, while the component parallel to the z axis remains constant.

The horizontal component of $\mathbf{L}_s$ is $L_s \sin\phi$. Hence

$$|d\mathbf{L}_s/dt| = \Omega L_s \sin\phi.$$

The torque due to gravity is horizontal and has magnitude

$$\tau = l \sin\phi\, W.$$

We have

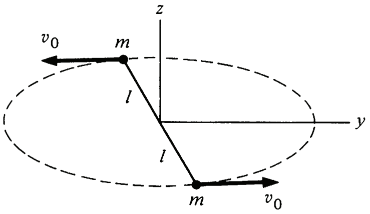
$$\Omega L_s \sin\phi = l \sin\phi\, W$$

$$\Omega = \frac{lW}{I_0\omega_s}.$$

The precessional velocity is independent of ϕ.

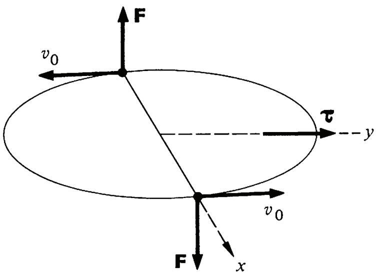
Our treatment shows that gyroscope precession is completely consistent with the dynamical equation $\boldsymbol{\tau} = d\mathbf{L}/dt$. The following example gives a more physical explanation of why a gyroscope precesses.

Example 7.8 **Why a Gyroscope Precesses**



Gyroscope precession is hard to understand because angular momentum is much less familiar to us than particle motion. However, the rotational dynamics of a simple rigid body can be understood directly in terms of Newton's laws. Rather than address ourselves specifically to the gyroscope, let us consider a rigid body consisting of two particles of mass m at either end of a rigid massless rod of length $2l$. Suppose that the rod is rotating in free space with its angular momentum $\mathbf{L}_s$ along the z direction. The speed of each mass is v_0. We shall show that an applied torque $\boldsymbol{\tau}$ causes $\mathbf{L}_s$ to precess with angular velocity $\Omega = \tau/L_s$.

To simplify matters, suppose that the torque is applied only during a short time Δt while the rod is instantaneously oriented along the x axis. We assume that the torque is due to two equal and opposite forces F, as shown. (The total force is zero, and the center of mass remains at rest.) The momentum of each mass changes by

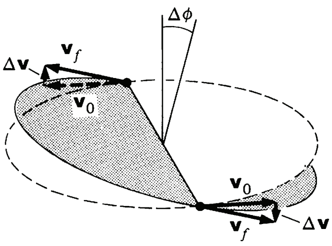


$$\Delta \mathbf{p} = m\,\Delta \mathbf{v} = \mathbf{F}\Delta t.$$

Since $\Delta \mathbf{v}$ is perpendicular to $\mathbf{v}_0$, the velocity of each mass changes direction, as shown at left below, and the rod rotates about a new direction.

The axis of rotation tilts by the angle

$$\Delta\phi \approx \frac{\Delta v}{v_0} = \frac{F\,\Delta t}{m v_0}.$$



The torque on the system is $\tau = 2Fl$, and the angular momentum is $L_s = 2mv_0 l$. Hence

$$\Delta\phi = \frac{F\,\Delta t}{m v_0} = \frac{2lF\,\Delta t}{2lmv_0} = \frac{\tau\,\Delta t}{L_s}.$$

The rate of precession while the torque is acting is therefore

$$\Omega = \frac{\Delta\phi}{\Delta t}$$

$$= \frac{\tau}{L_s},$$

which is identical to the result for gyroscope precession. Also, the change in the angular momentum, $\Delta\mathbf{L}_s$, is in the y direction parallel to the torque, as required.

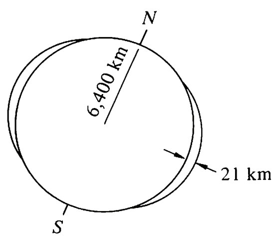
This model gives some insight into why a torque causes a tilt in the axis of rotation of a spinning body. Although the argument can be elaborated to apply to an extended body like a gyroscope, the final result is equivalent to using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

The discussion in this section applies to uniform precession, a very special case of gyroscope motion. We assumed at the beginning of our analysis that the gyroscope was executing this motion, but there are many other ways a gyroscope can move. For instance, if the free end of the axle is held at rest and suddenly released, the precessional velocity is instantaneously zero and the center of mass starts to fall. It is fascinating to see how this falling motion turns into uniform precession. We do this in Note 7.2 at the end of the chapter by a straightforward application of $\boldsymbol{\tau} = d\mathbf{L}/dt$. However, the treatment requires the general relation between $\mathbf{L}$ and $\boldsymbol{\omega}$ developed in Sec. 7.6.

7.4 Some Applications of Gyroscope Motion

In this section we present a few examples which show the application of angular momentum to rigid body motion.

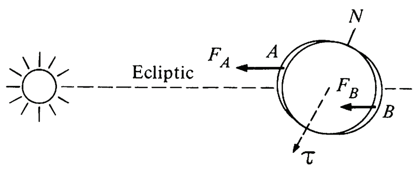
Example 7.9 Precession of the Equinoxes

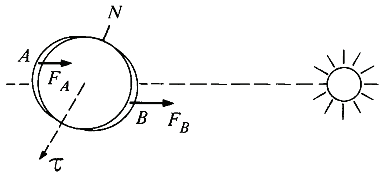


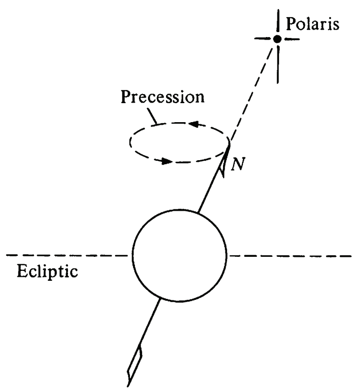
To a first approximation there are no torques on the earth and its angular momentum does not change in time. To this approximation, the earth's rotational speed is constant and its angular momentum always points in the same direction in space.

If we analyze the earth-sun system with more care, we find that there is a small torque on the earth. This causes the spin axis to slowly alter its direction, resulting in the phenomenon known as precession of the equinoxes.

The torque arises because of the interaction of the sun and moon with the nonspherical shape of the earth. The earth bulges slightly; its







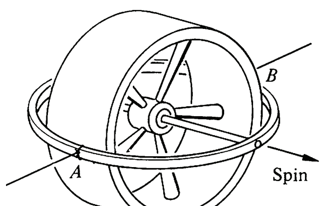
mean equatorial radius is 21 km greater than the polar radius. The gravitational force of the sun gives rise to a torque because the earth's axis of rotation is inclined with respect to the plane of the ecliptic (the orbital plane). During the winter, the part of the bulge above the ecliptic, A in the top sketch, is nearer the sun than the lower part B. The mass at A is therefore attracted more strongly by the sun than is the mass at B, as shown in the sketch. This results in a counterclockwise torque on the earth, out of the plane of the sketch. Six months later, when the earth is on the other side of the sun, B is attracted more strongly than A. However, the torque has the same direction in space as before. Midway between these extremes, the torque is zero. The average torque is perpendicular to the spin angular momentum and lies in the plane of the ecliptic. In a similar fashion, the moon exerts an average torque on the earth; this torque is about twice as great as that due to the sun.

The torque causes the spin axis to precess about a normal to the ecliptic. As the spin axis precesses, the torque remains perpendicular to it; the system acts like the gyroscope with tilted axis that we analyzed in Example 7.7.

The period of the precession is 26,000 years. 13,000 years from now, the polar axis will not point toward Polaris, the present north star; it will point $2 \times 23\frac{1}{2}° = 47°$ away. Orion and Sirius, those familiar winter guides, will then shine in the midsummer sky.

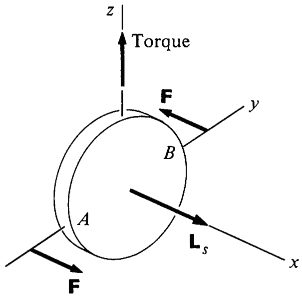
The spring equinox occurs at the instant the sun is directly over the equator in its apparent passage from south to north. Due to the precession of the earth's axis, the position of the sun at the equinox against the background of fixed stars shifts by 50 seconds of arc each year. This precession of the equinoxes was known to the ancients. It figures in the astrological scheme of cyclic history, which distinguishes twelve ages named by the constellation in which the sun lies at spring equinox. The present age is Pisces, and in 600 years it will be Aquarius.

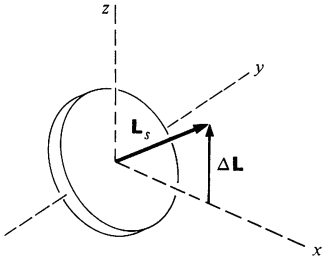
Example 7.10 The Gyrocompass Effect



Try the following experiment with a toy gyroscope. Tie strings to the frame of the gyroscope at points A and B on opposite sides midway between the bearings of the spin axis. Hold the strings taut at arm's length with the spin axis horizontal. Now slowly pivot so that the spinning gyroscope moves in a circle with arm length radius. The gyroscope suddenly flips and comes to rest with its spin axis vertical, parallel to your axis of rotation. Rotation in the opposite direction causes the gyro to flip by 180°, making its spin axis again parallel to the rotation axis. (The spin axis tends to oscillate about the vertical, but friction in the horizontal axle quickly damps this motion.)

The gyrocompass is based on this effect. A flywheel free to rotate about two perpendicular axes tends to orient its spin axis parallel to the axis of rotation of the system. In the case of a gyrocompass, the "sys-





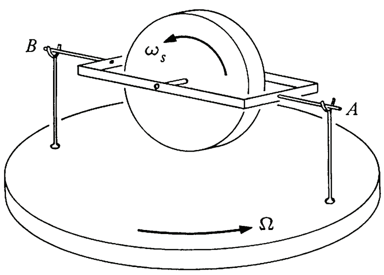
tem" is the earth; the compass comes to rest with its axis parallel to the polar axis.

We can understand the motion qualitatively by simple vector arguments. Assume that the axle is horizontal with $\mathbf{L}_s$ pointing along the x axis. Suppose that we attempt to turn the compass about the z axis. If we apply the forces shown, there is a torque along the z axis, τ_z, and the angular momentum along the z axis, L_z, starts to increase. If $\mathbf{L}_s$ were zero, L_z would be due entirely to rotation of the gyrocompass about the z axis: $L_z = I_z\omega_z$, where I_z is the moment of inertia about the z axis. However, when the flywheel is spinning, another way for L_z to change is for the gyrocompass to rotate around the AB axis, swinging $\mathbf{L}_s$ toward the z direction. Our experiment shows that if $\mathbf{L}_s$ is large, most of the torque goes into reorienting the spin angular momentum; only a small fraction goes toward rotating the gyrocompass about the z axis.

We can see why the effect is so pronounced by considering angular momentum along the y axis. The pivots at A and B allow the system to swing freely about the y axis, so there can be no torque along the y axis. Since L_y is initially zero, it must remain zero. As the gyrocompass starts to rotate about the z axis, L_s acquires a component in the y direction. At the same time, the gyrocompass and its frame begin to flip rapidly about the y axis. The angular momentum arising from this motion cancels the y component of $\mathbf{L}_s$. When $\mathbf{L}_s$ finally comes to rest parallel to the z axis, the motion of the frame no longer changes the direction of $\mathbf{L}_s$, and the spin axis remains stationary.

The earth is a rotating system, and a gyrocompass on the surface of the earth will line up with the polar axis, indicating true north. A practical gyrocompass is somewhat more complicated, however, since it must continue to indicate true north without responding to the motion of the ship or aircraft which it is guiding. In the next example we solve the dynamical equation for the gyrocompass and show how a gyrocompass fixed to the earth indicates true north.

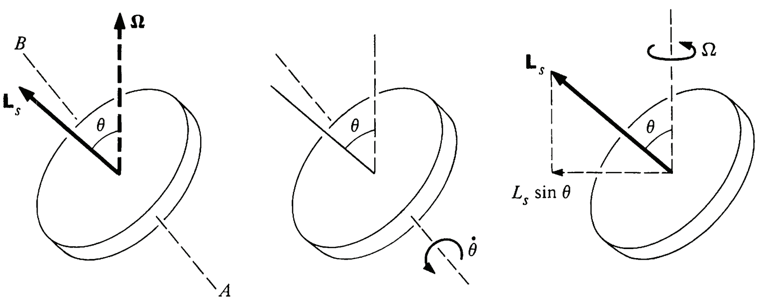
Example 7.11 Gyrocompass Motion



Consider a gyrocompass consisting of a balanced spinning disk held in a light frame supported by a horizontal axle. The assembly is on a turntable rotating at steady angular velocity Ω. The gyro has spin angular momentum $L_s = I_s\omega_s$ along the spin axis. In addition, it possesses angular momentum due to its bodily rotation about the vertical axis at rate Ω, and by virtue of rotation about the horizontal axle.

There cannot be any torque along the horizontal AB axis because that axle is pivoted. Hence, the angular momentum L_h along the AB direction is constant, and $dL_h/dt = 0$.

There are two contributions to dL_h/dt. If θ is the angle from the vertical to the spin axis, and $I_\perp$ is the moment of inertia about the AB axis, then $L_h = I_\perp\dot{\theta}$, and there is a contribution to dL_h/dt of $I_\perp\ddot{\theta}$.



In addition, L_h can change because of a change in direction of $\mathbf{L}_s$, as we have learned from analyzing the precessing gyroscope. The horizontal component of $\mathbf{L}_s$ is $L_s \sin\theta$, and its rate of increase along the AB axis is $\Omega L_s \sin\theta$.

We have considered the two changes in L_h independently. It is plausible that the total change in L_h is the sum of the two changes; a rigorous justification can be given based on arguments presented in Sec. 7.7.

Adding the two contributions to dL_h/dt gives

$$\frac{dL_h}{dt} = I_\perp\ddot{\theta} + \Omega L_s \sin\theta.$$

Since $dL_h/dt = 0$, the equation of motion becomes

$$\ddot{\theta} + \left(\frac{L_s\Omega}{I_\perp}\right)\sin\theta = 0.$$

This is identical to the equation for a pendulum discussed in Sec. 6.6. When the spin axis is near the vertical, $\sin\theta \approx \theta$ and the gyro executes simple harmonic motion in θ:

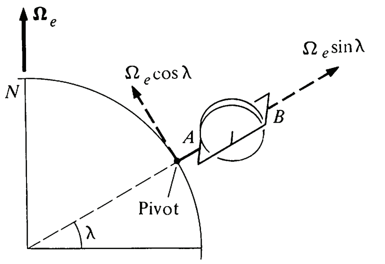
$$\theta = \theta_0 \sin\beta t$$

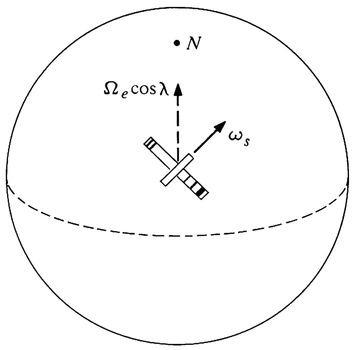
where

$$\beta = \sqrt{\frac{L_s\Omega}{I_\perp}} = \sqrt{\frac{\omega_s\Omega I_s}{I_\perp}}.$$

If there is a small amount of friction in the bearings at A and B, the amplitude of oscillation θ_0 will eventually become zero, and the spin axis comes to rest parallel to Ω.

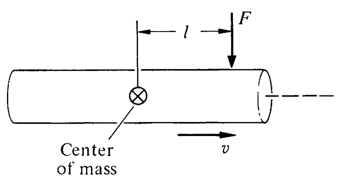
To use the gyro as a compass, fix it to the earth with the AB axle vertical, and the frame free to turn. As the drawing on the next page shows, if λ is the latitude of the gyro, the component of the earth's angular velocity Ω_e perpendicular to the AB axle is the horizontal component $\Omega_e \cos \lambda$. The spin axis oscillates in the horizontal plane about the direction of the north pole, and eventually comes to rest pointing north.

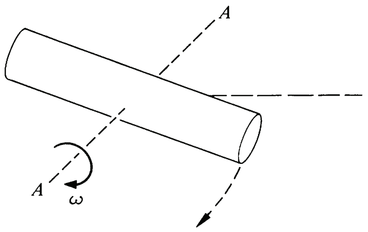




The period of small oscillations is $T = 2\pi/\beta = 2\pi\sqrt{I_\perp/(I_s\omega_s\Omega_e \cos \lambda)}$. For a thin disk $I_\perp/I_s = \frac{1}{2}$. $\Omega_e = 2\pi$ rad/day. With a gyro rotating at 20,000 rpm, the period at the equator is 11 s. Near the north pole the period becomes so long that the gyrocompass is not effective.

Example 7.12 The Stability of Rotating Objects





Angular momentum can make a freely moving object remarkably stable. For instance, spin angular momentum keeps a childs' rolling hoop upright even when it hits a bump; instead of falling, the hoop changes direction slightly and continues to roll. The effect of spin on a bullet provides another example. The spiral grooves, or rifling, in a gun's barrel give the bullet spin, which helps to stabilize it.

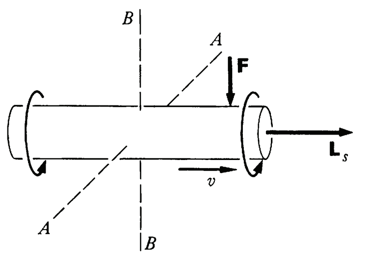
To analyze the effect of spin, consider a cylinder moving parallel to its axis. Suppose that a small perturbing force F acts on the cylinder for time Δt. F is perpendicular to the axis, and the point of application is a distance l from the center of mass.

We consider first the case where the cylinder has zero spin. The torque along the axis AA through the center of mass is $\tau = Fl$, and the "angular impulse" is $\tau\,\Delta t = Fl\,\Delta t$. The angular momentum acquired around the AA axis is

$$\Delta L_A = I_A(\omega - \omega_0) = Fl\,\Delta t.$$

Since ω_0, the initial angular velocity, is 0, the final angular velocity is given by

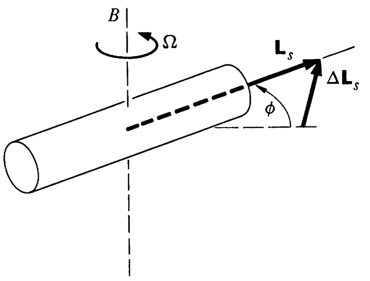
$$\omega = \frac{Fl\,\Delta t}{I_A}.$$



The effect of the blow is to give the cylinder angular velocity around the transverse axis; it starts to tumble.

Now consider the same situation, except that the cylinder is rapidly spinning with angular momentum $\mathbf{L}_s$. The situation is similar to that of the gyroscope: torque along the AA axis causes precession around the BB axis. The rate of precession while F acts is $dL_s/dt = \Delta L_s$, or

$$\Omega = \frac{Fl}{L_s}.$$



The angle through which the cylinder precesses is

$$\begin{aligned}\phi &= \Omega\,\Delta t\\ &= \frac{Fl\,\Delta t}{L_s}.\end{aligned}$$

Instead of starting to tumble, the cylinder slightly changes its orientation while the force is applied, and then stops precessing. The larger the spin, the smaller the angle and the less the effect of perturbations on the flight.

Note that spin has no effect on the center of mass motion. In both cases, the center of mass acquires velocity $\Delta\mathbf{v} = \mathbf{F}\,\Delta t/M$.

7.5 Conservation of Angular Momentum

Before tackling the general problem of rigid body motion, let us return to the question of whether or not the angular momentum of an isolated system is conserved. To start, we shall show that conservation of angular momentum does *not* follow from Newton's laws.

Consider a system of N particles with masses $m_1, m_2, \ldots, m_j, \ldots, m_N$. We assume that the system is isolated, so that the forces are due entirely to interactions between the particles. Let the force on particle j be

$$\mathbf{f}_j = \sum_{k=1}^{N} \mathbf{f}_{jk},$$

where $\mathbf{f}_{jk}$ is the force on particle j due to particle k. (In evaluating the sum, we can neglect the term with $k = j$, since $\mathbf{f}_{jj} = 0$, by Newton's third law.)

Let us choose an origin and calculate the torque $\boldsymbol{\tau}_j$ on particle j.

$$\begin{aligned}\boldsymbol{\tau}_j &= \mathbf{r}_j \times \mathbf{f}_j\\ &= \mathbf{r}_j \times \sum_k \mathbf{f}_{jk}.\end{aligned}$$

Let $\boldsymbol{\tau}_{jl}$ be the torque on j due to the particle l:

$$\boldsymbol{\tau}_{jl} = \mathbf{r}_j \times \mathbf{f}_{jl}.$$

Similarly, the torque on l due to j is

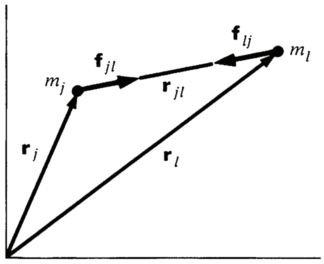
$$\boldsymbol{\tau}_{lj} = \mathbf{r}_l \times \mathbf{f}_{lj}.$$

The sum of these two torques is

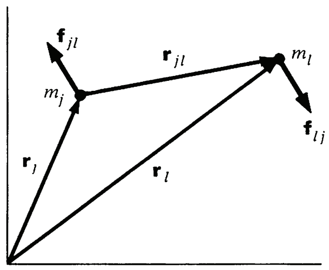
$$\boldsymbol{\tau}_{jl} + \boldsymbol{\tau}_{lj} = \mathbf{r}_l \times \mathbf{f}_{lj} + \mathbf{r}_j \times \mathbf{f}_{jl}.$$

Since $\mathbf{f}_{jl} = -\mathbf{f}_{lj}$, we have

$$\begin{aligned}\boldsymbol{\tau}_{jl} + \boldsymbol{\tau}_{lj} &= (\mathbf{r}_l \times \mathbf{f}_{lj}) - (\mathbf{r}_j \times \mathbf{f}_{lj}) \\ &= (\mathbf{r}_l - \mathbf{r}_j) \times \mathbf{f}_{lj} \\ &= \mathbf{r}_{jl} \times \mathbf{f}_{lj},\end{aligned}$$



(a)



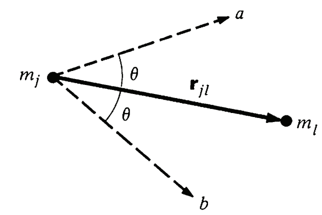
(b)

where $\mathbf{r}_{jl}$ is a vector from j to l. We would like to be able to prove that $\boldsymbol{\tau}_{jl} + \boldsymbol{\tau}_{lj} = 0$, since it would follow that the internal torques cancel in pairs, just as the internal forces do. The total internal torque would then be zero, proving that the angular momentum of an isolated system is conserved.

Since neither $\mathbf{r}_{jl}$ nor $\mathbf{f}_{lj}$ is zero, in order for the torque to vanish, $\mathbf{f}_{lj}$ must be parallel to $\mathbf{r}_{jl}$, as shown in figure (*a*). With respect to the situation in figure (*b*), however, the torque is not zero, and angular momentum is not conserved. Nevertheless, the forces are equal and opposite, and linear momentum is conserved.

The situation shown in figure (*a*) corresponds to the case of *central forces*, and we conclude that the conservation of angular momentum follows from Newton's laws in the case of central force motion. However, Newton's laws do not explicitly require forces to be central. We must conclude that Newton's laws have no direct bearing on whether or not the angular momentum of an isolated system is conserved, since these laws do not in themselves exclude the situation shown in figure (*b*).

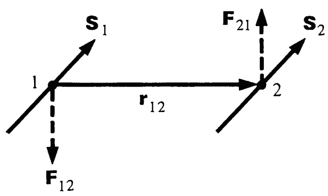
It is possible to take exception to the argument above on the following grounds: although Newton's laws do not explicitly require forces to be central, they implicitly make this requirement because in their simplest form Newton's laws deal with particles. Particles are idealized masses which have no size and no structure. In this case, the force between isolated particles must be central, since the only vector defined in a two particle system is the vector $\mathbf{r}_{jl}$ from one particle to the other. For instance, suppose that we try to invent a force which lies at angle θ with respect to the interparticle axis, as shown in the diagram. There is no way to dis-



tinguish direction a from b, however; both are at angle θ with respect to $\mathbf{r}_{jl}$. An angle-dependent force cannot be defined using only the single vector $\mathbf{r}_{jl}$; the force between the two particles must be central.

The difficulty in discussing angular momentum in the context of newtonian ideas is that our understanding of nature now encompasses entities vastly different from simple particles. As an example, perhaps the electron comes closest to the newtonian idea of a particle. The electron has a well-defined mass and, as far as present knowledge goes, zero radius. In spite of this, the electron has something analogous to internal structure; it possesses spin angular momentum. It is paradoxical that an object with zero size should have angular momentum, but we must accept this paradox as one of the facts of nature.

Because the spin of an electron defines an additional direction in space, the force between two electrons need not be central. As an example, there might be a force



$$\mathbf{F}_{12} = C\mathbf{r}_{12} \times (\mathbf{S}_1 + \mathbf{S}_2)$$
$$\mathbf{F}_{21} = C\mathbf{r}_{21} \times (\mathbf{S}_1 + \mathbf{S}_2),$$

where C is some constant and $\mathbf{S}_i$ is a vector parallel to the angular momentum of the ith electron. The forces are equal and opposite but not central, and they produce a torque.

There are other possibilities for noncentral forces. Experimentally, the force between two charged particles moving with respect to each other is not central; the velocity provides a second axis on which the force depends. The angular momentum of the two particles actually changes. The apparent breakdown of conservation of angular momentum is due to neglect of an important part of the system, the electromagnetic field. Although the concept of a field is alien to particle mechanics, it turns out that fields have mechanical properties. They can possess energy, momentum, and angular momentum. When the angular momentum of the field is taken into account, the angular momentum of the entire particle-field system is conserved.

The situation, in brief, is that newtonian physics is incapable of predicting conservation of angular momentum, but no isolated system has yet been encountered experimentally for which angular momentum is not conserved. We conclude that conservation of angular momentum is an independent physical law, and until a contradiction is observed, our physical understanding must be guided by it.

7.6 Angular Momentum of a Rotating Rigid Body

Angular Momentum and the Tensor of Inertia

The governing equation for rigid body motion, $\boldsymbol{\tau} = d\mathbf{L}/dt$, bears a formal resemblance to the translational equation of motion $\mathbf{F} = d\mathbf{P}/dt$. However, there is an essential difference between them. Linear momentum and center of mass motion are simply related by $\mathbf{P} = M\mathbf{V}$, but the connection between $\mathbf{L}$ and $\boldsymbol{\omega}$ is not so direct. For fixed axis rotation, $L = I\omega$, and it is tempting to suppose that the general relation is $\mathbf{L} = I\boldsymbol{\omega}$, where I is a scalar, that is, a simple number. However, this cannot be correct, since we know from our study of the rotating skew rod, Example 7.4, that $\mathbf{L}$ and $\boldsymbol{\omega}$ are not necessarily parallel.

In this section, we shall develop the general relation between angular momentum and angular velocity, and in the next section we shall attack the problem of solving the equations of motion.

As we discussed in Chap. 6, an arbitrary displacement of a rigid body can be resolved into a displacement of the center of mass plus a rotation about some instantaneous axis through the center of mass. The translational motion is easily treated. We start from the general expressions for the angular momentum and torque of a rigid body, Eqs. (6.11) and (6.14):

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j \qquad 7.3$$

$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} + \Sigma\mathbf{r}'_j \times \mathbf{f}_j, \qquad 7.4$$

where $\mathbf{r}'_j$ is the position vector of m_j relative to the center of mass. Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, we have

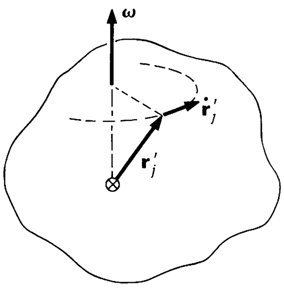
$$\begin{aligned}\mathbf{R} \times \mathbf{F} + \Sigma\mathbf{r}'_j \times \mathbf{f}_j &= \frac{d}{dt}(\mathbf{R} \times M\mathbf{V}) + \frac{d}{dt}(\Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j)\\ &= \mathbf{R} \times M\mathbf{A} + \frac{d}{dt}(\Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j).\end{aligned}$$

Since $\mathbf{F} = M\mathbf{A}$, the terms involving $\mathbf{R}$ cancel, and we are left with

$$\Sigma\mathbf{r}'_j \times \mathbf{f}_j = \frac{d}{dt}(\Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j). \qquad 7.5$$

The rotational motion can be found by taking torque and angular momentum about the center of mass, independent of the center of mass motion. The angular momentum $\mathbf{L}_0$ about the center of mass is

$$\mathbf{L}_0 = \Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j. \qquad 7.6$$



Our task is to express $\mathbf{L}_0$ in terms of the instantaneous angular velocity $\boldsymbol{\omega}$. Since $\mathbf{r}_j'$ is a rotating vector,

$$\dot{\mathbf{r}}_j' = \boldsymbol{\omega} \times \mathbf{r}_j'.$$

Therefore,

$$\mathbf{L}_0 = \Sigma \mathbf{r}_j' \times m_j(\boldsymbol{\omega} \times \mathbf{r}_j').$$

To simplify the notation, we shall write $\mathbf{L}$ for $\mathbf{L}_0$ and $\mathbf{r}_j$ for $\mathbf{r}_j'$. Our result becomes

$$\mathbf{L} = \Sigma \mathbf{r}_j \times m_j(\boldsymbol{\omega} \times \mathbf{r}_j). \qquad 7.7$$

This result looks complicated. As a matter of fact, it *is* complicated, but we can make it look simple. We will take the pedestrian approach of patiently evaluating the cross products in Eq. (7.7) using cartesian coordinates.[1]

Since $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$, we have

$$\boldsymbol{\omega} \times \mathbf{r} = (z\omega_y - y\omega_z)\hat{\mathbf{i}} + (x\omega_z - z\omega_x)\hat{\mathbf{j}} + (y\omega_x - x\omega_y)\hat{\mathbf{k}}. \qquad 7.8$$

Let us compute one component of $\mathbf{L}$, say L_x. Temporarily dropping the subscript j, we have

$$[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x = y(\boldsymbol{\omega} \times \mathbf{r})_z - z(\boldsymbol{\omega} \times \mathbf{r})_y. \qquad 7.9$$

If we substitute the results of Eq. (7.8) into Eq. (7.9), the result is

$$\begin{aligned}[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x &= y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x) \\ &= (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z. \qquad 7.10\end{aligned}$$

Hence,

$$L_x = \Sigma m_j(y_j^2 + z_j^2)\omega_x - \Sigma m_j x_j y_j \omega_y - \Sigma m_j x_j z_j \omega_z. \qquad 7.11$$

Let us introduce the following symbols:

$$\begin{aligned} I_{xx} &= \Sigma m_j(y_j^2 + z_j^2) \\ I_{xy} &= -\Sigma m_j x_j y_j \qquad 7.12 \\ I_{xz} &= -\Sigma m_j x_j z_j. \end{aligned}$$

I_{xx} is called a *moment of inertia*. It is identical to the moment of inertia introduced in the last chapter, $I = \Sigma m_j \rho_j^2$, provided that we take the axis in the x direction so that $\rho_j^2 = y_j^2 + z_j^2$. The quantities I_{xy} and I_{xz} are called *products of inertia*. They are symmetrical; for example, $I_{xy} = -\Sigma m_j x_j y_j = -\Sigma m_j y_j x_j = I_{yx}$.

To find L_y and L_z, we could repeat the derivation. However, a simpler method is to relabel the coordinates by letting $x \rightarrow y$,

[1] Another way is to use the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$.

$y \to z$, $z \to x$. If we make these substitutions in Eqs. (7.11) and (7.12), we obtain

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \qquad 7.13a$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \qquad 7.13b$$

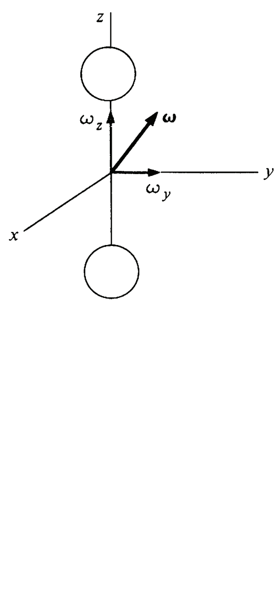
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z. \qquad 7.13c$$

This array of three equations is different from anything we have so far encountered. They include the results of the last chapter. For fixed axis rotation about the z direction, $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$ and Eq. (7.13c) reduces to

$$L_z = I_{zz}\omega$$
$$= \Sigma m_j(x_j^2 + y_j^2)\omega.$$

However, Eq. (7.13) also shows that angular velocity in the z direction can produce angular momentum about *any* of the three coordinate axes. For example, if $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, then $L_x = I_{xz}\omega$ and $L_y = I_{yz}\omega$. In fact, if we look at the set of equations for L_x, L_y, and L_z, we see that in each case the angular momentum about one axis depends on the angular velocity about *all three* axes. Both **L** and $\boldsymbol{\omega}$ are ordinary vectors, and **L** is proportional to $\boldsymbol{\omega}$ in the sense that doubling the components of $\boldsymbol{\omega}$ doubles the components of **L**. However, as we have already seen from the behavior of the rotating skew rod, Example 7.4, **L** does not necessarily point in the same direction as $\boldsymbol{\omega}$.

Example 7.13 Rotating Dumbbell



Consider a dumbbell made of two spheres of radius b and mass M separated by a thin rod. The distance between centers is $2l$. The body is rotating about some axis through its center of mass. At a certain instant the rod coincides with the z axis, and $\boldsymbol{\omega}$ lies in the yz plane, $\boldsymbol{\omega} = \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. What is **L**?

To find **L**, we need the moments and products of inertia. Fortunately, the products of inertia vanish for a symmetrical body lined up with the coordinate axes. For example, $I_{xy} = -\Sigma m_j x_j y_j = 0$, since for mass m_n located at (x_n,y_n) there is, in a symmetrical body, an equal mass located at $(x_n, -y_n)$; the contributions of these two masses to I_{xy} cancel. In this case Eq. (7.13) simplifies to

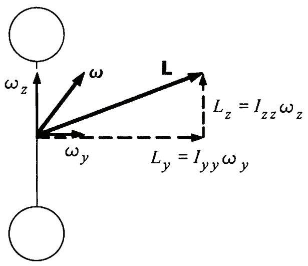
$$L_x = I_{xx}\omega_x$$
$$L_y = I_{yy}\omega_y$$
$$L_z = I_{zz}\omega_z.$$

The moment of inertia I_{zz} is just the moment of inertia of two spheres about their diameters.

$$I_{zz} = 2(\tfrac{2}{5}Mb^2) = \tfrac{4}{5}Mb^2.$$

In calculating I_{yy}, we can use the parallel axis theorem to find the moment of inertia of each sphere about the y axis.

$$\begin{aligned} I_{yy} &= 2(\tfrac{2}{5}Mb^2 + Ml^2) \\ &= \tfrac{4}{5}Mb^2 + 2Ml^2. \end{aligned}$$



We have assumed that the rod has negligible mass.

Since $\boldsymbol{\omega} = \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$,

$$\begin{aligned} L_x &= 0 \\ L_y &= I_{yy}\omega_y \\ L_z &= I_{zz}\omega_z. \end{aligned}$$

I_{yy} and I_{zz} are not equal; therefore $L_y/L_z \neq \omega_y/\omega_z$ and **L** is not parallel to $\boldsymbol{\omega}$, as the drawing shows.

Equations (7.13) are cumbersome, so that it is more convenient to write them in the following shorthand notation.

$$\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega}. \qquad 7.14$$

This vector equation represents three equations, just as $\mathbf{F} = m\mathbf{a}$ represents three equations. The difference is that m is a simple scalar while $\tilde{\mathbf{I}}$ is a more complicated mathematical entity called a *tensor*. $\tilde{\mathbf{I}}$ is the *tensor of inertia*.

We are accustomed to displaying the components of some vector **A** in the form

$$\mathbf{A} = (A_x,A_y,A_z).$$

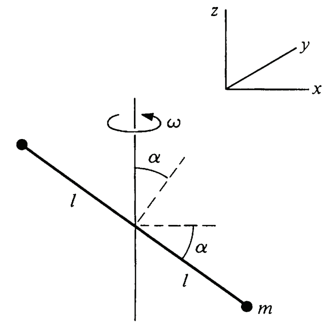
Similarly, the nine components of $\tilde{\mathbf{I}}$ can be tabulated in a 3×3 array:

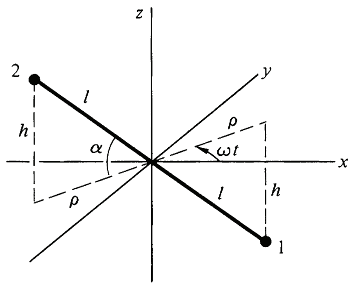
$$\tilde{\mathbf{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}. \qquad 7.15$$

Of the nine components, only six at most are different, since $I_{yx} = I_{xy}$, $I_{zx} = I_{xz}$, and $I_{yz} = I_{zy}$. The rule for multiplying $\boldsymbol{\omega}$ by $\tilde{\mathbf{I}}$ to find $\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega}$ is defined by Eq. (7.13).

The following example illustrates the tensor of inertia.

Example 7.14 **The Tensor of Inertia for a Rotating Skew Rod**





We found the angular momentum of a rotating skew rod from first principles in Example 7.3. Let us now find **L** for the same device by using $\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega}$.

A massless rod of length $2l$ separates two equal masses m. The rod is skewed at angle α with the vertical, and rotates around the z axis with angular velocity ω. At $t = 0$ it lies in the xz plane. The coordinates of the particles at any other time are:

Particle 1	Particle 2
$x_1 = \rho \cos \omega t$	$x_2 = -\rho \cos \omega t$
$y_1 = \rho \sin \omega t$	$y_2 = -\rho \sin \omega t$
$z_1 = -h$	$z_2 = h,$

when $\rho = l \cos \alpha$ and $h = l \sin \alpha$.

The components of $\tilde{\mathbf{I}}$ can now be calculated from their definitions. For instance,

$$
\begin{aligned}
I_{xx} &= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) \\
&= 2m(\rho^2 \sin^2 \omega t + h^2) \\
I_{zy} &= I_{yz} \\
&= -m_1 y_1 z_1 - m_2 y_2 z_2 \\
&= 2m\rho h \sin \omega t.
\end{aligned}
$$

The remaining terms are readily evaluated. We find:

$$
\mathbf{I} = 2m \begin{pmatrix} \rho^2 \sin^2 \omega t + h^2 & -\rho^2 \sin \omega t \cos \omega t & \rho h \cos \omega t \\ -\rho^2 \sin \omega t \cos \omega t & \rho^2 \cos^2 \omega t + h^2 & \rho h \sin \omega t \\ \rho h \cos \omega t & \rho h \sin \omega t & \rho^2 \end{pmatrix}.
$$

The common factor $2m$ multiplies each term.

Since $\boldsymbol{\omega} = (0,0,\omega)$, we have, from Eq. (7.13),

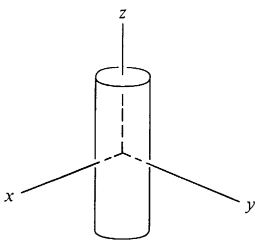
$$
\begin{aligned}
L_x &= 2m\rho h\omega \cos \omega t \\
L_y &= 2m\rho h\omega \sin \omega t \\
L_z &= 2m\rho^2\omega.
\end{aligned}
$$

We can differentiate **L** to find the applied torque:

$$
\begin{aligned}
\tau_x &= -2m\rho h\omega^2 \sin \omega t \\
\tau_y &= 2m\rho h\omega^2 \cos \omega t \\
\tau_z &= 0.
\end{aligned}
$$

The results are identical to those in Example 7.4, provided that we make the substitution $\rho h = l^2 \cos \alpha \sin \alpha$.

Principal Axes

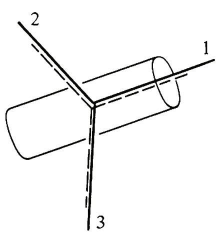


If the symmetry axes of a uniform symmetric body coincide with the coordinate axes, the products of inertia are zero, as we saw in Example 7.13. In this case the tensor of inertia takes a simple diagonal form:

$$\tilde{\mathbf{I}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \qquad 7.16$$

Remarkably enough, for a body of any shape and mass distribution, it is *always* possible to find a set of three orthogonal axes such that the products of inertia vanish. (The proof uses matrix algebra and is given in most texts on advanced dynamics.) Such axes are called *principal axes*. The tensor of inertia with respect to principal axes has a diagonal form.

For a uniform sphere, any perpendicular axes through the center are principal axes. For a body with cylindrical symmetry, the axis of revolution is a principal axis. The other two principal axes are mutually perpendicular and lie in a plane through the center of mass perpendicular to the axis of revolution.



Consider a rotating rigid body, and suppose that we introduce a coordinate system 1, 2, 3 which coincides instantaneously with the principal axes of the body. With respect to this coordinate system, the instantaneous angular velocity has components ω_1, ω_2, ω_3, and the components of **L** have the simple form

$$\begin{aligned} L_1 &= I_1\omega_1 \\ L_2 &= I_2\omega_2 \\ L_3 &= I_3\omega_3, \end{aligned} \qquad 7.17$$

where I_1, I_2, I_3 are the moments of inertia about the principal axes. In Sec. 7.7, we shall exploit Eq. (7.17) in our attack on the problem of rigid body dynamics.

Rotational Kinetic Energy

The kinetic energy of a rigid body is

$$K = \tfrac{1}{2}\Sigma m_j v_j^2.$$

To separate the translational and rotational contributions, we introduce center of mass coordinates:

$$\begin{aligned} \mathbf{r}_j &= \mathbf{R} + \mathbf{r}_j' \\ \mathbf{v}_j &= \mathbf{V} + \mathbf{v}_j'. \end{aligned}$$

We have

$$K = \tfrac{1}{2}\Sigma m_j(\mathbf{V} + \mathbf{v}_j')^2$$
$$= \tfrac{1}{2}MV^2 + \tfrac{1}{2}\Sigma m_j v_j'^2,$$

since the cross term $\mathbf{V} \cdot \Sigma m_j \mathbf{v}_j'$ is zero.

Using $\mathbf{v}_j' = \boldsymbol{\omega} \times \mathbf{r}_j'$, the kinetic energy of rotation becomes

$$K_{\text{rot}} = \tfrac{1}{2}\Sigma m_j \mathbf{v}_j'^2$$
$$= \tfrac{1}{2}\Sigma m_j(\boldsymbol{\omega} \times \mathbf{r}_j') \cdot (\boldsymbol{\omega} \times \mathbf{r}_j').$$

The right hand side can be simplified with the vector identity $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Let $\mathbf{A} = \boldsymbol{\omega}$, $\mathbf{B} = \mathbf{r}_j'$, and $\mathbf{C} = \boldsymbol{\omega} \times \mathbf{r}_j'$. We obtain

$$K_{\text{rot}} = \tfrac{1}{2}\Sigma m_j \boldsymbol{\omega} \cdot [\mathbf{r}_j' \times (\boldsymbol{\omega} \times \mathbf{r}_j')]$$
$$= \tfrac{1}{2}\boldsymbol{\omega} \cdot \Sigma m_j \mathbf{r}_j' \times (\boldsymbol{\omega} \times \mathbf{r}_j').$$

The sum in the last term is the angular momentum $\mathbf{L}$ by Eq. (7.7). Therefore,

$$K_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \qquad 7.18$$

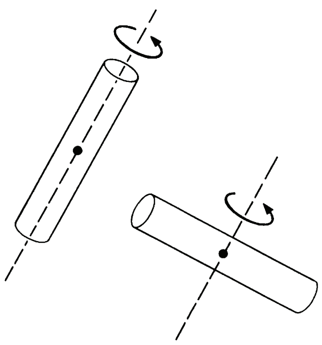
Rotational kinetic energy has a simple form when $\mathbf{L}$ and $\boldsymbol{\omega}$ are referred to principal axes. Using Eqs. (7.17) and (7.18) we have

$$K_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$$
$$= \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2. \qquad 7.19$$

Alternatively,

$$K_{\text{rot}} = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}. \qquad 7.20$$

Example 7.15 Why Flying Saucers Make Better Spacecraft than Do Flying Cigars



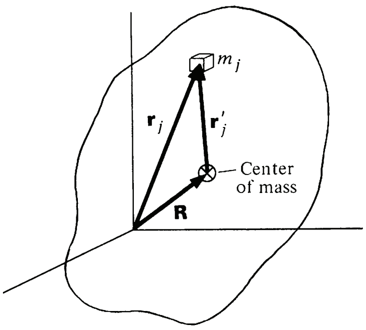
One of the early space satellites was cylindrical in shape and was put into orbit spinning around its long axis. To the designer's surprise, even though the spacecraft was torque-free, it began to wobble more and more, until finally it was spinning around a transverse axis.

The reason is that although $\mathbf{L}$ is strictly conserved for torque-free motion, kinetic energy of rotation can change if the body is not absolutely rigid. If the satellite is rotating slightly off the symmetry axis, each part of the body undergoes a time varying centripetal acceleration. The spacecraft warps and bends under the time varying force, and energy is dissipated by internal friction in the structure. The kinetic energy of rotation must therefore decrease. From Eq. (7.20), if the body is rotating about a single principal axis, $K_{\text{rot}} = L^2/2I$. K_{rot} is a minimum for the axis with greatest moment of inertia, and the motion is stable around that axis. For the cylindrical spacecraft, the initial axis of rotation had the minimum moment of inertia, and the motion was not stable.

A thin disk spinning about its cylindrical axis is inherently stable because the other two moments of inertia are only half as large. A cigar-shaped craft is unstable about its long axis and only neutrally stable about the transverse axes; there is no single axis of maximum moment of inertia.

Rotation about a Fixed Point

We showed at the beginning of this section that in analyzing the motion of a rotating and translating rigid body it is always correct to calculate torque and angular momentum about the center of mass. In some applications, however, one point of a body is fixed in space, like the pivot point of a gyroscope on a pylon. It is often convenient to analyze the motion using the fixed point as origin, since the center of mass motion need not be considered explicitly, and the constraint force at the pivot produces no torque.



Taking the origin at the fixed point, let $\mathbf{r}_j$ be the position vector of particle m_j and let $\mathbf{R} = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}} + Z\hat{\mathbf{k}}$ be the position vector of the center of mass. The torque about the origin is

$$\boldsymbol{\tau} = \Sigma \mathbf{r}_j \times \mathbf{f}_j,$$

where $\mathbf{f}_j$ is the force on m_j. If the angular velocity of the body is $\boldsymbol{\omega}$, the angular momentum about the origin is

$$\mathbf{L} = \Sigma \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$$
$$= \Sigma \mathbf{r}_j \times m_j(\boldsymbol{\omega} \times \mathbf{r}_j).$$

This has the same form as Eq. (7.6), which we evaluated earlier in this section. Taking over the results wholesale, we have

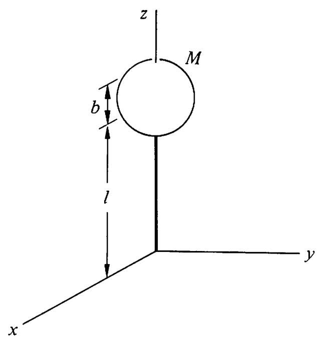
$$\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega}$$

where

$$I_{xx} = \Sigma m_j(y_j^2 + z_j^2)$$
$$I_{xy} = -\Sigma m_j x_j y_j$$

etc.

Although this result is identical in form to Eq. (7.13), the components of $\tilde{\mathbf{I}}$ are now calculated with respect to the pivot point rather than the center of mass.



Once the tensor of inertia about the center of mass, $\tilde{\mathbf{I}}_0$, is known, $\tilde{\mathbf{I}}$ about any other origin can be found from a generalization of the parallel axis theorem of Example 6.9. Typical results, the proof of which we leave as a problem, are

$$I_{xx} = (I_0)_{xx} + M(Y^2 + Z^2)$$
$$I_{xy} = (I_0)_{xy} - MXY$$
etc. 7.21

Consider, for example, a sphere of mass M and radius b centered on the z axis a distance l from the origin. We have $I_{xx} = \frac{2}{5}Mb^2 + Ml^2$, $I_{yy} = \frac{2}{5}Mb^2 + Ml^2$, $I_{zz} = \frac{2}{5}Mb^2$.

7.7 Advanced Topics in the Dynamics of Rigid Body Rotation

Introduction

In this section we shall attack the general problem of rigid body rotation. However, none of the results will be needed in subsequent chapters, and the section can be skipped without loss of continuity.

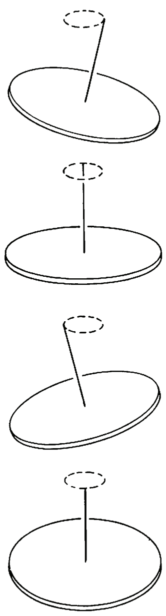
The fundamental problem of rigid body dynamics is to find the orientation of a rotating body as a function of time, given the torque. The problem is difficult because of the complicated relation $\mathbf{L} = \tilde{\mathbf{I}}\boldsymbol{\omega}$ between angular momentum and angular velocity. We can make the problem look simpler by taking our coordinate system coincident with the principal axes of the body. With respect to principal axes, the tensor of inertia $\tilde{\mathbf{I}}$ is diagonal in form, and the components of $\mathbf{L}$ are

$$L_x = I_{xx}\omega_x$$
$$L_y = I_{yy}\omega_y$$
$$L_z = I_{zz}\omega_z.$$

However, the crux of the problem is that the principal axes are fixed to the body, whereas we need the components of $\mathbf{L}$ with respect to axes having a fixed orientation in space. As the body rotates, its principal axes move out of coincidence with the space-fixed system. The products of inertia are no longer zero in the space-fixed system and, worse yet, the components of $\tilde{\mathbf{I}}$ vary with time.

The situation appears hopelessly tangled, but if the principal axes do not stray far from the space-fixed system, we can find the motion using simple vector arguments. Leaving the general case for later, we illustrate this approach by finding the torque-free motion of a rigid body.

Torque-free Precession: Why the Earth Wobbles

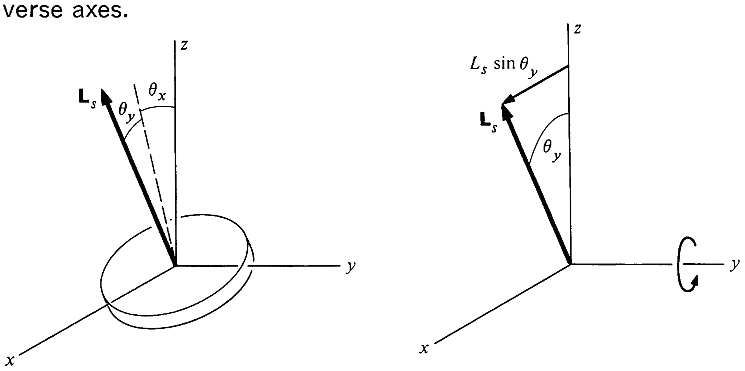


If you drop a spinning quarter with a slight flip, it will fall with a wobbling motion; the symmetry axis tends to rotate in space, as the sketch shows. Since there are no torques, the motion is known as torque-free precession.

Torque-free precession is a characteristic mode of rigid body motion. For example, the spin axis of the earth moves around the polar axis because of this effect. The physical explanation of the wobbling motion is related to our observation that **L** need not be parallel to $\boldsymbol{\omega}$. If there are no torques on the body, **L** is fixed in space, and $\boldsymbol{\omega}$ must move, as will be shown.

To avoid mathematical complexity, consider the special case of a cylindrically symmetric rigid body like a coin or an air suspension gyroscope. We shall assume that the precessional motion is small in amplitude, in order to apply small angle approximations.

Suppose that the body has a large spin angular momentum $L = I_s\omega_s$ along the main symmetry axis, where I_s is the moment of inertia and ω_s is the angular velocity about the symmetry axis. Let the body have small angular velocities about the other transverse axes.



Suppose that $\mathbf{L}_s$ is always close to the z axis and makes angles $\theta_x \ll 1$ and $\theta_y \ll 1$ with the x and y axes. Note 7.1 on infinitesimal rotations shows that to first order, rotations about each axis can be considered separately. The contribution to L_x from rotation about the x axis is $L_x = d(I_{xx}\theta_x)/dt = I_{xx}\,d\theta_x/dt$. We have treated I_{xx} as a constant. The justification is that moments of inertia about principal axes are constant to first order for small angular displacements. (The proof is left as a problem.) Rotation about y also contributes to L_x by giving L_s a component $L_s \sin \theta_y$ in the x direction. Adding the two contributions, we have

$$L_x = I_{xx} \frac{d\theta_x}{dt} + L_s \sin \theta_y.$$

Similarly,

$$L_y = I_{yy} \frac{d\theta_y}{dt} - L_s \sin \theta_x.$$

By symmetry, $I_{xx} = I_{yy} \equiv I_\perp$. For small angles, $\sin \theta = \theta$ and $\cos \theta = 1$, to first order. Hence

$$L_x = I_\perp \frac{d\theta_x}{dt} + L_s\theta_y \qquad 7.22a$$

$$L_y = I_\perp \frac{d\theta_y}{dt} - L_s\theta_x. \qquad 7.22b$$

To the same order of approximation,

$$\begin{aligned} L_z &= L_s \\ &= I_s\omega_s. \end{aligned} \qquad 7.23$$

Since the torque is zero, $d\mathbf{L}/dt = 0$. Equation (7.23) then gives L_s = constant, ω_s = constant, and Eqs. (7.22) yield

$$I_\perp \frac{d^2\theta_x}{dt^2} + L_s \frac{d\theta_y}{dt} = 0 \qquad 7.24a$$

$$I_\perp \frac{d^2\theta_y}{dt^2} - L_s \frac{d\theta_x}{dt} = 0. \qquad 7.24b$$

If we let $\omega_x = d\theta_x/dt$, $\omega_y = d\theta_y/dt$, Eqs. (7.24) become

$$I_\perp \frac{d\omega_x}{dt} + L_s\omega_y = 0 \qquad 7.25a$$

$$I_\perp \frac{d\omega_y}{dt} - L_s\omega_x = 0. \qquad 7.25b$$

If we differentiate Eq. (7.25a) and substitute the value for $d\omega_y/dt$ in Eq. (7.25b), we obtain

$$\frac{I_\perp{}^2}{L_s} \frac{d^2\omega_x}{dt^2} + L_s\omega_x = 0$$

or

$$\frac{d^2\omega_x}{dt^2} + \gamma^2\omega_x = 0, \qquad 7.26$$

where

$$\gamma = \frac{L_s}{I_\perp} = \omega_s \frac{I_s}{I_\perp}.$$

Equation (7.26) is the familiar equation for simple harmonic motion. The solution is

$$\omega_x = A \sin (\gamma t + \phi), \qquad 7.27$$

where A and ϕ are arbitrary constants. Substituting this in Eq. (7.25a) gives

$$\omega_y = -\frac{I_\perp}{L_s}\frac{d\omega_x}{dt} = \frac{I_\perp}{I_s\omega_s} A\gamma \cos (\gamma t + \phi),$$

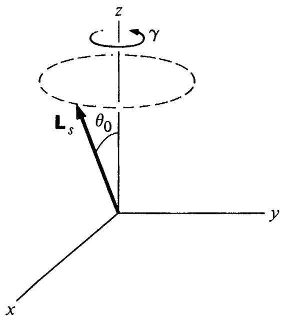
or

$$\omega_y = A \cos (\gamma t + \phi). \qquad 7.28$$

By integrating Eqs. (7.27) and (7.28) we obtain

$$\theta_x = \frac{A}{\gamma} \cos (\gamma t + \phi) + \theta_{x0}$$

$$\theta_y = -\frac{A}{\gamma} \sin (\gamma t + \phi) + \theta_{y0}, \qquad 7.29$$



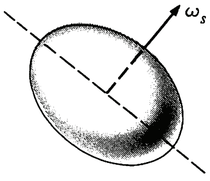
where θ_{x0} and θ_{y0} are constants of integration. The first terms of Eq. (7.29) reveal that the axis rotates around a fixed direction in space. If we take that direction along the z axis, then $\theta_{x0} = \theta_{y0} = 0$. Assuming that at $t = 0$ $\theta_x = \theta_0$, $\theta_y = 0$, we have

$$\theta_x = \theta_0 \cos \gamma t$$

$$\theta_y = \theta_0 \sin \gamma t, \qquad 7.30$$

where we have taken $A/\gamma = \theta_0$, $\phi = 0$.

Equation (7.30) describes torque-free precession. The frequency of the precessional motion is $\gamma = \omega_s I_s/I_\perp$. For a body flattened along the axis of symmetry, such as the oblate spheroid



shown, $I_s > I_\perp$ and $\gamma > \omega_s$. For a thin coin, $I_s = 2I_\perp$ and $\gamma = 2\omega_s$. Thus, the falling quarter described earlier wobbles twice as fast as it spins.

The earth is an oblate spheroid and exhibits torque-free precession. The amplitude of the motion is small; the spin axis wanders about the polar axis by about 5 m at the North Pole. Since the earth itself is spinning, the apparent rate of precession to an earthbound observer is

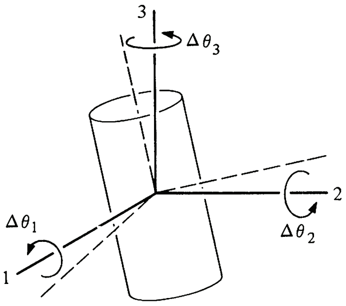
$$\begin{aligned}\gamma' &= \gamma - \omega_s \\ &= \omega_s\left(\frac{I_s - I_\perp}{I_\perp}\right). \end{aligned} \qquad 7.31$$

For the earth, $(I_s - I_\perp)/I_\perp = \frac{1}{300}$, and the precessional motion should have a period of 300 days. However, the motion is quite irregular with an apparent period of about 430 days. The fluctuations arise from the elastic nature of the earth, which is significant for motions this small.

Note 7.2 on the nutating gyroscope illustrates another application of the small angle approximation that we have used.

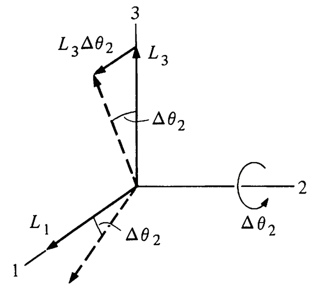
Euler's Equations

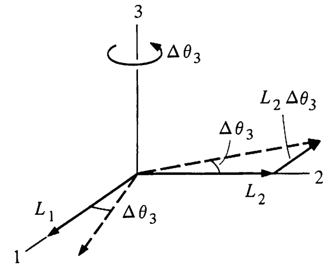
We turn now to the task of deriving the exact equations of motion for a rigid body. In order to find $d\mathbf{L}/dt$, we shall calculate the change in the components of $\mathbf{L}$ in the time interval from t to $t + \Delta t$, using the small angle approximation. The results are correct only to first order, but they become exact when we take the limit $\Delta t \to 0$.



Let us introduce an inertial coordinate system which coincides with the instantaneous position of the body's principal axes at time t. We label the axes of the inertial system 1, 2, 3. Let the components of the angular velocity $\boldsymbol{\omega}$ at time t relative to the 1, 2, 3 system be ω_1, ω_2, ω_3. At the same instant, the components of $\mathbf{L}$ are $L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$, $L_3 = I_3\omega_3$, where I_1, I_2, I_3 are the moments of inertia about the three principal axes.

In the time interval Δt, the principal axes rotate away from the 1, 2, 3 axes. To first order, the rotation angle about the 1 axis is $\Delta\theta_1 = \omega_1\,\Delta t$; similarly, $\Delta\theta_2 = \omega_2\,\Delta t$, $\Delta\theta_3 = \omega_3\,\Delta t$. The corresponding change $\Delta L_1 = L_1(t + \Delta t) - L_1(t)$ can be found to first order by treating the three rotations one by one, according to Note 7.1 on infinitesimal rotations. There are two ways L_1 can change. If ω_1 varies, $I_1\omega_1$ will change. In addition, rotations about the





other two axes cause L_2 and L_3 to change direction, and this can contribute to angular momentum along the first axis.

The first contribution to ΔL_1 is from $\Delta(I_1\omega_1)$. Since the moments of inertia are constant to the first order for small angular displacements about the principal axes, $\Delta(I_1\omega_1) = I_1\,\Delta\omega_1$.

To find the remaining contributions to ΔL_1, consider first rotation about the 2 axis through angle $\Delta\theta_2$. This causes L_1 and L_3 to rotate as shown. The rotation of L_1 causes no change along the 1 axis to first order. However, the rotation of L_3 contributes $L_3\,\Delta\theta_2 = I_3\omega_3\,\Delta\theta_2$ along the 1 axis. Similarly, rotation about the 3 axis contributes $-L_2\,\Delta\theta_3 = -I_2\omega_2\,\Delta\theta_3$ to ΔL_1.

Adding all the contributions gives

$$\Delta L_1 = I_1\,\Delta\omega_1 + I_3\omega_3\,\Delta\theta_2 - I_2\omega_2\,\Delta\theta_3.$$

Dividing by Δt and taking the limit $\Delta t \to 0$ yields

$$\frac{dL_1}{dt} = I_1\frac{d\omega_1}{dt} + (I_3 - I_2)\omega_3\omega_2.$$

The other components can be treated in a similar fashion, or we can simply relabel the subscripts by $1 \to 2$, $2 \to 3$, $3 \to 1$. We find

$$\frac{dL_2}{dt} = I_2\frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3$$

$$\frac{dL_3}{dt} = I_3\frac{d\omega_3}{dt} + (I_2 - I_1)\omega_2\omega_1.$$

Since $\boldsymbol{\tau} = d\mathbf{L}/dt$,

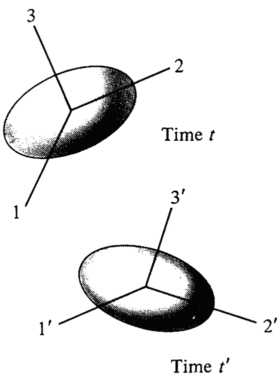
$$\tau_1 = I_1\frac{d\omega_1}{dt} + (I_3 - I_2)\omega_3\omega_2$$

$$\tau_2 = I_2\frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 \qquad 7.32$$

$$\tau_3 = I_3\frac{d\omega_3}{dt} + (I_2 - I_1)\omega_2\omega_1,$$

where τ_1, τ_2, τ_3 are the components of $\boldsymbol{\tau}$ along the axes of the inertial system 1, 2, 3. These equations were derived by Euler in the middle of the eighteenth century and are known as Euler's equations of rigid body motion.

Euler's equations are tricky to apply; thus, it is important to understand what they mean. At some time t we set up the 1,



2, 3 inertial system to coincide with the instantaneous directions of the body's principal axes. τ_1, τ_2, τ_3 are the components of torque along the 1, 2, 3 axes at time t. Similarly, ω_1, ω_2, ω_3 are the components of $\boldsymbol{\omega}$ along the 1, 2, 3 axes at time t, and $d\omega_1/dt$, $d\omega_2/dt$, $d\omega_3/dt$ are the instantaneous rates of change of these components. Euler's equations relate these quantities at time t. To apply Euler's equations at another time t', we have to resolve $\boldsymbol{\tau}$ and $\boldsymbol{\omega}$ along the axes of a new inertial system 1′, 2′, 3′ which coincides with the principal axes at t'.

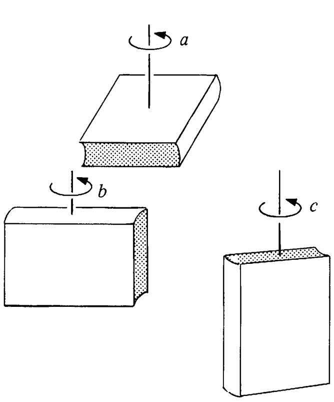
The difficulty is that Euler's equations do not show us how to find the orientation of these coordinate systems in space. Essentially, we have traded one problem for another; in the familiar x, y, z laboratory coordinate system, we know the disposition of the axes, but the components of the tensor of inertia vary in an unknown way. In the 1, 2, 3 system, the components of $\tilde{\mathbf{I}}$ are constant, but we do not know the orientation of the axes. Euler's equations cannot be integrated directly to give angles specifying the orientation of the body relative to the x, y, z laboratory system. Euler overcame this difficulty by expressing ω_1, ω_2, ω_3 in terms of a set of angles relating the principal axes to the axes of the x, y, z laboratory system.

In terms of these angles, Euler's equations are a set of coupled differential equations. The general equations are fairly complicated and are discussed in advanced texts. Fortunately, in many important applications we can find the motion from Euler's equations by using straightforward geometrical arguments. Here are a few examples.

Example 7.16 Stability of Rotational Motion

In principle, a pencil can be balanced on its point. In practice, the pencil falls almost immediately. Although a perfectly balanced pencil is in equilibrium, the equilibrium is not stable. If the pencil starts to tip because of some small perturbing force, the gravitational torque causes it to tip even further; the system continues to move away from equilibrium. A system is stable if displacement from equilibrium gives rise to forces which drive it back toward equilibrium. Similarly, a moving system is stable if it responds to a perturbing force by altering its motion only slightly. In contrast, an unstable system can have its motion drastically changed by a small perturbing force, possibly leading to catastrophic failure.

A rotating rigid body can exhibit either stable or unstable motion depending on the axis of rotation. The motion is stable for rotation about the axes of maximum or minimum moment of inertia but unstable for rotation about the axis with intermediate moment of inertia. The effect is easy to show: wrap a book with a rubber band and let it fall spinning about each of its principal axes in turn. I is maximum about axis



a and minimum about axis c; the motion is stable if the book is spun about either of these axes. However, if the book is spun about axis b, it tends to flop over as it spins, generally landing on its broad side.

To explain this behavior, we turn to Euler's equations. Suppose that the body is initially spinning with $\omega_1 =$ constant and $\omega_2 = 0$, $\omega_3 = 0$, and that immediately after a short perturbation, ω_2 and ω_3 are different from zero but very small compared with ω_1. Once the perturbation ends, the motion is torque-free and Euler's equations are:

$$I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 = 0 \qquad 1$$

$$I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3 = 0 \qquad 2$$

$$I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 = 0. \qquad 3$$

Since ω_2 and ω_3 are very small at first, we can initially neglect the second term in Eq. (1). Therefore $I_1\, d\omega_1/dt = 0$, and ω_1 is constant.

If we differentiate Eq. (2) and substitute the value of $d\omega_3/dt$ from Eq. (3), we have

$$I_2 \frac{d^2\omega_2}{dt^2} - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3}\omega_1^2\omega_2 = 0$$

or

$$\frac{d^2\omega_2}{dt^2} + A\omega_2 = 0 \qquad 4$$

where

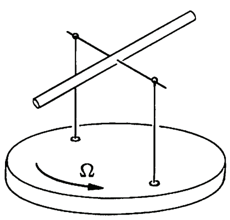
$$A = \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2.$$

If I_1 is the largest or the smallest moment of inertia, $A > 0$ and Eq. (4) is the equation for simple harmonic motion. ω_2 oscillates at frequency $\sqrt{A}$ with bounded amplitude. It is easy to show that ω_3 also undergoes simple harmonic motion. Since ω_2 and ω_3 are bounded, the motion is stable. (It corresponds to the torque-free precession we calculated earlier.)

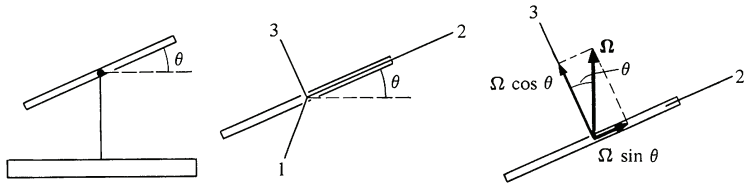
If I_1 is the intermediate moment of inertia, $A < 0$. In this case ω_2 and ω_3 tend to increase exponentially with time, and the motion is unstable.

Example 7.17 The Rotating Rod

Consider a uniform rod mounted on a horizontal frictionless axle through its center. The axle is carried on a turntable revolving with constant angular velocity Ω, with the center of the rod over the axis of the turntable. Let θ be the angle shown in the sketch. The problem is to find θ as a function of time.



To apply Euler's equations, let principal axis 1 of the rod be along the axle, principal axis 2 be along the length of the rod, and principal axis 3 be in the vertical plane perpendicular to the rod. $\omega_1 = \dot{\theta}$, and by resolving Ω along the 2 and 3 directions we find $\omega_2 = \Omega \sin \theta$, $\omega_3 = \Omega \cos \theta$.



Since there is no torque about the 1 axis, the first of Euler's equations gives

$$I_1\ddot{\theta} + (I_3 - I_2)\Omega^2 \sin \theta \cos \theta = 0$$

or

$$2\ddot{\theta} + \left(\frac{I_3 - I_2}{I_1}\right)\Omega^2 \sin 2\theta = 0. \qquad 1$$

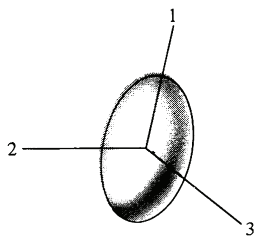
(We have used $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$.)

Since $I_3 > I_2$, this is the equation for pendulum motion in the variable 2θ. For oscillations near the horizontal, $\sin 2\theta \approx 2\theta$ and Eq. (1) becomes

$$\ddot{\theta} + \left(\frac{I_3 - I_2}{I_1}\right)\Omega^2\theta = 0.$$

The motion is simple harmonic with angular frequency $\sqrt{(I_3 - I_2)/I_1}\,\Omega$.

Example 7.18 Euler's Equations and Torque-free Precession



We discussed the torque-free motion of a cylindrically symmetric body earlier using the small angle approximation. In this example we shall obtain an exact solution by using Euler's equations.

Let the axis of cylindrical symmetry be principal axis 1 with moment of inertia I_1. The other two principal axes are perpendicular to the 1 axis, and $I_2 = I_3 = I_\perp$. From the first of Euler's equations

$$\tau_1 = I_1(d\omega_1/dt) + (I_3 - I_2)\omega_2\omega_3,$$

we have

$$0 = I_1 \frac{d\omega_1}{dt},$$

which gives

$$\omega_1 = \text{constant} = \omega_s.$$

Principal axes 2 and 3 revolve at the constant angular velocity ω_s about the 1 axis.

The remaining Euler's equations are

$$0 = I_\perp \frac{d\omega_2}{dt} + (I_1 - I_\perp)\omega_s\omega_3 \qquad 1$$

$$0 = I_\perp \frac{d\omega_3}{dt} + (I_\perp - I_1)\omega_s\omega_2. \qquad 2$$

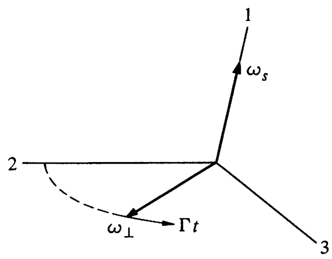
Differentiating the first equation and using the second to eliminate $d\omega_3/dt$ gives

$$\frac{d^2\omega_2}{dt^2} + \left(\frac{I_1 - I_\perp}{I_\perp}\right)^2 \omega_s{}^2\omega_2 = 0.$$

The angular velocity component ω_2 executes simple harmonic motion with angular frequency

$$\Gamma = \left|\frac{I_1 - I_\perp}{I_\perp}\right| \omega_s.$$

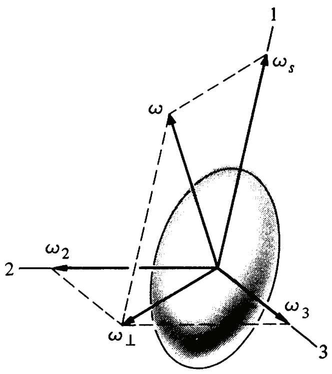
Thus, ω_2 is given by $\omega_2 = \omega_\perp \cos \Gamma t$ where the amplitude $\omega_\perp$ is determined by initial conditions. Then, if $I_1 > I_\perp$, Eq. (1) gives



$$\omega_3 = -\frac{1}{\Gamma}\frac{d\omega_2}{dt}$$
$$= \omega_\perp \sin \Gamma t.$$

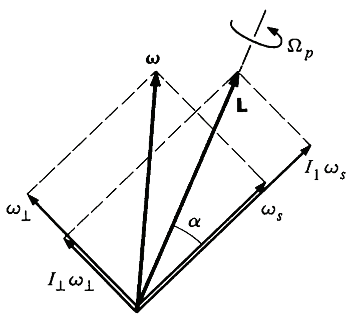
As the drawing shows, ω_2 and ω_3 are the components of a vector $\boldsymbol{\omega}_\perp$ which rotates in the 2-3 plane at rate Γ. Thus, an observer fixed to the body would see $\boldsymbol{\omega}$ rotate relative to the body about the 1 axis at angular frequency Γ. Since the 1, 2, 3 axes are fixed to the body and the body is rotating about the 1 axis at rate ω_s, the rotational speed of $\boldsymbol{\omega}$ to an observer fixed in space is

$$\Gamma + \omega_s = \frac{I_1}{I_\perp}\omega_s.$$



Euler's equations have told us how the angular velocity moves relative to the body, but we have yet to find the actual motion of the body in space. Here we must use our ingenuity. We know the motion of $\boldsymbol{\omega}$ relative to the body, and we also know that for torque-free motion, **L** is constant. As we shall show, this is enough to find the actual motion of the body.

The diagram at the top of the next page shows $\boldsymbol{\omega}$ and **L** at some instant of time. Since $L \cos \alpha = I_1\omega_s$, and ω_s and L are constant, α must be constant as well. Hence, the relative position of all the vectors in the diagram never changes. The only possible motion is for the



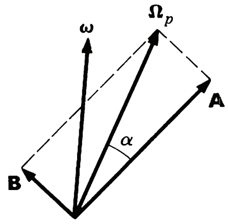


diagram to rotate about **L** with some "precessional" angular velocity Ω_p. (Bear in mind that the diagram is moving relative to the body; Ω_p is greater than ω_s.)

The remaining problem is to find Ω_p. We have shown that $\boldsymbol{\omega}$ precesses about $\boldsymbol{\omega}_s$ in space at rate $\Gamma + \omega_s$. To relate this to $\boldsymbol{\Omega}_p$, resolve $\boldsymbol{\Omega}_p$ into a vector **A** along $\boldsymbol{\omega}_s$ and a vector **B** perpendicular to $\boldsymbol{\omega}_s$. The magnitudes are $A = \Omega_p \cos\alpha$, $B = \Omega_p \sin\alpha$. The rotation **A** turns $\boldsymbol{\omega}$ about $\boldsymbol{\omega}_s$, but the rotation **B** does not. Hence the rate at which $\boldsymbol{\omega}$ precesses about $\boldsymbol{\omega}_s$ is $\Omega_p \cos\alpha$. Equating this to $\Gamma + \omega_s$,

$$\Omega_p \cos\alpha = \Gamma + \omega_s = \frac{I_1}{I_\perp}\omega_s$$

or

$$\Omega_p = \frac{I_1\omega_s}{I_\perp \cos\alpha}.$$

The precessional angular velocity Ω_p represents the rate at which the symmetry axis rotates about the fixed direction **L**. It is the frequency of wobble we observe when we flip a spinning coin. Earlier in this section we found that the rate at which the symmetry axis rotates about a space-fixed direction is $I_1\omega_s/I$ in the small angle approximation. The result agrees with Ω_p in the limit $\alpha \rightarrow 0$.

Note 7.1 Finite and Infinitesimal Rotations

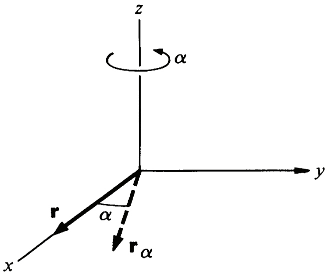
In this note we shall demonstrate that finite rotations do not commute, but that infinitesimal rotations do. By an infinitesimal rotation we mean one for which all powers of the rotation angle beyond the first can be neglected.

Consider rotation of an object through angle α about an axis $\hat{\mathbf{n}}_\alpha$ followed by a rotation through β about axis $\hat{\mathbf{n}}_\beta$. It is not possible to specify the orientation of the body by a vector because if the rotations are performed in opposite order, we do not obtain the same final orientation. To show this, we shall consider the effect of successive rotations on a vector **r**. Let $\mathbf{r}_\alpha$ be the result of rotating **r** through α about $\hat{\mathbf{n}}_\alpha$, and $\mathbf{r}_{\alpha\beta}$ be the result of rotating $\mathbf{r}_\alpha$ through β about $\hat{\mathbf{n}}_\beta$. We shall show that

$$\mathbf{r}_{\alpha\beta} \neq \mathbf{r}_{\beta\alpha}.$$

However, we shall find that for $\alpha \ll 1$, $\beta \ll 1$, $\mathbf{r}_{\alpha\beta} = \mathbf{r}_{\beta\alpha}$ to first order, and there is therefore no ambiguity in the orientation angle vector for infinitesimal rotations.

Consider the effect of successive rotation on a vector initially along the x axis, $\mathbf{r} = r\hat{\mathbf{i}}$, first through angle α about the z axis and then through angle β about the y axis. Although this is a special case, it illustrates the important features of a general proof.



First rotation: through angle α about z axis.

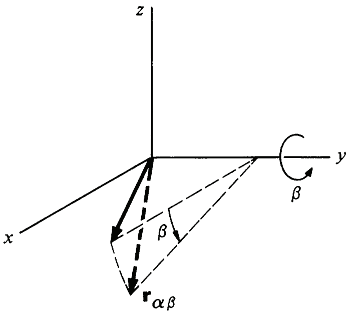
$$\mathbf{r} = r\hat{\mathbf{i}}$$

$$\mathbf{r}_\alpha = r\cos\alpha\hat{\mathbf{i}} + r\sin\alpha\hat{\mathbf{j}},$$

since $|\mathbf{r}_\alpha| = |\mathbf{r}| = r$.

Second rotation: through angle β about y axis.
The component $r\sin\alpha\hat{\mathbf{j}}$ is unchanged by this rotation.

$$\begin{aligned}\mathbf{r}_{\alpha\beta} &= r\cos\alpha\,(\cos\beta\hat{\mathbf{i}} - \sin\beta\hat{\mathbf{k}}) + r\sin\alpha\hat{\mathbf{j}} \\ &= r\cos\alpha\cos\beta\hat{\mathbf{i}} + r\sin\alpha\hat{\mathbf{j}} - r\cos\alpha\sin\beta\hat{\mathbf{k}} \qquad 1\end{aligned}$$



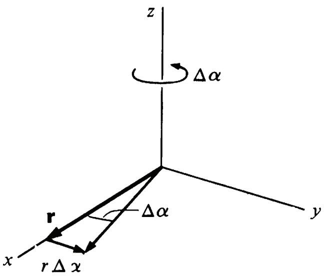
To find $\mathbf{r}_{\beta\alpha}$, we go through the same argument in reverse order. The result is

$$\mathbf{r}_{\beta\alpha} = r\cos\alpha\cos\beta\hat{\mathbf{i}} + r\cos\beta\sin\alpha\hat{\mathbf{j}} - r\sin\beta\hat{\mathbf{k}}. \qquad 2$$

From Eqs. (1) and (2), $\mathbf{r}_{\alpha\beta}$ and $\mathbf{r}_{\beta\alpha}$ differ in the y and z components. Suppose that we represent the angles by $\Delta\alpha$ and $\Delta\beta$, as in the lower two drawings, and take $\Delta\alpha \ll 1$, $\Delta\beta \ll 1$. If we neglect all terms of second order and higher, so that $\sin\Delta\theta \approx \Delta\theta$, $\cos\Delta\theta \approx 1$, Eq. (1) becomes

$$\mathbf{r}_{\alpha\beta} = r\hat{\mathbf{i}} + r\,\Delta\alpha\hat{\mathbf{j}} - r\,\Delta\beta\hat{\mathbf{k}}. \qquad 3$$

Equation (3) becomes



$$\mathbf{r}_{\beta\alpha} = r\hat{\mathbf{i}} + r\,\Delta\alpha\hat{\mathbf{j}} - r\,\Delta\beta\hat{\mathbf{k}}. \qquad 4$$

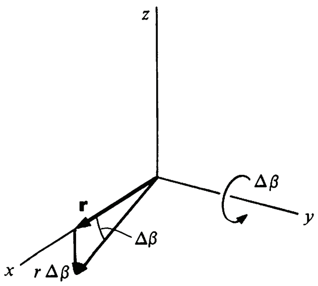
Hence $\mathbf{r}_{\alpha\beta} = \mathbf{r}_{\beta\alpha}$ to first order for small rotations, and the vector

$$\Delta\boldsymbol{\theta} = \Delta\beta\hat{\mathbf{j}} + \Delta\alpha\hat{\mathbf{k}}$$

is well defined. In particular, the displacement of $\mathbf{r}$ is

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}_{\text{final}} - \mathbf{r}_{\text{initial}} \\ &= \mathbf{r}_{\alpha\beta} - r\hat{\mathbf{i}} \\ &= r\,\Delta\alpha\hat{\mathbf{j}} - r\,\Delta\beta\hat{\mathbf{k}} = \Delta\boldsymbol{\theta}\times\mathbf{r}.\end{aligned}$$

If the displacement occurs in time Δt, the velocity is



$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{\theta}\times\mathbf{r}}{\Delta t} \\ &= \boldsymbol{\omega}\times\mathbf{r},\end{aligned}$$

where

$$\boldsymbol{\omega} = \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{\theta}}{\Delta t}.$$

In our example, $\boldsymbol{\omega} = (d\beta/dt)\hat{\mathbf{j}} + (d\alpha/dt)\hat{\mathbf{k}}$.

Our results in Eq. (3) or (4) indicate that the effect of infinitesimal rotations can be found by considering the rotations independently one at a time. To first order, the effect of rotating $\mathbf{r} = r\hat{\mathbf{i}}$ through $\Delta\alpha$ about z is to generate a y component $r\,\Delta\alpha\hat{\mathbf{j}}$. The effect of rotating $\mathbf{r}$ through $\Delta\beta$ about y is to generate a z component, $-r\,\Delta\beta\hat{\mathbf{k}}$. The total change in $\mathbf{r}$ to first order is the sum of the two effects,

$$\Delta\mathbf{r} = r\,\Delta\alpha\hat{\mathbf{j}} - r\,\Delta\beta\hat{\mathbf{k}},$$

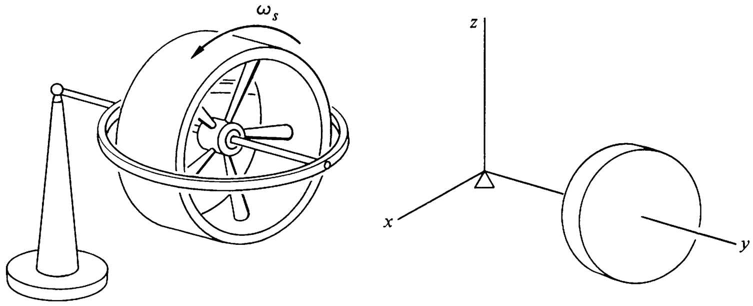
in agreement with Eq. (3) or (4).

Note 7.2 More about Gyroscopes

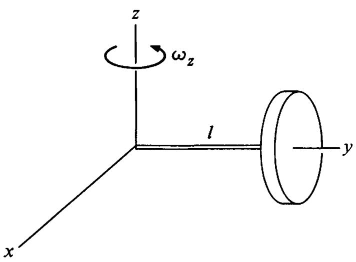
In Sec. 7.3 we used simple vector arguments to discuss the uniform precession of a gyroscope. However, uniform precession is not the most general form of gyroscope motion. For instance, a gyroscope released with its axle at rest horizontally does not instantaneously start to precess. Instead, the center of mass begins to fall. The falling motion is rapidly converted to an undulatory motion called *nutation*. If the undulations are damped out by friction in the bearings, the gyroscope eventually settles into uniform precession. The purpose of this note is to show how nutation occurs, using a small angle approximation. (The same method is used in Sec. 7.7 to explain torque-free precession.)

Consider a gyroscope consisting of a flywheel on a shaft of length l whose other end is attached to a universal pivot. The flywheel is set spinning rapidly and the axle is released from the horizontal. What is the motion?

Since it is natural to consider the motion in terms of rotation about the fixed pivot point, we introduce a coordinate system with its origin at the pivot.

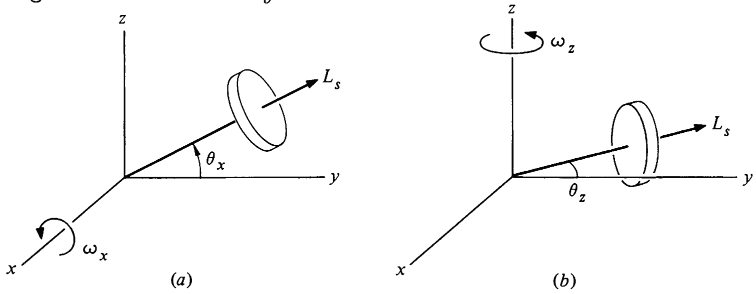


Assume for the moment that the gyroscope is not spinning but that the axle is rotating about the pivot. In order to calculate the angular momentum about the origin, we shall need a generalization of the parallel axis theorem of Example 6.9. Consider the angular momentum due to rotation of the axle about the z axis at rate ω_z. If the moment of inertia

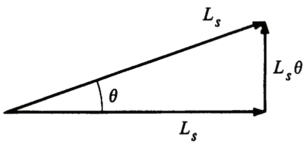


of the disk around a vertical axis through the center of mass is I_{zz}, then the moment of inertia about the z axis through the pivot is $I_{zz} + Ml^2$. The proof of this is straightforward, and we leave it as a problem. If we let $I_{zz} + Ml^2 = I_p$, then $L_z = \omega_z I_p$. By symmetry, the moment of inertia about the x axis is $I_{xx} + Ml^2 = I_p$, so that $L_x = \omega_x I_p$.

The results above are exact when the gyroscope lies along the y axis, as in the drawing, and they are true to first order in angle for small angles of tilt around the y axis.



Now suppose that the flywheel is set spinning at rate ω_s. If the moment of inertia along the axle is I_s, then the spin angular momentum is $L_s = I_s\omega_s$.



There are two kinds of contributions to the angular momentum associated with small angular displacements from the y axis. From rotation of the system as a whole with angular velocity ω, we have angular momentum contributions of the form $I_p\omega$. In addition, as the gyroscope moves away from the y axis, components of $\mathbf{L}_s$ can be generated in the x and z directions. For small angular displacements θ, such components will be of the form $L_s\theta$.

For small angular displacements, $\theta_x \ll 1$ about the x axis and $\theta_z \ll 1$ about the z axis, the rotations can be considered independently and their effects added.

a. *Rotation about the x Axis* (fig. *a*)
Suppose that the axle has rotated about the x axis through angle $\theta_x \ll 1$, and has instantaneous angular velocity ω_x. Then

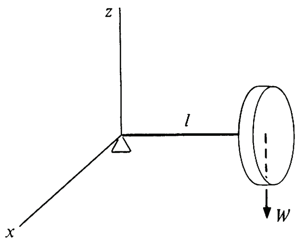
$$
\begin{aligned}
L_x &= I_p\omega_x \\
L_y &= L_s \cos\theta_x \approx L_s \\
L_z &= L_s \sin\theta_x \approx L_s\theta_x.
\end{aligned}
\tag{1}
$$

b. *Rotation about the z Axis* (fig. *b*)
For a rotation by $\theta_z \ll 1$ about the z axis, a similar argument gives

$$
\begin{aligned}
L_x &= -L_s \sin\theta_z \approx -L_s\theta_z \\
L_y &= L_s \cos\theta_z \approx L_s \\
L_z &= I_p\omega_z.
\end{aligned}
\tag{2}
$$

Equations (1) and (2) show that the rotations θ_x and θ_z leave L_y unchanged to first order. However, the rotations give rise to first order contributions to L_x and L_z. From Eqs. (1) and (2) we find

$$L_x = I_p\omega_x - L_s\theta_z$$
$$L_y = L_s$$
$$L_z = I_p\omega_z + L_s\theta_x. \qquad 3$$



The instantaneous torque about the origin is

$$\tau_x = -lW, \qquad 4$$

where l is the length of the axle and W is the weight of the gyro. Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, Eqs. (3) and (4) give

$$I_p\dot{\omega}_x - L_s\omega_z = -lW \qquad 5a$$
$$\dot{L}_s = 0 \qquad 5b$$
$$I_p\dot{\omega}_z + L_s\omega_x = 0, \qquad 5c$$

where we have used $\dot{\theta}_z = \omega_z$, $\dot{\theta}_x = \omega_x$.

Equation (5*b*) assures us that the spin is constant, as we expect for a flywheel with good bearings. If we differentiate Eq. (5*a*), we obtain

$$I_p\ddot{\omega}_x - L_s\dot{\omega}_z = 0.$$

Substituting the result $\dot{\omega}_z = -L_s\omega_x/I_p$ from Eq. (5*c*) gives

$$\ddot{\omega}_x + \frac{L_s^2}{I_p^2}\omega_x = 0.$$

If we let $\gamma = L_s/I_p = \omega_s I_s/I_p$, this becomes

$$\ddot{\omega}_x + \gamma^2\omega_x = 0.$$

We have the familiar equation for simple harmonic motion. The solution is

$$\omega_x = A\cos(\gamma t + \phi), \qquad 6$$

where A and ϕ are arbitrary constants.

We can use Eq. (5*a*) to find ω_z:

$$\omega_z = \frac{lW}{L_s} + \frac{I_p}{L_s}\dot{\omega}_x.$$

Substituting the result $\dot{\omega}_x = -A\gamma\sin(\gamma t + \phi)$ from Eq. (6) gives

$$\omega_z = \frac{lW}{L_s} - \frac{I_p}{L_s}A\gamma\sin(\gamma t + \phi)$$
$$= \frac{lW}{L_s} - A\sin(\gamma t + \phi). \qquad 7$$

We can integrate Eqs. (6) and (7) to obtain

$$\theta_x = B \sin (\gamma t + \phi) + C \qquad 8a$$

$$\theta_z = \frac{lW}{L_s} t + B \cos (\gamma t + \phi) + D, \qquad 8b$$

where $B = A/\gamma$, and C, D are constants of integration.

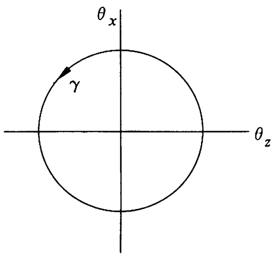
The motion of the gyroscope depends on the constants B, ϕ, C, and D in Eq. (8), and these depend on the initial conditions. We consider three separate cases.

CASE 1. UNIFORM PRECESSION
If we take $B = 0$, and $C = D = 0$, Eq. (8) gives

$$\theta_x = 0$$

$$\theta_z = lW \frac{t}{L_s}. \qquad 9$$

This corresponds to the case of uniform precession we treated in Sec. 7.3. The rate of precession is $d\theta_z/dt = lW/L_s$, as in Eq. (7.2). If the gyroscope is moving in uniform precession at $t = 0$, it will continue to do so.



CASE 2. TORQUE-FREE PRECESSION
If we "turn off" gravity so that W is zero, then Eq. (8) gives, with $C = D = 0$,

$$\theta_x = B \sin (\gamma t + \phi) \qquad 10$$

$$\theta_z = B \cos (\gamma t + \phi).$$

The tip of the axle moves in a circle about the y axis. The amplitude of the motion depends on the initial conditions. This is identical to the torque-free precession discussed in Sec. 7.7.

CASE 3. NUTATION
Suppose that the axle is released from rest along the y axis at $t = 0$. The initial conditions at $t = 0$ on the x motion are $(\theta_x)_0 = (d\theta_x/dt)_0 = 0$. From Eq. (8$a$) we obtain

$$B \sin \phi + C = 0$$

$$B\gamma \cos \phi = 0.$$

Assuming for the moment that B is not zero, we have $\phi = \pi/2$, $C = -B$. Equation (8b) then becomes

$$\theta_z = \frac{lW}{L_s} t - B \sin \gamma t + D.$$

From the initial conditions on the z motion, $(\theta_z)_0 = (d\theta_z/dt)_0 = 0$, we obtain

$$D = 0$$

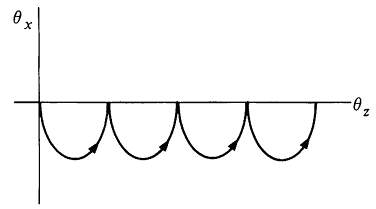
$$-B\gamma + \frac{lW}{L_s} = 0$$

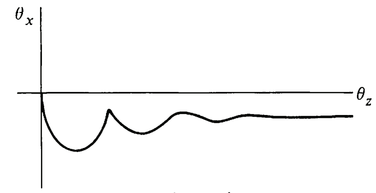
or

$$B = \frac{lW}{\gamma L_s}.$$

Inserting these results in Eq. (8) gives

$$\theta_x = \frac{lW}{\gamma L_s}(\cos \gamma t - 1)$$
$$\theta_z = \frac{lW}{\gamma L_s}(\gamma t - \sin \gamma t). \qquad 11$$

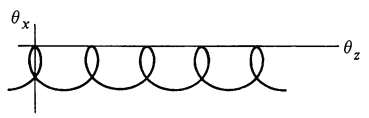




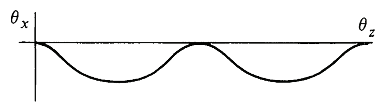
Damped nutation

The motion described by Eq. (11) is illustrated in the sketch. As time increases, the tip of the axle traces out a cycloidal path. The dipping motion of the axle is called *nutation*. The motion is easy to see with a well-made gyroscope. Note that the initial motion of the axle is vertically down; the gyro starts to fall when it is released. Eventually the nutation dies out due to friction in the pivot, and the motion turns into uniform precession, as shown in the second sketch. The axle is left with a slight dip after the nutation is damped; this keeps the total angular momentum about the z axis zero. The rotational energy of precession comes from the fall of the center of mass. Other nutational motions are also possible, depending on the initial conditions; the lower two sketches show two possible cases. These can all be described by Eq. (8) by suitable choices of the constants.

We made the approximation that $\theta_x \ll 1$, $\theta_z \ll 1$, but because of precession, θ_z increases linearly with time, so that the approximation inevitably breaks down. This is not a problem if we examine the motion for one period of nutation. The nutational motion repeats itself whenever $\gamma t = 2\pi$. The period of the nutation is $T = 2\pi/\gamma$. If θ_z is small during one period, then we can mentally start the problem over at the end of the period with a new coordinate system having its y axis again along the direction of the axle. The restriction on θ_z is then that $\Omega T \ll 1$, or

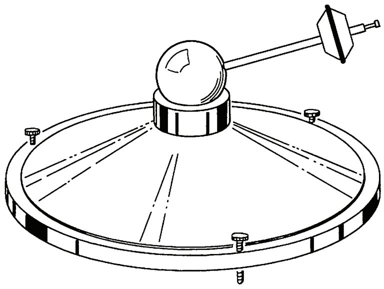


$$\frac{2\pi\Omega}{\gamma} \ll 1.$$

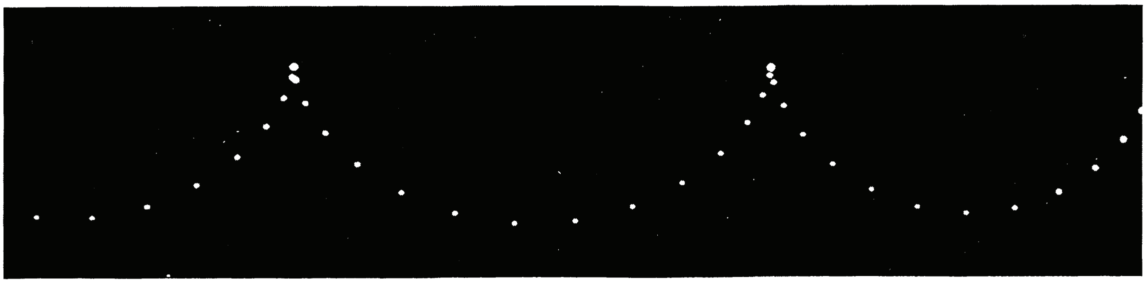


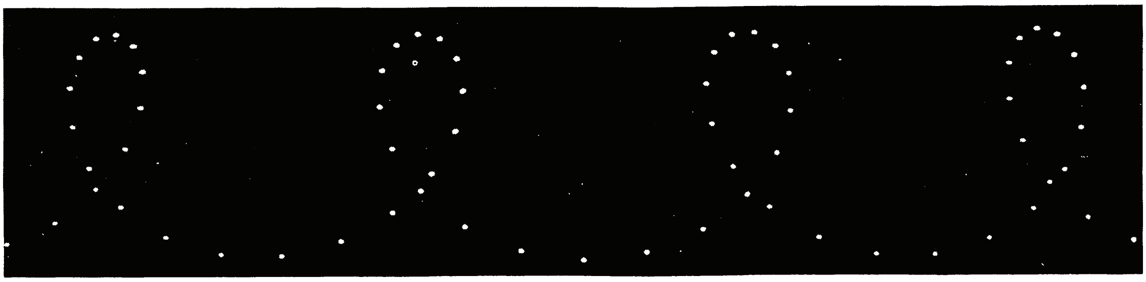
Our solution breaks down if the rate of precession becomes comparable to the rate of nutation. More vividly, we require the gyroscope to nutate many times as it precesses through a full turn.

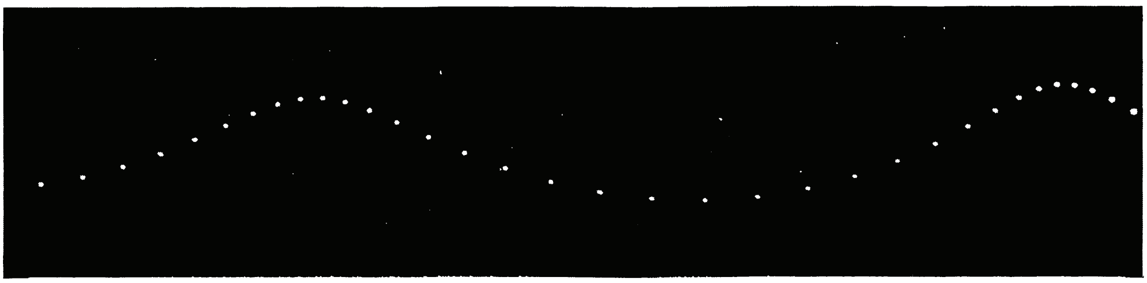
In a toy gyroscope, friction is so large that it is practically impossible to observe nutation. However, in the air suspension gyroscope, friction is so small that nutation is easy to observe. The rotor of this gyroscope



is a massive metal sphere which rests in a close fitting cup. The sphere is suspended on a film of air which flows from an orifice at the bottom of the cup. Torque is applied by the weight of a small mass on a rod protruding radially from the sphere. The pictures below are photographs of a stroboscopic light source reflected from a small bead on the end of the rod. The three modes of precession are apparent; by studying the distance between the dots you can discern the variation in speed of the rod through the precession cycle.





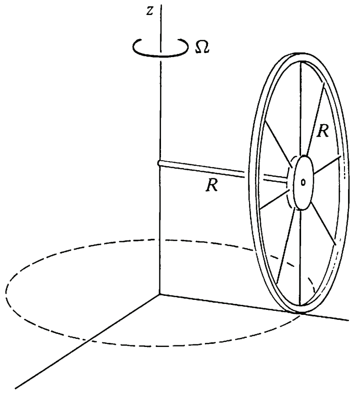


Problems

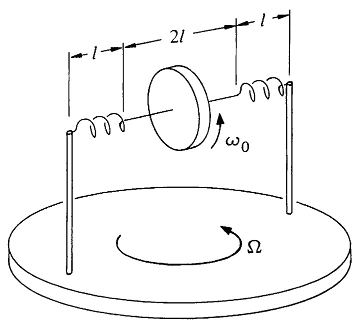
7.1 A thin hoop of mass M and radius R rolls without slipping about the z axis. It is supported by an axle of length R through its center, as shown. The hoop circles around the z axis with angular speed Ω.

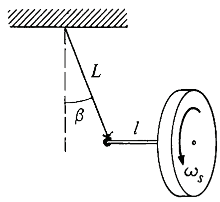
a. What is the instantaneous angular velocity $\boldsymbol{\omega}$ of the hoop?

b. What is the angular momentum $\mathbf{L}$ of the hoop? Is $\mathbf{L}$ parallel to $\boldsymbol{\omega}$? (*Note:* the moment of inertia of a hoop for an axis along its diameter is $\frac{1}{2}MR^2$.)



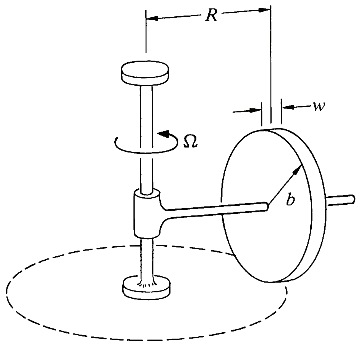
7.2 A flywheel of moment of inertia I_0 rotates with angular velocity ω_0 at the middle of an axle of length $2l$. Each end of the axle is attached to a support by a spring which is stretched to length l and provides tension T. You may assume that T remains constant for small displacements of the axle. The supports are fixed to a table which rotates at constant angular velocity, Ω, where $\Omega \ll \omega_0$. The center of mass of the flywheel is directly over the center of rotation of the table. Neglect gravity and assume that the motion is completely uniform so that nutational effects are absent. The problem is to find the direction of the axle with respect to a straight line between the supports.





7.3 A gyroscope wheel is at one end of an axle of length l. The other end of the axle is suspended from a string of length L. The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of inertia about its center of mass I_0. Its spin angular velocity is ω_s. Neglect the mass of the shaft and of the string.

Find the angle β that the string makes with the vertical. Assume that β is so small that approximations like $\sin\beta \approx \beta$ are justified.



7.4 In an old-fashioned rolling mill, grain is ground by a disk-shaped millstone which rolls in a circle on a flat surface driven by a vertical shaft. Because of the stone's angular momentum, the contact force with the surface can be considerably greater than the weight of the wheel.

Assume that the millstone is a uniform disk of mass M, radius b, and width w, and that it rolls without slipping in a circle of radius R with angular velocity Ω. Find the contact force. Assume that the millstone is closely fitted to the axle so that it cannot tip, and that $w \ll R$. Neglect friction.

Ans. clue. If $\Omega^2 b = 2\,g$, the force is twice the weight

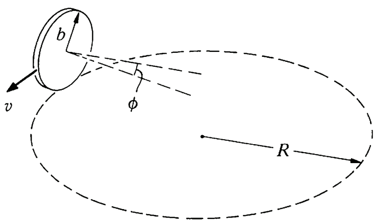
7.5 When an automobile rounds a curve at high speed, the loading (weight distribution) on the wheels is markedly changed. For sufficiently high speeds the loading on the inside wheels goes to zero, at which point the car starts to roll over. This tendency can be avoided by mounting a large spinning flywheel on the car.

a. In what direction should the flywheel be mounted, and what should be the sense of rotation, to help equalize the loading? (Be sure that your method works for the car turning in either direction.)

b. Show that for a disk-shaped flywheel of mass m and radius R, the requirement for equal loading is that the angular velocity of the flywheel, ω, is related to the velocity of the car v by

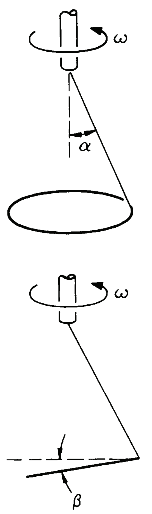
$$\omega = 2v\frac{ML}{mR^2},$$

where M is the total mass of the car and flywheel, and L is the height of the center of mass of the car (including the flywheel) above the road. Assume that the road is unbanked.



7.6 If you start a coin rolling on a table with care, you can make it roll in a circle. As you can see from the drawing, the coin "leans" inward, with its axis tilted. The radius of the coin is b, the radius of the circle it follows on the table is R, and its velocity is v. Assume that there is no slipping. Find the angle ϕ that the axis makes with the horizontal.

Ans. $\tan\phi = 3v^2/2gR$



7.7 A thin hoop of mass M and radius R is suspended from a string through a point on the rim of the hoop. If the support is turned with high angular velocity ω, the hoop will spin as shown, with its plane nearly horizontal and its center nearly on the axis of the support. The string makes angle α with the vertical.

a. Find, approximately, the small angle β between the plane of the hoop and the horizontal.

b. Find, approximately, the radius of the small circle traced out by the center of mass about the vertical axis. (With skill you can demonstrate this motion with a rope. It is a favorite cowboy lariat trick.)

7.8 A child's hoop of mass M and radius b rolls in a straight line with velocity v. Its top is given a light tap with a stick at right angles to the direction of motion. The impulse of the blow is I.

a. Show that this results in a deflection of the line of rolling by angle $\phi = I/Mv$, assuming that the gyroscope approximation holds and neglecting friction with the ground.

b. Show that the gyroscope approximation is valid provided $F \ll Mv^2/b$, where F is the peak applied force.

7.9 This problem involves investigating the effect of the angular momentum of a bicycle's wheels on the stability of the bicycle and rider. Assume that the center of mass of the bike and rider is height $2l$ above the ground. Each wheel has mass m, radius l, and moment of inertia ml^2. The bicycle moves with velocity V in a circular path of radius R. Show that it leans through an angle given by

$$\tan\phi = \frac{V^2}{Rg}\left(1 + \frac{m}{M}\right),$$

where M is the total mass.

The last term in parentheses would be absent if angular momentum were neglected. Do you think that it is important? How important is it for a bike without a rider?

7.10 Latitude can be measured with a gyro by mounting the gyro with its axle horizontal and lying along the east-west axis.

a. Show that the gyro can remain stationary when its spin axis is parallel to the polar axis and is at the latitude angle λ with the horizontal.

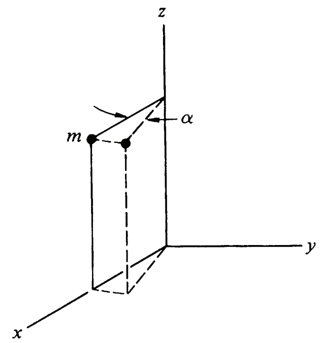
b. If the gyro is released with the spin axis at a small angle to the polar axis show that the gyro spin axis will oscillate about the polar axis with a frequency $\omega_{osc} = \sqrt{I_1\omega_s\Omega_e/I_\perp}$, where I_1 is the moment of inertia of the gyro about its spin axis, $I_\perp$ is its moment of inertia about the fixed horizontal axis, and Ω_e is the earth's rotational angular velocity.

What value of ω_{osc} is expected for a gyro rotating at 40,000 rpm, assuming that it is a thin disk and that the mounting frame makes no contribution to the moment of inertia?

7.11 A particle of mass m is located at $x = 2$, $y = 0$, $z = 3$.

a. Find its moments and products of inertia relative to the origin.

b. The particle undergoes pure rotation about the z axis through a small angle α. Show that its moments of inertia are unchanged to first order in α if $\alpha \ll 1$.



8 NONINERTIAL SYSTEMS AND FICTITIOUS FORCES

8.1 Introduction

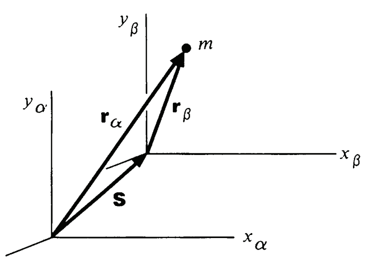
In discussing the principles of dynamics in Chap. 2, we stressed that Newton's second law $\mathbf{F} = m\mathbf{a}$ holds true only in inertial coordinate systems. We have so far avoided noninertial systems in order not to obscure our goal of understanding the physical nature of forces and accelerations. Since that goal has largely been realized, in this chapter we turn to the use of noninertial systems. Our purpose is twofold. By introducing noninertial systems we can simplify many problems; from this point of view, the use of noninertial systems represents one more computational tool. However, consideration of noninertial systems enables us to explore some of the conceptual difficulties of classical mechanics, and the second goal of this chapter is to gain deeper insight into Newton's laws, the properties of space, and the meaning of inertia.

We start by developing a formal procedure for relating observations in different inertial systems.

8.2 The Galilean Transformations

In this section we shall show that any coordinate system moving uniformly with respect to an inertial system is also inertial. This result is so transparent that it hardly warrants formal proof. However, the argument will be helpful in the next section when we analyze noninertial systems.

Suppose that two physicists, α and β, set out to observe a series of events such as the position of a body of mass m as a function of time. Each has his own set of measuring instruments and each works in his own laboratory. α has confirmed by separate experiments that Newton's laws hold accurately in his laboratory. His reference frame is therefore inertial. How can he predict whether or not β's system is also inertial?



For simplicity, α and β agree to use cartesian coordinate systems with identical scale units. In general, their coordinate systems do not coincide. Leaving rotations for later, we suppose for the time being that the systems are in relative motion but that corresponding axes are parallel. Let the position of mass m be given by $\mathbf{r}_\alpha$ in α's system, and $\mathbf{r}_\beta$ in β's system. If the origins of the two systems are displaced by $\mathbf{S}$, as shown in the sketch, then

$$\mathbf{r}_\beta = \mathbf{r}_\alpha - \mathbf{S}. \tag{8.1}$$

If physicist α sees the mass accelerating at rate $\mathbf{a}_\alpha = \ddot{\mathbf{r}}_\alpha$, he concludes from Newton's second law that there is a force on m given by

$$\mathbf{F}_\alpha = m\mathbf{a}_\alpha.$$

Physicist β observes m to be accelerating at rate $\mathbf{a}_\beta$, as if it were acted on by a force

$$\mathbf{F}_\beta = m\mathbf{a}_\beta.$$

What is the relation between $\mathbf{F}_\beta$ and the true force $\mathbf{F}_\alpha$ measured in an inertial system?

It is a simple matter to relate the accelerations in the two systems. Successive differentiation with respect to time of Eq. (8.1) yields

$$\mathbf{v}_\beta = \mathbf{v}_\alpha - \mathbf{V}$$
$$\mathbf{a}_\beta = \mathbf{a}_\alpha - \mathbf{A}. \qquad 8.2$$

If $\mathbf{V} = \dot{\mathbf{S}}$ is constant, the relative motion is uniform and $\mathbf{A} = 0$. In this case $\mathbf{a}_\beta = \mathbf{a}_\beta$, and

$$\mathbf{F}_\beta = m\mathbf{a}_\beta = m\mathbf{a}_\alpha$$
$$= \mathbf{F}_\alpha.$$

The force is the same in both systems. The equations of motion in a system moving uniformly with respect to an inertial system are identical to those in the inertial system. It follows that all systems translating uniformly relative to an inertial system are inertial. This simple result leads to something of an enigma. Although it would be appealing to single out a coordinate system absolutely at rest, there is no dynamical way to distinguish one inertial system from another. Nature provides no clue to absolute rest.

We have tacitly made a number of plausible assumptions in the above argument. In the first place, we have assumed that both observers use the same scale for measuring distance. To assure this, α and β must calibrate their scales with the same standard of length. If α determines that the length of a certain rod at rest in his system is L_α, we expect that β will measure the same length. This is indeed the case if there is no motion between the two systems. However, it is not generally true. If β moves parallel to the rod with uniform velocity v, he will measure a length $L_\beta = L_\alpha(1 - v^2/c^2)^{\frac{1}{2}}$, where c is the velocity of light. This result follows from the theory of special relativity. The contraction of the moving rod, known as the Lorentz contraction, is discussed in Sec. 12.3.

A second assumption we have made is that time is the same in both systems. That is, if α determines that the time between two events is T_α, then we assumed that β will observe the same interval. Here again the assumption breaks down at high velocities. As discussed in Sec. 13.3, β finds that the interval he measures is $T_\beta = T_\alpha/(1 - v^2/c^2)^{\frac{1}{2}}$. Once again nature provides an unexpected result.

The reason these results are so unexpected is that our notions of space and time come chiefly from immediate contact with the world around us, and this never involves velocities remotely near the velocity of light. If we normally moved with speeds approaching the velocity of light, we would take these results for granted. As it is, even the highest "everyday" velocities are low compared with the velocity of light. For instance, the velocity of an artificial satellite around the earth is about 8 km/s. In this case $v^2/c^2 \approx 10^{-9}$, and length and time are altered by only one part in a billion.

A third assumption is that the observers agree on the value of the mass. However, mass is defined by experiments which involve both time and distance, and so this assumption must also be examined. As mentioned in our discussion of momentum, if an object at rest has mass m_0, the most useful quantity corresponding to mass for an observer moving with velocity v is $m = m_0/(1 - v^2/c^2)^{\frac{1}{2}}$.

Now that we are aware of some of the complexities, let us defer consideration of special relativity until Chaps. 11 to 14 and for the time being limit our discussion to situations where $v \ll c$. In this case the classical ideas of space, time, and mass are valid to high accuracy. The following equations then relate measurements made by α and β, provided that their coordinate systems move with uniform relative velocity $\mathbf{V}$. We choose the origins of the coordinate systems to coincide at $t = 0$ so that $\mathbf{S} = \mathbf{V}t$. Then from Eq. (8.1) we have

$$\begin{aligned} \mathbf{r}_\beta &= \mathbf{r}_\alpha - \mathbf{V}t \\ t_\beta &= t_\alpha. \end{aligned} \qquad 8.3$$

The time relation is generally assumed implicitly.

This set of relations, called *transformations*, gives the prescription for transforming coordinates of an event from one coordinate system to another. Equations (8.3) transform coordinates between inertial systems and are known as the *Galilean transformations*. Since force is unchanged by the Galilean transformations, observers in different inertial systems obtain the same dynamical equations. It follows that the forms of the laws of physics are the same in all inertial systems. Otherwise, different observers would make different predictions; for instance, if one observer predicts the collision of two particles, another observer might not. The assertion that the forms of the laws of physics are the same in all inertial systems is known as the *principle of relativity*. Although the principle of relativity played only a minor role in the development of classical mechanics, its role in Einstein's theory of relativity is crucial. This is discussed further in Chap. 11, where it is also shown that the Galilean transformations are not universally valid but must be replaced by a more general transformation law, the Lorentz transformation. However, the Galilean transformations are accurate for $v \ll c$, and we shall take them to be exact in this chapter.

8.3 Uniformly Accelerating Systems

Next we turn our attention to the appearance of physical laws to an observer in a system accelerating at rate $\mathbf{A}$ with respect to an inertial system. To simplify notation we shall drop the subscripts α and β and label quantities in noninertial systems by primes. Thus, Eq. (8.2), $\mathbf{a}_\beta = \mathbf{a}_\alpha - \mathbf{A}$, becomes

$$\mathbf{a}' = \mathbf{a} - \mathbf{A},$$

where $\mathbf{A}$ is the acceleration of the primed system as measured in the inertial system.

In the accelerating system the apparent force is

$$\begin{aligned}\mathbf{F}' &= m\mathbf{a}' \\ &= m\mathbf{a} - m\mathbf{A}.\end{aligned}$$

$m\mathbf{a}$ is the true force $\mathbf{F}$ due to physical interactions. Hence,

$$\mathbf{F}' = \mathbf{F} - m\mathbf{A}.$$

We can write this as

$$\mathbf{F}' = \mathbf{F} + \mathbf{F}_{\text{fict}},$$

where

$$\mathbf{F}_{\text{fict}} \equiv -m\mathbf{A}.$$

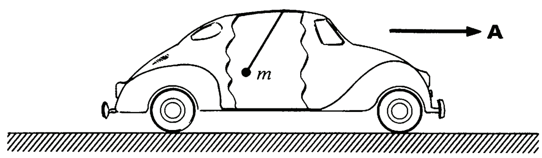
$\mathbf{F}_{fict}$ is called a *fictitious force.*[1] The fictitious force experienced in a uniformly accelerating system is uniform and proportional to the mass, like a gravitational force. However, fictitious forces originate in the acceleration of the coordinate system, not in interaction between bodies.

Here are two examples illustrating the use of fictitious forces.

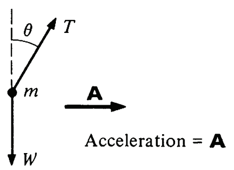
Example 8.1 The Apparent Force of Gravity

A small weight of mass m hangs from a string in an automobile which accelerates at rate A. What is the static angle of the string from the vertical, and what is its tension?

We shall analyze the problem both in an inertial frame and in a frame accelerating with the car.

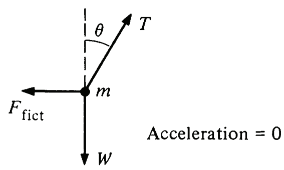


Inertial system



$$T\cos\theta - W = 0$$
$$T\sin\theta = MA$$
$$\tan\theta = \frac{MA}{W} = \frac{A}{g}$$
$$T = M(g^2 + A^2)^{1/2}$$

System accelerating with auto



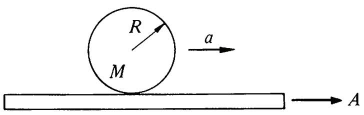
$$T\cos\theta - W = 0$$
$$T\sin\theta - F_{fict} = 0$$
$$F_{fict} = -MA$$
$$\tan\theta = \frac{A}{g}$$
$$T = M(g^2 + A^2)^{1/2}$$

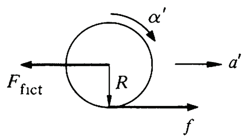
From the point of view of a passenger in the accelerating car, the fictitious force acts like a horizontal gravitational force. The effective gravitational force is the vector sum of the real and fictitious forces. How would a helium-filled balloon held on a string in the accelerating car behave?

[1] Sometimes $\mathbf{F}_{fict}$ is called an *inertial force*. However, the term fictitious force more clearly emphasizes that $\mathbf{F}_{fict}$ does not arise from physical interactions.

The fictitious force in a uniformly accelerating system behaves exactly like a constant gravitational force; the fictitious force is constant and is proportional to the mass. The fictitious force on an extended body therefore acts at the center of mass.

Example 8.2 Cylinder on an Accelerating Plank





A cylinder of mass M and radius R rolls without slipping on a plank which is accelerated at the rate A. Find the acceleration of the cylinder.

The force diagram for the horizontal force on the cylinder as viewed in a system accelerating with the plank is shown in the sketch. a' is the acceleration of the cylinder as observed in a system fixed to the plank. f is the friction force, and $F_{\text{fict}} = MA$ with the direction shown.

The equations of motion in the system fixed to the accelerating plank are

$$f - F_{\text{fict}} = Ma'$$
$$Rf = -I_0\alpha'.$$

The cylinder rolls on the plank without slipping, so

$$\alpha' R = a'.$$

These yield

$$Ma' = -I_0 \frac{a'}{R^2} - F_{\text{fict}}$$
$$a' = -\frac{F_{\text{fict}}}{M + I_0/R^2}.$$

Since $I_0 = MR^2/2$, and $F_{\text{fict}} = MA$, we have

$$a' = -\tfrac{2}{3}A.$$

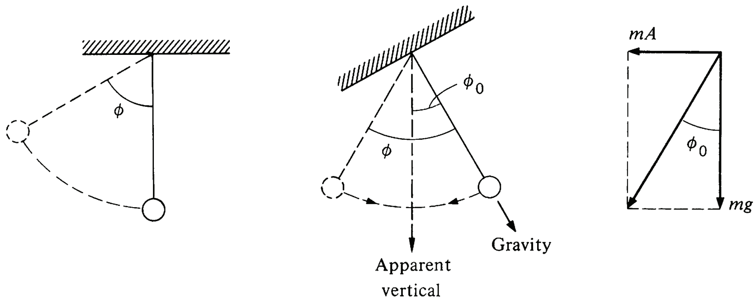
The acceleration of the cylinder in an inertial system is

$$a = A + a'$$
$$= \tfrac{1}{3}A.$$

Example 8.1 and 8.2 can be worked with about the same ease in either an inertial or an accelerating system. Here is a problem which is rather complicated to solve in an inertial system (try it), but which is almost trivial in an accelerating system.

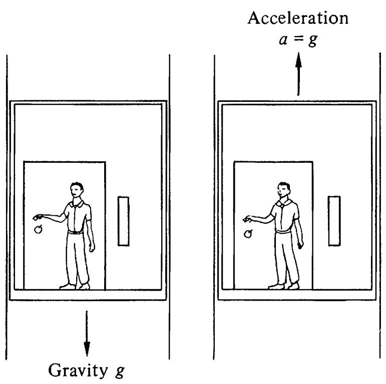
Example 8.3 Pendulum in an Accelerating Car

Consider again the car and weight on a string of Example 8.1, but now assume that the car is at rest with the weight hanging vertically. The car suddenly accelerates at rate A. The problem is to find the maximum angle ϕ through which the weight swings. ϕ is larger than the equilibrium position due to the sudden acceleration.



In a system accelerating with the car, the bob behaves like a pendulum in a gravitational field in which "down" is at an angle ϕ_0 from the true vertical. From Example 8.1, $\phi_0 = \arctan(A/g)$. The pendulum is initially at rest, so that it swings back and forth with amplitude ϕ_0 about the apparent vertical direction. Hence, $\phi = 2\phi_0 = 2\arctan(A/g)$.

8.4 The Principle of Equivalence



The laws of physics in a uniformly accelerating system are identical to those in an inertial system provided that we introduce a fictitious force on each particle, $\mathbf{F}_{\text{fict}} = -m\mathbf{A}$. $\mathbf{F}_{\text{fict}}$ is indistinguishable from the force due to a uniform gravitational field $\mathbf{g} = -\mathbf{A}$; both the gravitational force and the fictitious force are constant forces proportional to the mass. In a local gravitational field $\mathbf{g}$, a free particle of mass m experiences a force $\mathbf{F} = m\mathbf{g}$. Consider the same particle in a noninertial system uniformly accelerating at rate $\mathbf{A} = -\mathbf{g}$, with no gravitational field nor any other interaction. The apparent force is $\mathbf{F}_{\text{fict}} = -m\mathbf{A} = m\mathbf{g}$, as before. Is there any way to distinguish physically between these different situations?

The significance of this question was first pointed out by Einstein, who illustrated the problem with the following "gedanken" experiment. (A gedanken, or thought, experiment is meant to be thought about rather than carried out.)

A man is holding an apple in an elevator at rest in a gravitational field g. He lets go of the apple, and it falls with a downward acceleration $a = g$. Now consider the same man in the same elevator, but let the elevator be in free space accelerating upward at rate $a = g$. The man again lets go of the apple, and it again appears to him to accelerate down at rate g. From his point of view the two situations are identical. He cannot distinguish between acceleration of the elevator and a gravitational field.

The point becomes even more apparent in the case of the elevator freely falling in the gravitational field. The elevator and all its contents accelerate downward at rate g. If the man releases the apple, it will float as if the elevator were motionless in free space. Einstein pointed out that the downward acceleration of the elevator exactly cancels the local gravitational field. From the point of view of an observer in the elevator, there is no way to determine whether the elevator is in free space or whether it is falling in a gravitational field.

This apparently simple idea, known as the *principle* of *equivalence*, underlies Einstein's general theory of relativity, and all other theories of gravitation. We summarize the principle of equivalence as follows: there is no way to distinguish locally between a uniform gravitational acceleration $\mathbf{g}$ and an acceleration of the coordinate system $\mathbf{A} = -\mathbf{g}$. By saying that there is no way to distinguish *locally*, we mean that there is no way to distinguish from within a sufficiently confined system. The reason that Einstein put his observer in an elevator was to define such an enclosed system. For instance, if you are in an elevator and observe that free objects accelerate toward the floor at rate a, there are two possible explanations:

1. There is a gravitational field down, $g = a$, and the elevator is at rest (or moving uniformly) in the field.
2. There is no gravitational field, but the elevator is accelerating up at rate a.

To distinguish between these alternatives, you must look out of the elevator. Suppose, for instance, that you see an apple suddenly drop from a nearby tree and fall down with acceleration a. The most likely explanation is that you and the tree are at rest in a downward gravitational field of magnitude $g = a$. However, it is conceivable that your elevator and the tree are both at rest on a giant elevator which is accelerating up at rate a.

To choose between these alternatives you must look farther off. If you see that you have an upward acceleration a relative to the fixed stars, that is, if the stars appear to accelerate down at rate a, the only possible explanation is that you are in a noninertial system; your elevator and the tree are actually accelerating up. The alternative is the impossible conclusion that you are at rest in a gravitational field which extends uniformly *through all of space*. But such fields do not exist; real forces arise from interactions between real bodies, and for sufficiently large separations the forces always decrease. Hence it is most unphysical to invoke a uniform gravitational field extending throughout space.

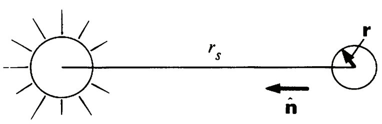
This, then, is the difference between a gravitational field and an accelerating coordinate system. Real fields are local; at large distances they decrease. An accelerating coordinate system is nonlocal; the acceleration extends uniformly throughout space. Only for small systems are the two indistinguishable.

Although these ideas may sound somewhat abstract, the next two examples show that they have direct physical consequences.

Example 8.4 **The Driving Force of the Tides**

The earth is in free fall toward the sun, and according to the principle of equivalence it should be impossible to observe the sun's gravitational force in an earthbound system. However, the equivalence principle applies only to local systems. The earth is so large that appreciable nonlocal effects like the tides can be observed. In this example we shall discuss the origin of the tides to see what is meant by a nonlocal effect.

The tides arise because of variations in the apparent gravitational field of the sun and the moon at different points on the earth's surface. Although the moon's effect is larger than the sun's, we shall consider only the sun for purposes of illustration.



The gravitational field of the sun at the center of the earth is

$$\mathbf{G}_0 = GM_s \frac{\hat{\mathbf{n}}}{r_s^2},$$

where M_s is the sun's mass, r_s is the distance between the center of the sun and the center of the earth, and $\hat{\mathbf{n}}$ is the unit vector from the earth toward the sun. The earth accelerates toward the sun at rate $\mathbf{A} = \mathbf{G}_0$.

If $\mathbf{G}(\mathbf{r})$ is the gravitational field of the sun at some point $\mathbf{r}$ on the earth, where the origin of $\mathbf{r}$ is the center of the earth, then the force on mass m at $\mathbf{r}$ is

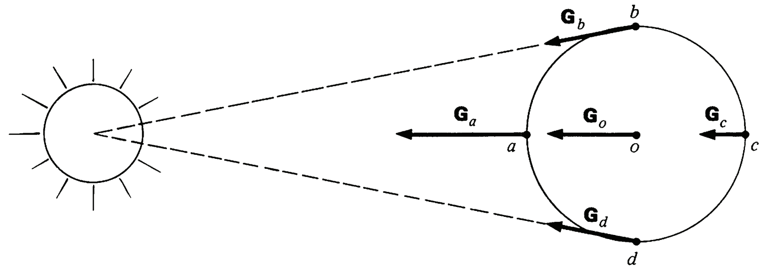
$$\mathbf{F} = m\mathbf{G}(\mathbf{r}).$$

The apparent force to an earthbound observer is

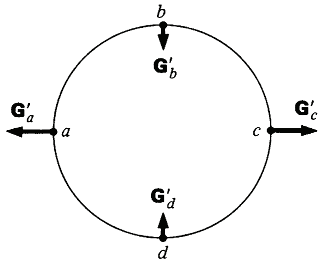
$$\mathbf{F}' = \mathbf{F} - m\mathbf{A} = m[\mathbf{G}(\mathbf{r}) - \mathbf{G}_0].$$

The apparent field is

$$\mathbf{G}'(\mathbf{r}) = \frac{\mathbf{F}'}{m} = \mathbf{G}(\mathbf{r}) - \mathbf{G}_0.$$



The drawing above shows the true field **G(r)** at different points on the earth's surface. (The variations are exaggerated.) G_a is larger than G_0 since a is closer to the sun than the center of the earth. Similarly, G_c is less than G_0. The magnitudes of $\mathbf{G}_b$ and $\mathbf{G}_c$ are approximately the same as the magnitude of $\mathbf{G}_0$, but their directions are slightly different.



The apparent field $\mathbf{G}' = \mathbf{G} - \mathbf{G}_0$ is shown in the drawing at left. We now evaluate $\mathbf{G}'$ at each of the points indicated.

1. $\mathbf{G}'_a$ AND $\mathbf{G}'_c$

The distance from a to the center of the sun is $r_s - R_e$ where R_e is the earth's radius. The magnitude of the sun's field at a is

$$G_a = \frac{GM_s}{(r_s - R_e)^2}.$$

$\mathbf{G}_a$ is parallel to $\mathbf{G}_0$. The magnitude of the apparent field at a is

$$\begin{aligned} G'_a &= G_a - G_0 \\ &= \frac{GM_s}{(r_s - R_e)^2} - \frac{GM_s}{r_s^2} \\ &= \frac{GM_s}{r_s^2}\left[\frac{1}{[1 - (R_e/r_s)]^2} - 1\right]. \end{aligned}$$

Since $R_e/r_s = 6.4 \times 10^3 \text{ km}/1.5 \times 10^8 \text{ km} = 4.3 \times 10^{-5} \ll 1$, we have

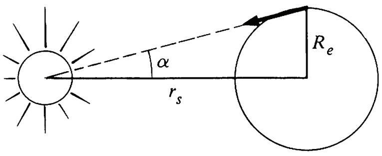
$$\begin{aligned} G'_a &= G_0\left[\left(1 - \frac{R_e}{r_s}\right)^{-2} - 1\right] \\ &= G_0\left[1 + 2\frac{R_e}{r_s} + \cdots - 1\right] \\ &= 2G_0\frac{R_e}{r_s}, \end{aligned}$$

where we have neglected terms of order $(R_e/r_s)^2$ and higher.

The analysis at c is similar, except that the distance to the sun is $r_s + R_e$ instead of $r_s - R_e$. We obtain

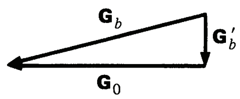
$$G_c' = -2G_0\frac{R_e}{r_s}.$$

Note that $\mathbf{G}_a'$ and $\mathbf{G}_c'$ point radially out from the earth.



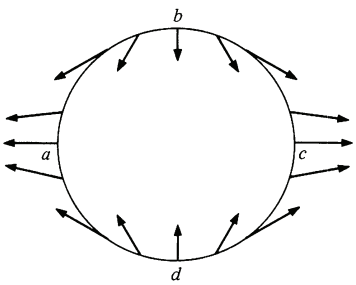
2. $\mathbf{G}_b'$ AND $\mathbf{G}_d'$

Points b and d are, to excellent approximation, the same distance from the sun as the center of the earth. However, $\mathbf{G}_b$ is not parallel to $\mathbf{G}_0$; the angle between them is $\alpha \approx R_e/r_s = 4.3 \times 10^{-5}$. To this approximation



$$G_b' = G_0\alpha$$
$$= G_0\frac{R_e}{r_s}.$$

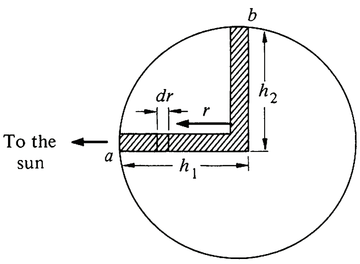
By symmetry, $\mathbf{G}_d'$ is equal and opposite to $\mathbf{G}_b'$. Both $\mathbf{G}_b'$ and $\mathbf{G}_d'$ point toward the center of the earth.



The sketch shows $\mathbf{G}'(\mathbf{r})$ at various points on the earth's surface. This diagram is the starting point for analyzing the tides. The forces at a and c tend to lift the oceans, and the forces at b and d tend to depress them. If the earth were uniformly covered with water, the tangential force components would cause the two tidal bulges to sweep around the globe with the sun. This picture explains the twice daily ebb and flood of the tides, but the actual motions depend in a complicated way on the response of the oceans as the earth rotates, and on features of local topography.

We can estimate the magnitude of tidal effects quite easily, as the next example shows.

Example 8.5 **Equilibrium Height of the Tide**



The following argument is based on a model devised by Newton. Pretend that two wells full of water run from the surface of the earth to the center, where they join. One is along the earth-sun axis and the other is perpendicular. For equilibrium, the pressures at the bottom of the wells must be identical.

The pressure due to a short column of water of height dr is $\rho g(r)dr$, where ρ is the density and $g(r)$ is the effective gravitational field at r. The condition for equilibrium is

$$\int_0^{h_1} \rho g_1(r)\, dr = \int_0^{h_2} \rho g_2(r)\, dr.$$

h_1 and h_2 are the distances from the center of the earth to the surface of the respective water columns. If we assume that the water is incompressible, so that ρ is constant, then the equilibrium condition becomes

$$\int_0^{h_1} g_1(r)\, dr = \int_0^{h_2} g_2(r)\, dr.$$

The problem is to calculate the difference $h_1 - h_2 = \Delta h_s$, the height of the tide due to the sun. We shall assume that the earth is spherical and neglect effects due to its rotation.

The effective field toward the center of the earth along column 1 is $g_1(r) = g(r) - G_1'(r)$, where $g(r)$ is the gravitational field of the earth and $G_1'(r)$ is the effective field of the sun along column 1. (The negative sign indicates that $G_1'(r)$ is directed radially out.) In the last example we evaluated $G_1'(R_e) = G_a' = 2GM_sR_e/r_s^3$. The effective field along column 1 is obtained by substituting r for R_e. Hence,

$$G_1'(r) = \frac{2GM_sr}{r_s^3}$$
$$= 2Cr,$$

where $C = GM_s/r_s^3$.

Putting these together, we obtain

$$g_1(r) = G(r) - 2Cr.$$

By the same reasoning we obtain

$$g_2(r) = g(r) + G_2'(r)$$
$$= g(r) + Cr.$$

The condition for equilibrium is

$$\int_0^{h_1} [g(r) - 2Cr]\, dr = \int_0^{h_2} [g(r) + Cr]\, dr,$$

or, rearranging,

$$\int_0^{h_1} g(r)\, dr - \int_0^{h_2} g(r)\, dr = \int_0^{h_1} 2Cr\, dr + \int_0^{h_2} Cr\, dr.$$

We can combine the integrals on the left hand side to give $\int_{h_1}^{h_2} g(r)\, dr$. Since h_1 and h_2 are close to the earth's radius, $g(r)$ can be taken as constant in the integral. $g(r) = g(R_e) = g$, the acceleration due to gravity at the earth's surface. The integrals on the left become $g(h_1 - h_2) = g\,\Delta h_s$. The integrals on the right can be combined by taking $h_1 \approx h_2 \approx R_e$, and they yield $\int_0^{R_e} 3Cr\, dr = \frac{3}{2}CR_e^2$. The final result is

$$g\,\Delta h_s = \tfrac{3}{2}CR_e^2.$$

By using $g = GM_e/R_e^2$, $C = GM_s/r_s^3$, we find

$$\Delta h_s = \frac{3}{2}\frac{M_s}{M_e}\left(\frac{R_e}{r_s}\right)^3 R_e.$$

From the numerical values

$M_s = 1.99 \times 10^{33}$ g $\qquad r_s = 1.49 \times 10^{13}$ cm

$M_e = 5.98 \times 10^{27}$ g $\qquad R_e = 6.37 \times 10^8$ cm, we obtain

$$\Delta h_s = 24.0 \text{ cm}.$$

The identical argument for the moon gives

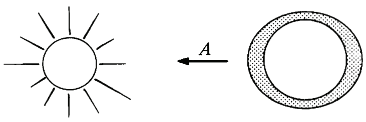
$$\Delta h_m = \frac{3}{2}\frac{M_m}{M_e}\left(\frac{R_e}{r_m}\right)^3 R_e.$$

Inserting $M_m = 7.34 \times 10^{25}$ g, $r_m = 3.84 \times 10^{10}$ cm, we obtain $\Delta h_m = 53.5$ cm. We see that the moon's effect is about twice as large as the sun's, even though the sun's gravitational field at the earth is about 200 times stronger than the moon's. The reason is that the tidal force depends on the gradient of the gravitational field. The moon is so close that its field varies considerably across the earth, whereas the field of the distant sun is more nearly constant.

The strongest tides, called the spring tides, occur at the new and full moon when the moon and sun act together. Midway between, at the quarters of the moon, occur the weak neap tides. The ratio of the driving forces in these two cases is

$$\frac{\Delta h_{\text{spring}}}{\Delta h_{\text{neap}}} = \frac{\Delta h_m + \Delta h_s}{\Delta h_m - \Delta h_s} \approx 3.$$

Earth not accelerating



Earth in free fall

The tides offer convincing evidence that the earth is in free fail toward the sun. If the earth were attracted by the sun but not in free fall, there would be only a single tide, whereas free fall results in two tides a day, as the sketches illustrate. The fact that we can sense the sun's gravitational field from a body in free fall does not contradict the principle of equivalence. The height of the tide depends on the ratio of the earth's radius to the sun's distance, R_e/r_s. However, for a system to be local with respect to a gravitational field, the variation of the field must be negligible over the dimensions of the system. The earth would be a local system if R_e were negligible compared with r_s, but then there would be no tides. Hence, the tides demonstrate that the earth is too large to constitute a local system in the sun's field.

There have been a number of experimental investigations of the principle of equivalence, since in spite of its apparent simplicity, far-reaching conclusions follow from it. For example, the principle of equivalence demands that gravitational force be strictly proportional to inertial mass. An alternative statement is that the ratio of gravitational mass to inertial mass must be the same for all matter, where the gravitational mass is the mass which enters the gravitational force equation and the inertial mass is the mass which appears in Newton's second law. Hence, if an object with gravitational mass M_{gr} and inertial mass M_{in} interacts with an object of gravitational mass M_0, we have

$$\mathbf{F} = -\frac{GM_0M_{gr}\hat{\mathbf{r}}}{r^2}.$$

Since the acceleration is $\mathbf{F}/M_{in}$,

$$\mathbf{a} = -\frac{GM_0}{r^2}\left(\frac{M_{gr}}{M_{in}}\right)\hat{\mathbf{r}}. \qquad 8.4$$

The equivalence principle requires M_{gr}/M_{in} to be the same for all objects, since otherwise it would be possible to distinguish locally between a gravitational field and an acceleration. For instance, suppose that for object A, M_{gr}/M_{in} is twice as large as for object B. If we release both objects in an Einstein elevator and they fall with the same acceleration, the only possible conclusion is that the elevator is actually accelerating up. On the other hand, if A falls with twice the acceleration of B, we know that the acceleration must be due to a gravitational field. The upward acceleration of the elevator would be distinguishable from a downward gravitational field, in defiance of the principle of equivalence.

The ratio M_{gr}/M_{in} is taken to be 1 in Newton's law of gravitation. Any other choice for the ratio would be reflected in a different value for G, since experimentally the only requirement is that $G(M_{gr}/M_{in}) = 6.67 \times 10^{-11}$ N·m²/kg².

Newton investigated the equivalence of inertial and gravitational mass by studying the period of a pendulum with interchangeable bobs. The equation of motion for the bob in the small angle approximation is

$$M_{in}l\ddot{\theta} + M_{gr}g\theta = 0.$$

The period of the pendulum is

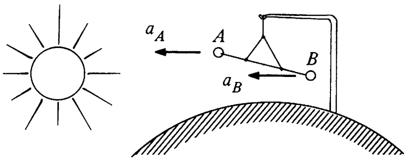
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{M_{in}}{M_{gr}}}.$$

Newton's experiment consisted of looking for a variation in T using bobs of different composition. He found no such change and, from an estimate of the sensitivity of the method, concluded that M_{gr}/M_{in} is constant to better than one part in a thousand for common materials.

The most compelling evidence for the principle of equivalence comes from an experiment devised by the Hungarian physicist Baron Roland von Eötvös at the turn of the century. (The experiments were completed in 1908 but the results were not published until 1922, three years after von Eötvös' death.) The method and technique of von Eötvös' experiment were refined by R. H. Dicke and his collaborators at Princeton University, and it is this work, completed in 1963, which we shall now outline.[1]

Consider a torsion balance consisting of two masses A and B of different composition at each end of a bar which hangs from a thin fiber so that it can rotate only about the vertical axis. The masses are attracted by the earth and also by the sun. The gravitational force due to the earth is vertical and causes no rotation of the balance, but as we now show, the sun's attraction will cause a rotation if the principle of equivalence is violated.

Assume that the sun is on the horizon, as shown in the sketch, and that the horizontal bar is perpendicular to the sun-earth axis. According to Eq. (8.4) the accelerations of the masses due to the sun are



$$a_A = \frac{GM_s}{r_s^2}\left[\frac{M_{gr}(A)}{M_{in}(A)}\right]$$
$$a_B = \frac{GM_s}{r_s^2}\left[\frac{M_{gr}(B)}{M_{in}(B)}\right],$$

where M_s is the gravitational mass of the sun, and r_s is the distance between sun and earth. The acceleration of the masses in a coordinate system fixed to the earth are

$$a'_A = a_A - a_0$$
$$a'_B = a_B - a_0,$$

where a_0 is the acceleration of the earth toward the sun. (Acceleration due to the rotation of the earth plays no role and we neglect it.)

If the principle of equivalence is obeyed, $a'_A = a'_B$ and the bar has no tendency to rotate about the fiber. However, if the two masses A and B have different ratios of gravitational to inertial mass, then one will accelerate more than the other. The balance

[1] An account of the experiment is given in an article by R. H. Dicke in *Scientific American*, vol. 205, no. 84, December, 1961.

will rotate until the restoring torque of the suspension fiber brings it to rest. As the earth rotates, the apparent direction of the sun changes; the equilibrium position of the balance moves with a 24-h period.

Dicke's apparatus was capable of detecting the deflection caused by a variation of 1 part in 10^{11} in the ratio of gravitational to inertial mass, but no effect was found to this accuracy.

The principle of equivalence is generally regarded as a fundamental law of physics. We have used it to discuss the ratio of gravitational to inertial mass. Surprisingly enough, it can also be used to show that clocks run at different rates in different gravitational fields. A simple argument showing how the principle of equivalence forces us to give up the classical notion of time is presented in Note 8.1.

8.5 Physics in a Rotating Coordinate System

The transformation from an inertial coordinate system to a rotating system is fundamentally different from the transformation to a translating system. A coordinate system translating uniformly relative to an inertial system is also inertial; the transformation leaves the laws of motion unaffected. In contrast, a uniformly rotating system is intrinsically noninertial. Rotational motion is accelerating motion, and the laws of physics always involve fictitious forces when referred to a rotating reference frame. The fictitious forces do not have the simple form of a uniform gravitational field, as in the case of a uniformly accelerating system, but involve several terms, including one which is velocity dependent. However, in spite of these complications, rotating coordinate systems can be very helpful. In certain cases the fictitious forces actually simplify the form of the equations of motion. In other cases it is more natural to introduce the fictitious forces than to describe the motion with inertial coordinates. A good example is the physics of airflow over the surface of the earth. It is easier to explain the rotational motion of weather systems in terms of fictitious forces than to use inertial coordinates which must then be related to coordinates on the rotating earth.

If a particle of mass m is accelerating at rate $\mathbf{a}$ with respect to inertial coordinates and at rate $\mathbf{a}_{\text{rot}}$ with respect to a rotating coordinate system, then the equation of motion in the inertial system is

$$\mathbf{F} = m\mathbf{a}.$$

We would like to write the equation of motion in the rotating system as

$$\mathbf{F}_{\text{rot}} = m\mathbf{a}_{\text{rot}}.$$

If the accelerations of m in the two systems are related by

$$\mathbf{a} = \mathbf{a}_{\text{rot}} + \mathbf{A},$$

where $\mathbf{A}$ is the relative acceleration, then

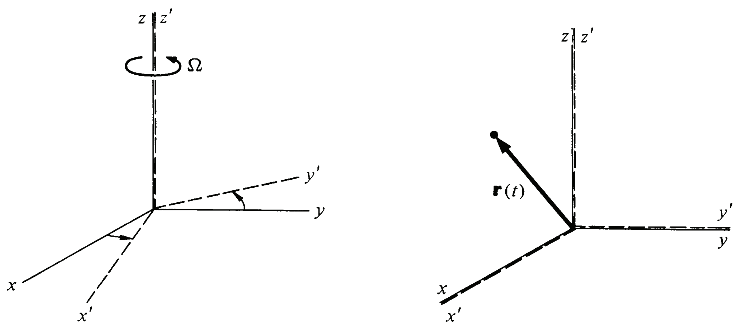
$$\begin{aligned}\mathbf{F}_{\text{rot}} &= m(\mathbf{a} - \mathbf{A}) \\ &= \mathbf{F} + \mathbf{F}_{\text{fict}},\end{aligned}$$

where $\mathbf{F}_{\text{fict}} = -m\mathbf{A}$. So far the argument is identical to that in Sec. 8.3. Our task now is to find $\mathbf{A}$ for a rotating system.

One way of evaluating $\mathbf{A}$ is to find the transformation connecting the inertial and rotating coordinates and then to differentiate. However, there is a much simpler and more general method, which consists of finding a transformation rule relating the time derivatives of any vector in inertial and rotating coordinates. In order to motivate the derivation, we proceed by first finding the relation between the velocity of a particle measured in an inertial system, $\mathbf{v}_{\text{in}}$, and the velocity measured in a rotating system, $\mathbf{v}_{\text{rot}}$.

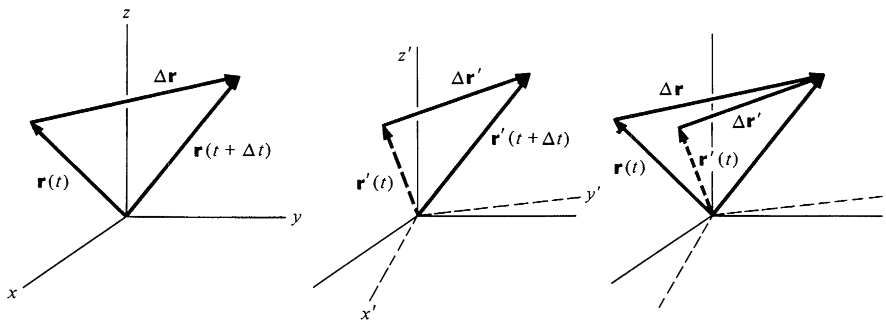
Time Derivatives and Rotating Coordinates

We are interested in pure rotation without translation, and so we consider a rotating system x', y', z' whose origin coincides with the origin of an inertial system x, y, z. Suppose, for the sake of the argument, that the x', y', z' system is rotating so that the z and z' axes always coincide. Thus, the angular velocity of the rotating system, $\mathbf{\Omega}$, lies along the z axis. Furthermore, let the x and x' axes coincide instantaneously at time t. Imagine now that a particle has position vector $\mathbf{r}(t)$ in the xz plane (and $x'z'$ plane) at time t.



At time $t + \Delta t$, the position vector is $\mathbf{r}(t + \Delta t)$, and, from the figure at left below the displacement of the particle in the inertial system is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$



The situation is different for an observer in the rotating coordinate system. He also notes the same final position vector $\mathbf{r}(t + \Delta t)$, but in calculating the displacement he remembers that the initial position vector in his coordinate system $\mathbf{r}'(t)$ was in the $x'z'$ plane. The displacement he measures relative to his coordinates is $\Delta\mathbf{r}' = \mathbf{r}(t + \Delta t) - \mathbf{r}'(t)$, as in the figure at right above however, the $x'z'$ plane is now rotated away from its earlier position and, as we see from the drawing at left, $\Delta\mathbf{r}$ and $\Delta\mathbf{r}'$ are not the same

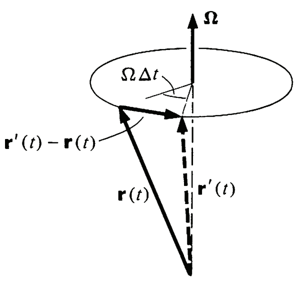
$$\Delta\mathbf{r} = \Delta\mathbf{r}' + \mathbf{r}'(t) - \mathbf{r}(t).$$

Consequently, the velocity is different in the two frames.

Since $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ differ only by a pure rotation, we can use the result of Sec. 7.2 to write

$$\mathbf{r}'(t) - \mathbf{r}(t) = (\mathbf{\Omega} \times \mathbf{r})\,\Delta t.$$

Hence,



$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta\mathbf{r}'}{\Delta t} + \mathbf{\Omega} \times \mathbf{r}.$$

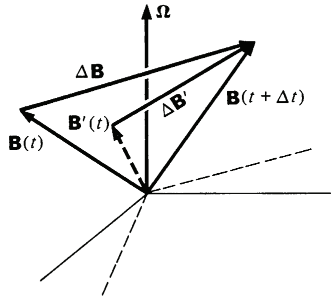
Taking the limit $\Delta t \to 0$ yields

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r}. \qquad 8.5$$

It is important to realize that Eq. (8.5) is a general vector relation; the proof did not employ the special arrangement of axes we used to illustrate the derivation.

An alternative way to write Eq. (8.5) is

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r}. \qquad 8.6$$



Since our proof used only the geometric properties of **r**, Eq. (8.6) can immediately be generalized for any vector **B**, as the sketch indicates.

$$\left(\frac{d\mathbf{B}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{B}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{B}. \qquad 8.7$$

When applying Eq. (8.7), keep in mind that **B** is instantaneously the same in both systems; it is only the time rates of change which differ. Note 8.2 presents an alternative derivation of Eq. (8.7).

Acceleration Relative to Rotating Coordinates

We can use Eq. (8.7) to relate the acceleration observed in a rotating system, $\mathbf{a}_{\text{rot}} = (d\mathbf{v}_{\text{rot}}/dt)_{\text{rot}}$, to the acceleration in an inertial system, $\mathbf{a}_{\text{in}} = (d\mathbf{v}_{\text{in}}/dt)_{\text{in}}$. Applying Eq. (8.7) to $\mathbf{v}_{\text{in}}$ gives

$$\mathbf{a}_{\text{in}} = \left(\frac{d\mathbf{v}_{\text{in}}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{v}_{\text{in}}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{v}_{\text{in}}.$$

Using

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r}$$

we have

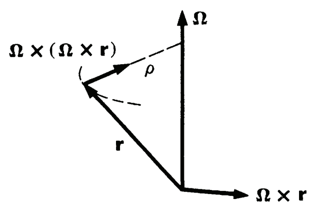
$$\mathbf{a}_{\text{in}} = \left[\frac{d}{dt}(\mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r})\right]_{\text{rot}} + \mathbf{\Omega} \times \mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$

We shall assume that $\mathbf{\Omega}$ is constant, since this is the case generally needed in practice. Hence

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + \mathbf{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}),$$

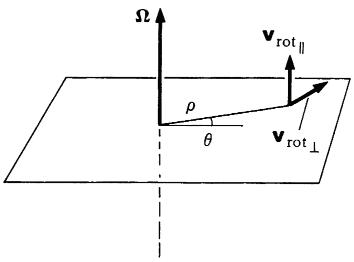
or

$$\mathbf{a}_{\text{in}} = \mathbf{a}_{\text{rot}} + 2\mathbf{\Omega} \times \mathbf{v}_{\text{rot}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}). \qquad 8.8$$



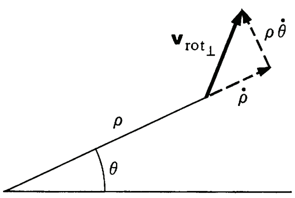
Let us examine the various contributions to $\mathbf{a}_{\text{in}}$ in Eq. (8.8). The term $\mathbf{a}_{\text{rot}}$ is simply the acceleration measured in the rotating coordinate system; there is nothing mysterious here. For example, if we measure the acceleration of a car or plane in a coordinate system fixed to the rotating earth, we are measuring $\mathbf{a}_{\text{rot}}$.

To see the origin of the term $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, note first that $\mathbf{\Omega} \times \mathbf{r}$ is perpendicular to the plane of $\mathbf{\Omega}$ and **r** and has magnitude $\Omega\rho$, where ρ is the perpendicular distance from the axis of rotation

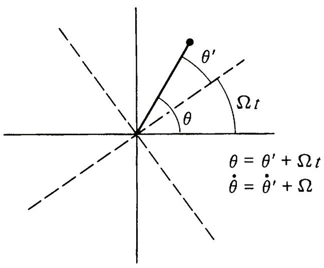


to the tip of **r**. Hence $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is directed radially inward toward the axis of rotation and has magnitude $\Omega^2\rho$. It is a centripetal acceleration, arising because every point at rest in the rotating system is actually moving in a circular path in inertial space.

The term $2\mathbf{\Omega} \times \mathbf{v}_{rot}$ is the general vector expression for the Coriolis acceleration in three dimensions. If $\mathbf{v}_{rot}$ is resolved into components $\mathbf{v}_{rot_\parallel}$ and $\mathbf{v}_{rot_\perp}$, parallel and perpendicular to $\mathbf{\Omega}$, respectively, only $\mathbf{v}_{rot_\perp}$ contributes to $2\mathbf{\Omega} \times \mathbf{v}_{rot}$. Hence, the coriolis acceleration is perpendicular to $\mathbf{\Omega}$. Here is how it arises:



The radial component $\dot{\rho}$ of $\mathbf{v}_{rot_\perp}$ contributes $2\Omega\dot{\rho}$ in the tangential direction to $\mathbf{a}_{in}$. This is simply the Coriolis term we found in Sec. 1.9 for motion in inertial space with angular velocity Ω and radial velocity $\dot{\rho}$. The tangential component $\rho\dot{\theta}'$ of $\mathbf{v}_{rot_\perp}$ contributes $2\Omega\rho\dot{\theta}'$ toward the rotation axis. To see the origin of this term, note that in inertial space the instantaneous angular velocity is $\dot{\theta} = \dot{\theta}' + \Omega$ and the centripetal acceleration term in $\mathbf{a}_{in}$ is



$$\begin{aligned}\rho\dot{\theta}^2 &= \rho(\dot{\theta}' + \Omega)^2\\ &= \rho\dot{\theta}'^2 + 2\Omega\rho\dot{\theta}' + \rho\Omega^2.\end{aligned}$$

The three terms on the right correspond to the three terms on the right of Eq. (8.8). $\rho\dot{\theta}'^2$ is part of $\mathbf{a}_{rot}$, $2\Omega\rho\dot{\theta}'$ follows from $2\mathbf{\Omega} \times \mathbf{v}_{rot}$ as we have shown, and $\rho\Omega^2$ comes from $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$.

The Apparent Force in a Rotating Coordinate System

From Eq. (8.8) we have

$$\mathbf{a}_{rot} = \mathbf{a}_{in} - 2\mathbf{\Omega} \times \mathbf{v}_{rot} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$

The force observed in the rotating system is

$$\begin{aligned}\mathbf{F}_{rot} &= m\mathbf{a}_{rot} = m\mathbf{a}_{in} - m[2\mathbf{\Omega} \times \mathbf{v}_{rot} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})]\\ &= \mathbf{F} + \mathbf{F}_{fict},\end{aligned}$$

where the fictitious force is

$$\mathbf{F}_{fict} = -2m\mathbf{\Omega} \times \mathbf{v}_{rot} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$

The first term on the right is called the *Coriolis force*, and the second term, which points outward from the rotation axis, is called the *centrifugal force*.

The Coriolis and centrifugal forces are nonphysical; they arise from kinematics and are not due to physical interactions. For instance, the centrifugal force actually increases with ρ, whereas real forces always decrease with distance. Nevertheless, the

Coriolis and centrifugal forces seem quite real to an observer in a rotating frame. When we drive a car too fast around a curve, it skids outward as if pushed by the centrifugal force. From the standpoint of an observer in an inertial frame, however, what has happened is that the sideward force exerted by the road on the tires is not adequate to keep the car turning with the road.

There is a natural human tendency to describe rotational motion with a rotating system. For instance, if we whirl a rock on a string, we instinctively say that centrifugal force is pulling the rock outward. In a coordinate system rotating with the rock, this is correct; the rock is stationary and the centrifugal force is in balance with the tension in the string. In an inertial system there is no centrifugal force; the rock is accelerating radially due to the force exerted by the string. Either system is valid for analyzing the problem. However, it is essential not to confuse the systems by trying to use fictitious forces in inertial frames.

Here are some examples to illustrate the use of rotating coordinates.

Example 8.6 **Surface of a Rotating Liquid**

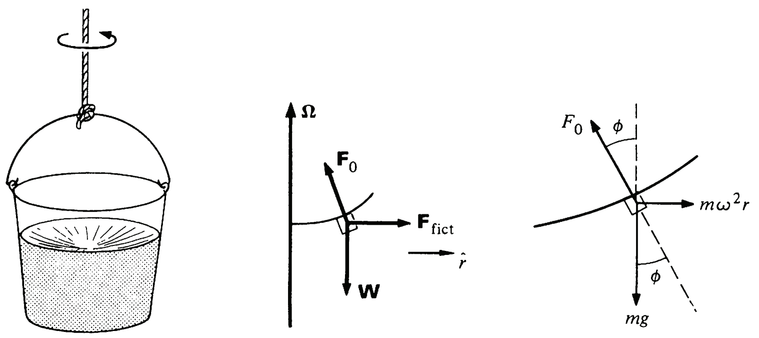
A bucket of water spins with angular speed ω. What shape does the water's surface assume?

In a coordinate system rotating with the bucket, the problem is purely static. Consider the force on a small volume of water of mass m at the surface of the liquid. For equilibrium, the total force on m must be zero. The forces are the contact force $\mathbf{F}_0$, the weight $\mathbf{W}$, and the fictitious force $\mathbf{F}_{\text{fict}}$, which is radial.

$$F_0 \cos\phi - W = 0$$

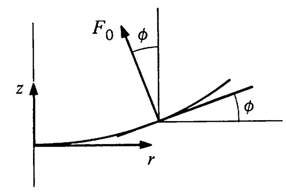
$$-F_0 \sin\phi + F_{\text{fict}} = 0,$$

where $F_{\text{fict}} = m\Omega^2 r = m\omega^2 r$, since $\Omega = \omega$ for a coordinate system rotating with the bucket.



Solving these equations for ϕ yields

$$\phi = \arctan \frac{\omega^2 r}{g}.$$



Unlike solids, liquids cannot exert a static force tangential to the surface. Hence $\mathbf{F}_0$, the force on m due to the neighboring liquid, must be perpendicular to the surface. The slope of the surface at any point is therefore

$$\frac{dz}{dr} = \tan \phi$$

$$= \frac{\omega^2 r}{g}.$$

We can integrate this relation to find the equation of the surface $z = f(r)$.

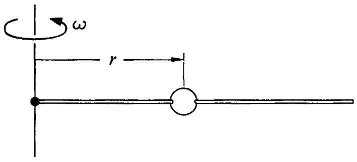
We have

$$\int dz = \frac{\omega^2}{g} \int r\, dr$$

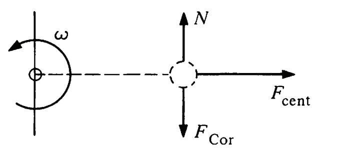
$$z = \frac{1}{2} \frac{\omega^2}{g} r^2,$$

where we have taken $z = 0$ on the axis at the surface of the liquid. The surface is a paraboloid of revolution.

Example 8.7 **The Coriolis Force**



A bead slides without friction on a rigid wire rotating at constant angular speed ω. The problem is to find the force exerted by the wire on the bead.



In a coordinate system rotating with the wire the motion is purely radial. The sketch shows the force diagram in the rotating system. F_{cent} is the centrifugal force and F_{Cor} is the Coriolis force. Since the wire is frictionless, the contact force N is normal to the wire. (We neglect gravity.) In the rotating system the equations of motion are

$$F_{cent} = m\ddot{r}$$

$$N - F_{Cor} = 0.$$

Using $F_{cent} = m\omega^2 r$, the first equation gives

$$m\ddot{r} - m\omega^2 r = 0,$$

which has the solution

$$r = Ae^{\omega t} + Be^{-\omega t},$$

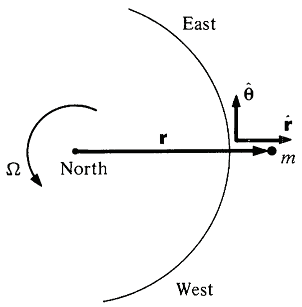
where A and B are constants depending on the initial conditions.

The tangential equation of motion, which expresses the fact that there is no tangential acceleration in the rotating system, gives

$$N = F_{\text{Cor}} = 2m\dot{r}\omega$$
$$= 2m\omega^2(Ae^{\omega t} - Be^{-\omega t}).$$

To complete the problem, we must be given the initial conditions which specify A and B.

Example 8.8 Deflection of a Falling Mass



Because of the Coriolis force, falling objects on the earth are deflected horizontally. For instance, a mass dropped from a tower lands to the east of a plumb line from the release point. In this example we shall calculate the deflection for a mass m dropped from a tower of height h at the equator.

In the coordinate system r, θ fixed to the earth (with the tangential direction toward the east) the apparent force on m is

$$\mathbf{F} = -mg\hat{\mathbf{r}} - 2m\mathbf{\Omega} \times \mathbf{v}_{\text{rot}} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$
$$F_\theta = -2m\dot{r}\Omega.$$

The gravitational and centrifugal forces are radial, and if m is dropped from rest, the Coriolis force is in the equatorial plane. Thus the motion of m is confined to the equatorial plane, and we have

$$\mathbf{v}_{\text{rot}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Using $\mathbf{\Omega} \times \mathbf{v}_{\text{rot}} = \Omega\dot{r}\hat{\boldsymbol{\theta}} - r\Omega\dot{\theta}\hat{\mathbf{r}}$, and $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\Omega^2 r\hat{\mathbf{r}}$, we obtain

$$F_r = -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r,$$

$$F_\theta = -2m\dot{r}\Omega.$$

The radial equation of motion is

$$m\ddot{r} - mr\dot{\theta}^2 = -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r.$$

To an excellent approximation, m falls vertically and $\dot{\theta} \ll \Omega$. We can therefore omit the terms $mr\dot{\theta}^2$ and $2m\Omega\dot{\theta}r$ in comparison with $m\Omega^2 r$. Thus

$$\ddot{r} = -g + \Omega^2 r. \qquad 1$$

The tangential equation of motion is

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -2m\dot{r}\Omega.$$

To the same approximation $\dot{\theta} \ll \Omega$ we have

$$r\ddot{\theta} = -2\dot{r}\Omega. \qquad 2$$

During the fall, r changes only slightly, from $R_e + h$ to R_e, where R_e is the radius of the earth, and we can take g to be constant and $r \approx R_e$. Equation (1) becomes

$$\begin{aligned}\ddot{r} &= -g + \Omega^2 R_e \\ &= -g',\end{aligned}$$

where $g' = g - \Omega^2 R_e$ is the acceleration due to the gravitational force minus a centrifugal term. g' is the apparent acceleration due to gravity, and since this is customarily denoted by g, we shall henceforth drop the prime. The solution of the radial equation of motion $\ddot{r} = -g$ is

$$\begin{aligned}\dot{r} &= -gt \\ r &= r_0 - \tfrac{1}{2}gt^2.\end{aligned} \qquad 3$$

If we insert $\dot{r} = -gt$ in the tangential equation of motion, Eq. (2), we have

$$r\ddot{\theta} = 2gt\Omega$$

or

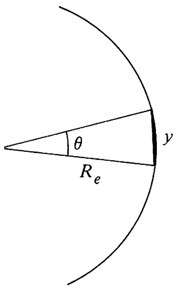
$$\ddot{\theta} = \frac{2g\Omega}{R_e}t,$$

where we have used $r \approx R_e$. Hence

$$\dot{\theta} = \frac{g\Omega}{R_e}t^2$$

and

$$\theta = \frac{1}{3}\frac{g\Omega}{R_e}t^3. \qquad 4$$



The horizontal deflection of m is $y \approx R_e\theta$ or

$$y = \tfrac{1}{3}g\Omega t^3.$$

The time T to fall distance h is given by

$$\begin{aligned}r - r_0 &= -h \\ &= -\tfrac{1}{2}gT^2\end{aligned}$$

so that

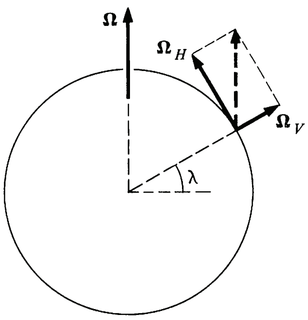
$$T = \sqrt{\frac{2h}{g}} \qquad \text{and} \qquad y = \frac{1}{3}g\Omega\left(\frac{2h}{g}\right)^{\frac{3}{2}}.$$

For a tower 50 m high,

$$y \approx 0.77 \text{ cm}.$$

θ is positive, and the deflection is toward the east.

Example 8.9 Motion on the Rotating Earth



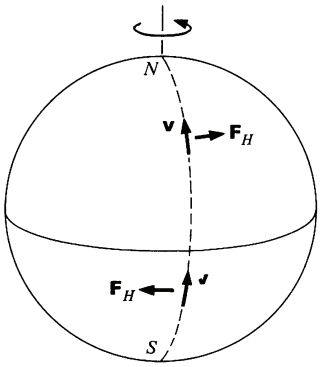
A surprising effect of the Coriolis force is that it turns straight line motion on a rotating sphere into circular motion. As we shall show in this example, for a velocity $\mathbf{v}$ tangential to the sphere (like the velocity of a wind over the earth's surface) the horizontal component of the Coriolis force is perpendicular to $\mathbf{v}$ and its magnitude is independent of the direction of $\mathbf{v}$.

Consider a particle of mass m moving with velocity $\mathbf{v}$ at latitude λ on the surface of a sphere. The sphere is rotating with angular velocity $\mathbf{\Omega}$. If we decompose $\mathbf{\Omega}$ into a vertical part $\mathbf{\Omega}_V$ and a horizontal part $\mathbf{\Omega}_H$, the Coriolis force is

$$\mathbf{F} = -2m\mathbf{\Omega} \times \mathbf{v}$$
$$= -2m(\mathbf{\Omega}_V \times \mathbf{v} + \mathbf{\Omega}_H \times \mathbf{v}).$$

$\mathbf{\Omega}_H$ and $\mathbf{v}$ are horizontal, so that $\mathbf{\Omega}_H \times \mathbf{v}$ is vertical. Thus the horizontal Coriolis force, $\mathbf{F}_H$, arises solely from the term $\mathbf{\Omega}_V \times \mathbf{v}$. $\mathbf{\Omega}_V$ is perpendicular to $\mathbf{v}$ and $\mathbf{\Omega}_V \times \mathbf{v}$ has magnitude $v\Omega_V$, independent of the direction of $\mathbf{v}$, as we wished to prove.

We can write the result in a more explicit form. If $\hat{\mathbf{r}}$ is a unit vector perpendicular to the surface at latitude λ, $\mathbf{\Omega}_V = \Omega \sin \lambda \hat{\mathbf{r}}$ and



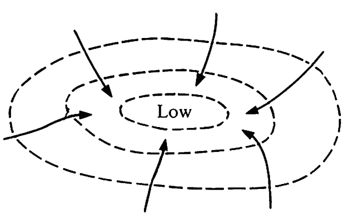
$$\mathbf{F}_H = -2m\Omega \sin \lambda \, \hat{\mathbf{r}} \times \mathbf{v}.$$

The magnitude of $\mathbf{F}_H$ is

$$F_H = 2mv\Omega \sin \lambda.$$

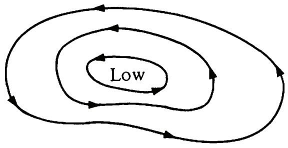
$\mathbf{F}_H$ is always perpendicular to $\mathbf{v}$, and in the absence of other horizontal forces it would produce circular motion, clockwise in the northern hemisphere and counterclockwise in the southern. Air flow on the earth is strongly influenced by the Coriolis force and without it stable circular weather patterns could not form. However, to understand the dynamics of weather systems we must also include other forces, as the next example discusses.

Example 8.10 Weather Systems

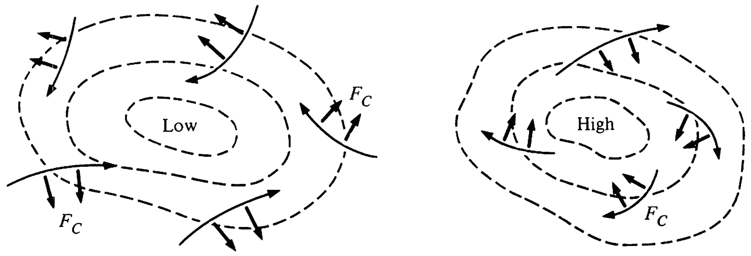


Imagine that a region of low pressure occurs in the atmosphere, perhaps because of differential heating of the air. The closed curves in the sketch represent lines of constant pressure, or *isobars*. There is a force on each element of air due to the pressure gradient, and in the absence of other forces winds would blow inward, quickly equalizing the pressure difference.

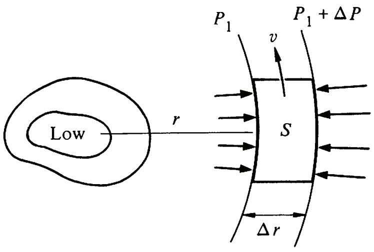
However, the wind pattern is markedly altered by the Coriolis force. As the air begins to flow inward, it is deflected sideways by the Coriolis



force, as shown in figure *a*. (The drawing is for the northern hemisphere.) The result is that the wind circulates counterclockwise about the low along the isobars, as in the sketch at left. Similarly, wind circulates clockwise about regions of high pressure in the northern hemisphere. Since the Coriolis force is essentially zero near the equator, circular weather systems cannot form there and the weather tends to be uniform.



(*a*) (*b*)



In order to analyze the motion, consider the forces on a parcel of air which is rotating about a low. The pressure force on the face along the isobar P_1 is P_1S, where S is the area of the inner face, as shown in the sketch. The force on the outer face is $(P_1 + \Delta P)S$, and the net pressure force is $(\Delta P)S$ inward. The Coriolis force is $2mv\Omega \sin \lambda$, where m is the mass of the parcel and v its velocity. The air is rotating counterclockwise about the low, so that the Coriolis force is outward. Hence, the radial equation of motion for steady circular flow is

$$\frac{mv^2}{r} = (\Delta P)S - 2mv\Omega \sin \lambda.$$

The volume of the parcel is $\Delta r\, S$, where Δr is the distance between the isobars, and the mass is $w\, \Delta r\, S$, where w is the density of air, assumed constant. Inserting this in the equation of motion and taking the limit $\Delta r \to 0$ yields

$$\frac{v^2}{r} = \frac{1}{w}\frac{dP}{dr} - 2v\Omega \sin \lambda. \qquad 1$$

Air masses do not rotate as rigid bodies. Near the center of the low, where the pressure gradient dP/dr is large, wind velocities are highest. Far from the center, v^2/r is small and can be neglected. Equation (1) predicts that far from the center the wind speed is

$$v = \frac{1}{2\Omega \sin \lambda}\frac{1}{w}\frac{dP}{dr}. \qquad 2$$

The density of air at sea level is 1.3 kg/m^3 and atmospheric pressure is $P_{at} = 10^5$ N/m^2. dP/dr can be estimated by looking at a weather map.

Far from a high or low, a typical gradient is 3 millibars over 100 km $\approx$ 3×10^{-3} N/m^3, and at latitude 45° Eq. (2) gives

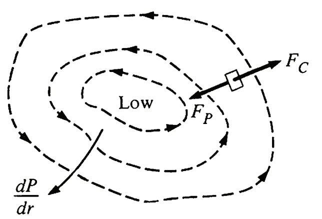
$$v = 22 \text{ m/s}$$
$$= 50 \text{ mi/h}.$$

Near the ground this speed is reduced by friction with the land, but at higher altitudes Eq. (2) can be applied with good accuracy.

A hurricane is an intense compact low in which the pressure gradient can be as high as 30×10^{-3} N/m^3. Hurricane winds are so strong that the v^2/r term in Eq. (1) cannot be neglected. Solving Eq. (1) for v we find

$$v = \sqrt{(r\Omega \sin \lambda)^2 + \frac{r}{w}\frac{dP}{dr}} - r\Omega \sin \lambda. \qquad 3$$

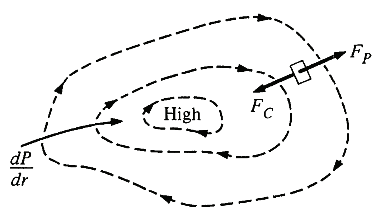
At a distance 100 km from the eye of a hurricane at latitude 20°, Eq. (3) predicts a wind speed of 45 m/s $\approx$ 100 mi/h for a pressure gradient of 30×10^{-3} N/m^3. This is in reasonable agreement with weather observations. At larger radii, the wind speed drops because of a decrease in the pressure gradient.



There is an interesting difference between lows and highs. In a low, the pressure force is inward and the Coriolis force is outward, whereas in a high, the directions of the forces are reversed. The radial equation of motion for air circulating around a high is

$$\frac{v^2}{r} = 2v\Omega \sin \lambda - \frac{1}{w}\left|\frac{dP}{dr}\right|. \qquad 4$$

Solving Eq. (4) for v yields



$$v = r\Omega \sin \lambda - \sqrt{(r\Omega \sin \lambda)^2 - \frac{r}{w}\left|\frac{dP}{dr}\right|}. \qquad 5$$

We see from Eq. (5) that if $1/w|dP/dr| > r(\Omega \sin \lambda)^2$, the high cannot form; the Coriolis force is too weak to supply the needed centripetal acceleration against the large outward pressure force. For this reason, storms like hurricanes are always low pressure systems; the strong inward pressure force helps hold a low together.

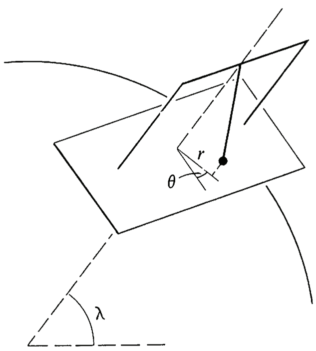
The Foucault pendulum provides one of the most dramatic demonstrations that the earth is a noninertial system. The pendulum is simply a heavy bob hanging from a long wire mounted to swing freely in any direction. As the pendulum swings back and forth, the plane of motion precesses slowly about the vertical, taking about a day and a half for a complete rotation in the mid-latitudes. The precession is a result of the earth's rotation.

The plane of motion tends to stay fixed in inertial space while the earth rotates beneath it.

In the 1850s Foucault hung a pendulum 67 m long from the dome of the Pantheon in Paris. The bob precessed almost a centimeter on each swing, and it presented the first direct evidence that the earth is indeed rotating. The pendulum became the rage of Paris.

The next example uses our analysis of the Coriolis force to calculate the motion of the Foucault pendulum in a simple way.

Example 8.11 **The Foucault Pendulum**



Consider a pendulum of mass m which is swinging with frequency $\gamma = \sqrt{g/l}$, where l is the length of the pendulum. If we describe the position of the pendulum's bob in the horizontal plane by coordinates r, θ, then

$$r = r_0 \sin \gamma t,$$

where r_0 is the amplitude of the motion. In the absence of the Coriolis force, there are no tangential forces and θ is constant.

The horizontal Coriolis force $\mathbf{F}_{\text{CH}}$ is

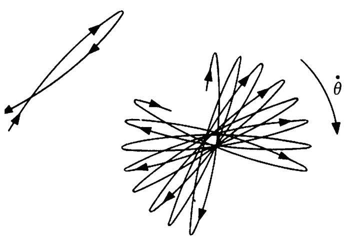
$$\mathbf{F}_{\text{CH}} = -2m\Omega \sin \lambda \dot{r}\hat{\boldsymbol{\theta}}.$$

Hence, the tangential equation of motion, $ma_\theta = F_{\text{CH}}$, becomes

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -2m\Omega \sin \lambda\, \dot{r}$$

or

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -2\Omega \sin \lambda\, \dot{r}.$$



The simplest solution to this equation is found by taking $\dot{\theta}$ = constant. In this case the term $r\ddot{\theta}$ vanishes, and we have

$$\dot{\theta} = -\Omega \sin \lambda.$$

The pendulum precesses uniformly in a clockwise direction. The time for the plane of oscillation to rotate once is

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi}{\Omega \sin \lambda} = \frac{24 \text{ h}}{\sin \lambda}.$$

Thus, at a latitude of 45°, the Foucault pendulum rotates once in 34 h.

At the North Pole the period of precession is 24 h; the pendulum rotates clockwise with respect to the earth at the same rate as the earth rotates counterclockwise. With respect to inertial space the plane of motion remains fixed.

In addition to its dramatic display of the earth's rotation, the Foucault pendulum embodies a profound mystery. Consider, for instance, a Foucault pendulum at the North Pole. The precession is obviously an artifact; the plane of motion stays fixed while the earth rotates beneath it. The plane of the pendulum remains fixed relative to the fixed stars. Why should this be? How does the pendulum "know" that it must swing in a plane which is stationary relative to the fixed stars instead of, say, in a plane which rotates at some uniform rate?

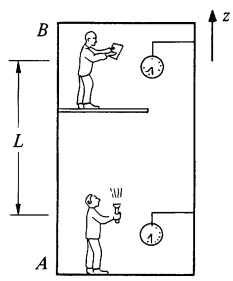
This question puzzled Newton, who described it in terms of the following experiment: if a bucket contains water at rest, the surface of the water is flat. If the bucket is set spinning at a steady rate, the water at first lags behind, but gradually, as the water's rotational speed increases, the surface takes on the form of the parabola of revolution discussed in Example 8.6. If the bucket is suddenly stopped, the concavity of the water's surface persists for some time. It is evidently not motion relative to the bucket that is important in determining the shape of the liquid surface. So long as the water rotates, the surface is depressed. Newton concluded that rotational motion is absolute, since by observing the water's surface it is possible to detect rotation without reference to outside objects.

From one point of view there is really no paradox to the absolute nature of rotational motion. The principle of galilean invariance asserts that there is no way to detect locally the uniform translational motion of a system. However, this does not limit our ability to detect the *acceleration* of a system. A rotating system accelerates in a most nonuniform way. At every point the acceleration is directed toward the axis of rotation; the acceleration points out the axis. Our ability to detect such an acceleration in no way contradicts galilean invariance.

Nevertheless, there is an engima. Both the rotating bucket and the Foucault pendulum maintain their motion *relative to the fixed stars*. How do the fixed stars determine an inertial system? What prevents the plane of the pendulum from rotating with respect to the fixed stars? Why is the surface of the water in the rotating bucket flat only when the bucket is at rest with respect to the fixed stars? Ernst Mach, who in 1883 wrote the first incisive critique of newtonian physics, put the matter this way. Suppose that we keep a bucket of water fixed and rotate all the stars. Physically there is no way to distinguish this from the original case where the bucket is rotated, and we expect the surface of the water to again assume a parabolic shape. Apparently the motion of the water in the bucket depends on the motion of matter far off in the universe. To put it more dramatically, suppose that we eliminate the stars, one by one, until only our bucket remains. What will happen now if we rotate the bucket? There is no way for us to predict the motion of the water in the bucket—the inertial properties of space might be totally different. We have a most peculiar situation. The local properties of space depend on far-off matter, yet when we rotate the water, the surface *immediately* starts to deflect. There is no time for signals to travel to the distant stars and return. How does the water in the bucket "know" what the rest of the universe is doing?

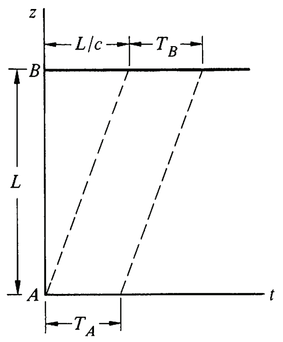
The principle that the inertial properties of space depend on the existence of far-off matter is known as Mach's principle. The principle is accepted by many physicists, but it can lead to strange conclusions. For instance, there is no reason to believe that matter in the universe is uniformly distributed around the earth; the solar system is located well out in the limb of our galaxy, and matter in our galaxy is concentrated predominantly in a very thin plane. If inertia is due to far-off matter, then we might well expect it to be different in different directions so that the value of mass would depend on the direction of acceleration. No such effects have ever been observed. Inertia remains a mystery.

Note 8.1 The Equivalence Principle and the Gravitational Red Shift

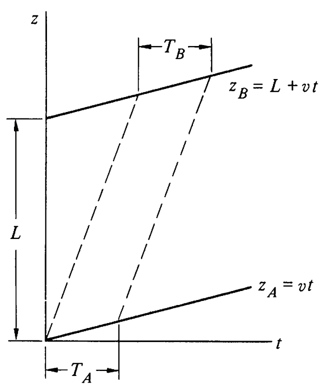


(a)

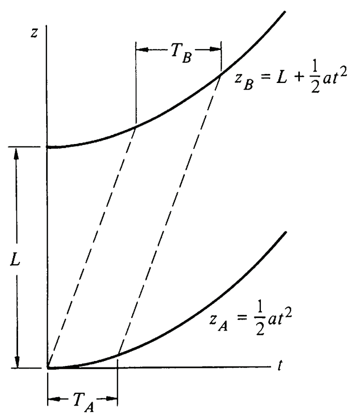
Radiating atoms emit light at only certain characteristic wavelengths. If light from atoms in the strong gravitational field of dense stars is analyzed spectroscopically, the characteristic wavelengths are observed to be slightly increased, shifted toward the red. We can visualize atoms as clocks which "tick" at characteristic frequencies. The shift toward longer wavelengths, known as the gravitational red shift, corresponds to a slowing of the clocks. The gravitational red shift implies that clocks in a gravitational field appear to run slow when viewed from outside the field. As we shall show, the origin of the effect lies in the nature of space, time, and gravity, not in the trivial effect of gravity on mechanical clocks.



(b)



(c)



(d)

It is rather startling to see how the equivalence principle, which is so simple and nonmathematical, leads directly to a connection between space, time, and gravity. To show the connection we must use an elementary result from the theory of relativity; it is impossible to transmit information faster than the velocity of light, $c = 3 \times 10^8$ m/s. However, this is the only relativistic idea needed; aside from this, our argument is completely classical.

Consider two scientists, A and B, separated by distance L as shown in sketch (a). A has a clock and a light which he flashes at intervals separated by time T_A. The signals are received by B, who notes the interval between pulses, T_B, with his own clock. A plot of vertical distance versus time is shown for two light pulses in (b). The pulses are delayed by the transit time, L/c, but the interval T_B is the same as T_A. Hence, if A transmits the pulses at, say, 1-s intervals, so that $T_A = 1$ s, then B's clock will read 1 s between the arrival of successive pulses.

Now consider the situation if both observers move upward uniformly with speed v, as shown in sketch (c). Although both scientists move during the time interval, they move equally, and we still have $T_B = T_A$.

The situation is entirely different if both observers are accelerating upward at uniform rate a as shown in sketch (d). A and B start from rest, and the graph of distance versus time is a parabola. Since A and B have the same acceleration, the curves are parallel, separated by distance L at each instant. It is apparent from the sketch that $T_B > T_A$, since the second pulse travels farther than the first and has a longer transit time. The effect is purely kinematical.

Now, by the principle of equivalence, A and B cannot distinguish between their upward accelerating system and a system at rest in a downward gravitational field with magnitude $g = a$. Thus, if the experiment is repeated in a system at rest in a gravitational field, the equivalence principle requires that $T_B > T_A$, as before. If $T_A = 1$ s, B will observe an interval greater than 1 s between successive pulses. B will conclude that A's clock is running slow. This is the origin of the gravitational red shift.

By applying the argument quantitatively, the following approximate result is readily obtained:

$$\frac{\Delta T}{T} = \frac{T_B - T_A}{T_A} = \frac{gL}{c^2},$$

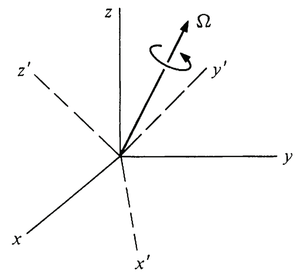
where it is assumed that $\Delta T/T \ll 1$.

On earth the gravitational red shift is $\Delta T/T = 10^{-16} L$, where L is in meters. In spite of its small size, the effect has been measured and confirmed to an accuracy of 1 percent. The experiment was done by Pound, Rebka, and Snyder at Harvard University. The "clock" was the frequency of a gamma ray, and by using a technique known as Mössbauer absorption they were able to measure accurately the gravitational red shift due to a vertical displacement of 25 m.

Note 8.2 Rotating Coordinate Transformation

In this note we present an analytical derivation of Eq. (8.7) relating the time derivative of any vector **B** as observed in a rotating coordinate system to the time derivative observed in an inertial system. If the system x', y', z' rotates at rate $\mathbf{\Omega}$ with respect to the inertial system x, y, z, we shall prove that the time derivatives in the two systems of any vector **B** are related by

$$\left(\frac{d\mathbf{B}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{B}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{B}. \tag{1}$$



Consider an inertial coordinate system x, y, z and a coordinate system x', y', z' which rotates with respect to the inertial system at angular velocity $\mathbf{\Omega}$. The origins coincide. We can describe an arbitrary vector **B** by components along base vectors of either coordinate system. Thus, we have

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{2}$$

or, alternatively,

$$\mathbf{B} = B_x'\hat{\mathbf{i}}' + B_y'\hat{\mathbf{j}}' + B_z'\hat{\mathbf{k}}', \tag{3}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are the base vectors along the inertial axes and $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, $\hat{\mathbf{k}}'$ are the base vectors along the rotating axes.

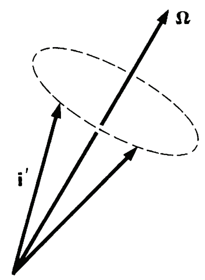
We now find an expression for the time derivative of **B** in each coordinate system. By differentiating Eq. (2) we have

$$\left(\frac{d\mathbf{B}}{dt}\right) = \frac{d}{dt}(\mathbf{B}_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}).$$

The x, y, z system is inertial so that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are fixed in space. We have

$$\frac{d\mathbf{B}}{dt} = \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}}, \tag{4}$$

which is the familiar expression for the time derivative of a vector in cartesian coordinates. We designate this expression by $(d\mathbf{B}/dt)_{\text{in}}$.



If we differentiate Eq. (3) we obtain

$$\left(\frac{d\mathbf{B}}{dt}\right) = \left(\frac{dB_x'}{dt}\hat{\mathbf{i}}' + \frac{dB_y'}{dt}\hat{\mathbf{j}}' + \frac{dB_z'}{dt}\hat{\mathbf{k}}'\right) + \left(B_x'\frac{d\hat{\mathbf{i}}'}{dt} + B_y'\frac{d\hat{\mathbf{j}}'}{dt} + B_z'\frac{d\hat{\mathbf{k}}'}{dt}\right). \tag{5}$$

The first term is the time derivative of **B** with respect to the $x'y'z'$ axes; this is the rate of change of **B** which would be measured by an observer in the rotating system, $(d\mathbf{B}/dt)_{\text{rot}}$. To evaluate the second term, note that since $\hat{\mathbf{i}}'$ is a unit vector, it can change only in direction, not in magnitude; thus $\hat{\mathbf{i}}'$ undergoes pure rotation. In Sec. 7.2 we found that the time derivative of a vector **r** of constant magnitude rotating with angular velocity $\boldsymbol{\omega}$ is $d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$. We can use this result to evaluate $d\hat{\mathbf{i}}'/dt$. Let $\mathbf{r}$ lie along the x' axis and have unit magnitude: $\mathbf{r} = \hat{\mathbf{i}}'$. Hence

$$\frac{d\hat{\mathbf{i}}'}{dt} = \mathbf{\Omega} \times \hat{\mathbf{i}}'.$$

Similarly,

$$\frac{d\hat{\mathbf{j}}'}{dt} = \mathbf{\Omega} \times \hat{\mathbf{j}}' \qquad \text{and} \qquad \frac{d\hat{\mathbf{k}}'}{dt} = \mathbf{\Omega} \times \hat{\mathbf{k}}'.$$

The second term in Eq. (5) becomes

$$\begin{aligned} B'_x(\mathbf{\Omega} \times \hat{\mathbf{i}}') + B'_y(\mathbf{\Omega} \times \hat{\mathbf{j}}') + B'_z(\mathbf{\Omega} \times \hat{\mathbf{k}}') &= \mathbf{\Omega} \times (B'_x\hat{\mathbf{i}}' + B'_y\hat{\mathbf{j}}' + B'_z\hat{\mathbf{k}}') \\ &= \mathbf{\Omega} \times \mathbf{B}. \end{aligned}$$

Equation (5) becomes

$$\left(\frac{d\mathbf{B}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{B}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{B}, \qquad 6$$

which is the desired result.

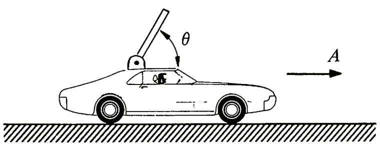
Since $\mathbf{B}$ is an arbitrary vector, this result is quite general; it can be applied to any vector we choose. It is important to be clear on the meaning of Eq. (6). The vector $\mathbf{B}$ itself is the same in both the inertial and the rotating coordinate systems. (For this reason there is no subscript to $\mathbf{B}$ in the term $\mathbf{\Omega} \times \mathbf{B}$.) It is only the time derivative of $\mathbf{B}$ which depends on the coordinate system. For instance, a vector which is constant in one system will change with time in the other.

Problems

8.1 A uniform thin rod of length L and mass M is pivoted at one end. The pivot is attached to the top of a car accelerating at rate A, as shown.

a. What is the equilibrium value of the angle θ between the rod and the top of the car?

b. Suppose that the rod is displaced a small angle ϕ from equilibrium. What is its motion for small ϕ?

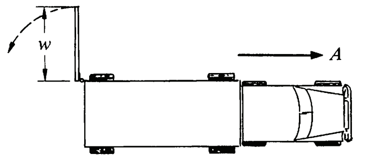


8.2 A truck at rest has one door fully open, as shown. The truck accelerates forward at constant rate A, and the door begins to swing shut.

The door is uniform and solid, has total mass M, height h, and width w. Neglect air resistance.

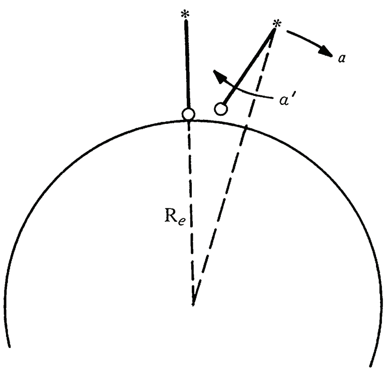
a. Find the instantaneous angular velocity of the door about its hinges when it has swung through 90°.

b. Find the horizontal force on the door when it has swung through 90°.



8.3 A pendulum is at rest with its bob pointing toward the center of the earth. The support of the pendulum starts to move horizontally with uniform acceleration a, and the pendulum starts to swing. Find the angular acceleration α' of the pendulum. Find the period of the pendulum for which the bob continues to point toward the center of the earth. Neglect rotation of the earth. This is the principle of a device known as a Schuler pendulum which is used to suspend the gyroscope stage in inertial guidance systems.)

Ans. clue. $T \approx 1\frac{1}{2}$ h



8.4 The center of mass of a 3,200-lb car is midway between the wheels and 2 ft above the ground. The wheels are 8 ft apart.

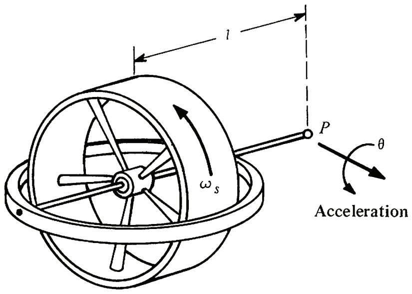
a. What is the minimum acceleration A of the car so that the front wheels just begin to lift off the ground?

b. If the car decelerates at rate g, what is the normal force on the front wheels and on the rear wheels?

8.5 Many applications for gyroscopes have been found in navigational systems. For instance, gyroscopes can be used to measure acceleration. Consider a gyroscope spinning at high speed ω_s. The gyroscope is attached to a vehicle by a universal pivot P. If the vehicle accelerates in the direction perpendicular to the spin axis at rate a, then the gyroscope will precess about the acceleration axis, as shown in the sketch. The total angle of precession, θ, is measured. Show that if the system starts from rest, the final velocity of the vehicle is given by

$$v = \frac{I_s\omega_s}{Ml}\,\theta,$$

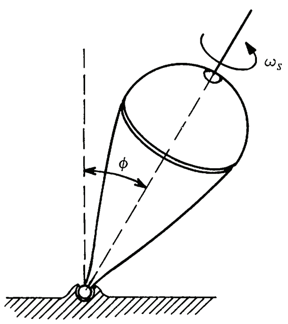
where $I_s\omega_s$ is the gyroscope's spin angular momentum, M is the total mass of the pivoted portion of the gyroscope, and l is the distance from the pivot to the center of mass. (Such a system is called an integrating gyro, since it automatically integrates the acceleration to give the velocity.)



8.6 A top of mass M spins with angular speed ω_s about its axis, as shown. The moment of inertia of the top about the spin axis is I_0, and the center of mass of the top is a distance l from the point. The axis is inclined at angle ϕ with respect to the vertical, and the top is undergoing uniform precession. Gravity is directed downward. The top is in an elevator, with its tip held to the elevator floor by a frictionless pivot. Find the rate of precession, Ω, clearly indicating its direction, in each of the following cases:

a. The elevator at rest

b. The elevator accelerating down at rate $2g$



8.7 Find the difference in the apparent acceleration of gravity at the equator and the poles, assuming that the earth is spherical.

8.8 Derive the familiar expression for velocity in plane polar coordinates, $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$, by examining the motion of a particle in a rotating coordinate system in which the velocity is instantaneously radial.

8.9 A 400-ton train runs south at a speed of 60 mi/h at a latitude of 60° north.

a. What is the horizontal force on the tracks?

b. What is the direction of the force?

Ans. (*a*) Approximately 300 lb

8.10 The acceleration due to gravity measured in an earthbound coordinate system is denoted by g. However, because of the earth's rotation, g differs from the true acceleration due to gravity, g_0. Assuming that the earth is perfectly round, with radius R_e and angular velocity Ω_e, find g as a function of latitude λ. (Assuming the earth to be round is actually not justified—the contributions to the variation of g with latitude due to the polar flattening is comparable to the effect calculated here.)

Ans. $g = g_0[1 - (2x - x^2)\cos^2\lambda]^{\frac{1}{2}}$, where $x = R_e\Omega_e{}^2/g_0$

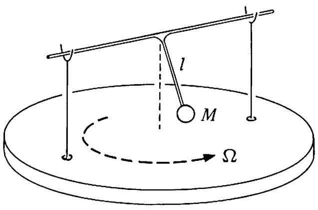
8.11 A high speed hydrofoil races across the ocean at the equator at a speed of 200 mi/h. Let the acceleration of gravity for an observer at rest on the earth be g. Find the fractional change in gravity, $\Delta g/g$, measured by a passenger on the hydrofoil when the hydrofoil heads in the following directions:

a. East

b. West

c. South

d. North



8.12 A pendulum is rigidly fixed to an axle held by two supports so that it can swing only in a plane perpendicular to the axle. The pendulum consists of a mass M attached to a massless rod of length l. The supports are mo inted on a platform which rotates with constant angular velocity Ω. Find the pendulum's frequency assuming that the amplitude is small.

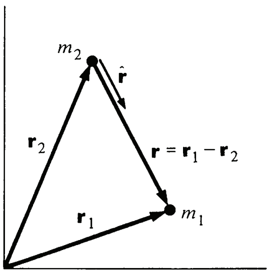
9 CENTRAL FORCE MOTION

9.1 Introduction

It was Newton's fascination with planetary motion that led him to formulate his laws of motion and the law of universal gravitation. His success in explaining Kepler's empirical laws of planetary motion was an overwhelming argument in favor of the new mechanics and marked the beginning of modern mathematical physics. Planetary motion and the more general problem of motion under a central force continue to play an important role in most branches of physics and turn up in such topics as particle scattering, atomic structure, and space navigation.

In this chapter we apply newtonian physics to the general problem of central force motion. We shall start by looking at some of the general features of a system of two particles interacting with a central force $f(r)\hat{\mathbf{r}}$, where $f(r)$ is any function of the distance r between the particles and $\hat{\mathbf{r}}$ is a unit vector along the line of centers. After making a simple change of coordinates, we shall show how to find a complete solution by using the conservation laws of angular momentum and energy. Finally, we shall apply these results to the case of planetary motion, $f(r) \propto 1/r^2$, and show how they predict Kepler's empirical laws.

9.2 Central Force Motion as a One Body Problem



Consider an isolated system consisting of two particles interacting under a central force $f(r)$. The masses of the particles are m_1 and m_2 and their position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. We have

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ r &= |\mathbf{r}| \\ &= |\mathbf{r}_1 - \mathbf{r}_2|. \end{aligned} \qquad 9.1$$

The equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}} \qquad 9.2a$$

$$m_2\ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}. \qquad 9.2b$$

The force is attractive for $f(r) < 0$ and repulsive for $f(r) > 0$. Equations (9.2*a* and *b*) are coupled together by $\mathbf{r}$; the behavior of $\mathbf{r}_1$ and $\mathbf{r}_2$ depends on $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. We shall show that the problem is easier to handle if we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the center of mass vector $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$. The equation of motion for $\mathbf{R}$ is trivial since there are no external forces. The equation for $\mathbf{r}$ turns out to be like the equation of motion of a single particle and has a straightforward solution.

The equation of motion for $\mathbf{R}$ is

$$\ddot{\mathbf{R}} = 0,$$

which has the simple solution

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{V}t. \tag{9.3}$$

The constant vectors $\mathbf{R}_0$ and $\mathbf{V}$ depend on the choice of coordinate system and the initial conditions. If we are clever enough to take the origin at the center of mass, $\mathbf{R}_0 = 0$ and $\mathbf{V} = 0$.

To find the equation of motion for $\mathbf{r}$ we divide Eq. (9.2a) by m_1 and Eq. (9.2b) by m_2 and subtract. This gives

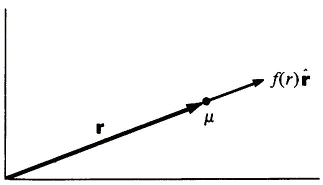
$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\mathbf{r}}$$

or

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = f(r)\hat{\mathbf{r}}.$$

Denoting $m_1 m_2/(m_1 + m_2)$ by μ, the *reduced mass*, and using $\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}$, we have

$$\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}. \tag{9.4}$$



Equation (9.4) is identical to the equation of motion for a particle of mass μ acted on by a force $f(r)\hat{\mathbf{r}}$; no trace of the two particle problem remains. The two particle problem has been transformed to a one particle problem. (Unfortunately, the method cannot be generalized. There is no way to reduce the equations of motion for three or more particles to equivalent one body equations, and for this reason the exact solution of the three body problem is unknown.)

The problem now is to find $\mathbf{r}$ as a function of time from Eq. (9.4). Once we know $\mathbf{r}$, we can easily find $\mathbf{r}_1$ and $\mathbf{r}_2$ by using the relations

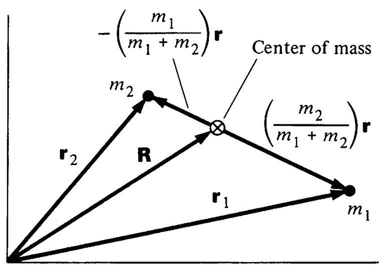
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{9.5a}$$

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{9.5b}$$

Solving for $\mathbf{r}_1$ and $\mathbf{r}_2$ gives

$$\mathbf{r}_1 = \mathbf{R} + \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r} \tag{9.6a}$$

$$\mathbf{r}_2 = \mathbf{R} - \left(\frac{m_1}{m_1 + m_2}\right)\mathbf{r}. \tag{9.6b}$$



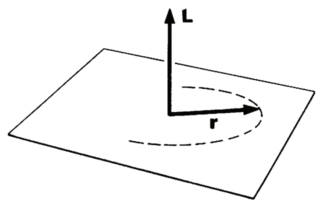
$m_2\mathbf{r}/(m_1 + m_2)$ and $-m_1\mathbf{r}/(m_1 + m_2)$ are the position vectors of m_1 and m_2 relative to the center of mass, as the sketch shows.

The complete solution of $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ depends on the particular form of $f(r)$. However, a number of the properties of central force motion hold true in general regardless of the form of $f(r)$, and we turn next to investigate these.

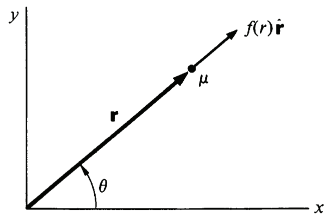
9.3 General Properties of Central Force Motion

The equation $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ is a vector equation, and although only a single particle is involved, there are three components to be considered. In this section we shall see how to use the conservation laws to find some general properties of the solution and to reduce the equation to an equation in a single scalar variable.

The Motion Is Confined to a Plane



The central force $f(r)\,\hat{\mathbf{r}}$ is along $\mathbf{r}$ and can exert no torque on the reduced mass μ. Hence, the angular momentum $\mathbf{L}$ of μ is constant. It is easy to show that this implies that the motion of μ is confined to a plane. Since $\mathbf{L} = \mathbf{r} \times \mu\mathbf{v}$, where $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{r}$ is always perpendicular to $\mathbf{L}$ by the properties of the cross product. However, $\mathbf{L}$ is fixed in space, and it follows that $\mathbf{r}$ can only move in the plane perpendicular to $\mathbf{L}$ through the origin.



Since the motion is confined to a plane, we can, without loss of generality, choose our coordinate system so that the motion is in the xy plane. Introducing polar coordinates, the equation of motion $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ becomes

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad 9.7a$$

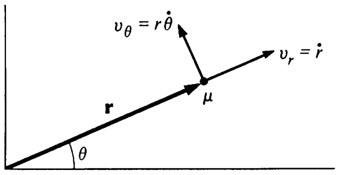
$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \qquad 9.7b$$

The Energy and Angular Momentum Are Constants of the Motion

We have reduced the problem to two dimensions by using the fact that the direction of $\mathbf{L}$ is constant. There are two other important constants of central force motion: the magnitude of the angular momentum $|\mathbf{L}| \equiv l$, and the total energy E. Using l and E, we can solve the problem of central force motion more easily and with greater physical insight than by working with Eqs. (9.7a and b).

The angular momentum of μ has magnitude

$$l = \mu r v_\theta = \mu r^2\dot{\theta}. \qquad 9.8a$$



The total energy of μ is

$$E = \tfrac{1}{2}\mu v^2 + U(r)$$
$$= \tfrac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) + U(r), \qquad 9.8b$$

where the potential energy $U(r)$ is given by

$$U(r) - U(r_a) = -\int_{r_a}^{r} f(r)\,dr.$$

The constant $U(r_a)$ is not physically significant and so we can leave r_a unspecified; adding a constant to the energy has no effect on the motion.

We can eliminate $\dot\theta$ from Eq. (9.8b) by using Eq. (9.8a). The result is

$$E = \frac{1}{2}\mu\dot r^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r). \qquad 9.9$$

This looks like the equation of motion of a particle moving in one dimension; all reference to θ is gone. We can press the parallel further by introducing

$$U_{\text{eff}}(r) = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r), \qquad 9.10$$

so that

$$E = \tfrac{1}{2}\mu\dot r^2 + U_{\text{eff}}(r). \qquad 9.11$$

U_{eff} is called the *effective potential energy*. Often it is referred to simply as the *effective potential* v_{eff} differs from the true potential $U(r)$ by the term $l^2/2\mu r^2$, called the *centrifugal potential*.

The formal solution of Eq. (9.11) is

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}})} \qquad 9.12$$

or

$$\int_{r_0}^{r} \frac{dr}{\sqrt{(2/\mu)(E - U_{\text{eff}})}} = t - t_0. \qquad 9.13$$

Equation (9.13) gives us r as a function of t, although the integral may have to be done numerically in some cases. To find θ as a function of t, we can use the solution for r in Eq. (9.8a):

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2}. \qquad 9.14$$

Since r is known as a function of t from Eq. (9.13), it is possible to integrate to find θ:

$$\theta - \theta_0 = \int_{t_0}^{t} \frac{l}{\mu r^2}\,dt. \qquad 9.15$$

Often we are interested in the path of the particle, which means knowing r as a function of θ rather than as a function of time. We call $r(\theta)$ the *orbit* of the particle. (The term is used even if the trajectory does not close on itself.) Dividing Eq. (9.14) by Eq. (9.12) gives

$$\frac{d\theta}{dr} = \frac{l}{\mu r^2} \frac{1}{\sqrt{(2/\mu)(E - U_{\text{eff}})}}. \qquad 9.16$$

This completes the formal solution of the central force problem. We can obtain $r(t)$, $\theta(t)$, or $r(\theta)$ as we please; all we need to do is evaluate the appropriate integrals.

You may have noticed that we found the solution without using the equations of motion, Eqs. (9.7a and b). Actually, we did use them, but in a disguised form. For instance, differentiating $l = \mu r^2\dot{\theta}$ with respect to time gives $0 = \mu r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}$ or

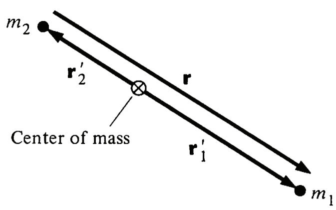
$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0,$$

which is identical to the tangential equation of motion, Eq. (9.7b). Similarly, differentiation of the energy equation with respect to time gives the radial equation of motion, Eq. (9.7a).

The Law of Equal Areas

We have already shown in Example 6.3 that for any central force, **r** sweeps out equal areas in equal times. This general property of central force motion is a direct consequence of the fact that the angular momentum is constant.

9.4 Finding the Motion in Real Problems



In order to apply the solution for the motion which we found in the last section, we need to relate the position vectors of m_1 and m_2 to **r** and evaluate l and E.

From Eqs. (9.6a and b) the position vectors of m_1 and m_2 relative to the center of mass are

$$\mathbf{r}_1' = \frac{m_2}{m_1 + m_2}\mathbf{r} \qquad 9.17a$$

$$\mathbf{r}_2' = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \qquad 9.17b$$

$\mathbf{r}_1'$ and $\mathbf{r}_2'$ lie along **r**. They remain back to back in the plane of motion. Hence, m_1 and m_2 move about their center of mass in the fixed plane, separated by distance r.

In many problems, like the motion of a planet around the sun, the masses of the two particles are very different. If $m_2 \gg m_1$, Eqs. (9.17a and b) become

$$\mathbf{r}'_1 \approx \mathbf{r}$$

$$\mathbf{r}'_2 \approx 0.$$

The reduced mass μ is approximately m_1, and the center of mass lies at m_2. In this case the more massive particle is essentially fixed at the origin, and there is no important difference between the actual two particle problem and the equivalent one particle problem.

In the one particle problem the angular momentum is

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v}.$$

It is easy to show that $\mathbf{L}$ is simply the angular momentum of m_1 and m_2 about the center of mass, $\mathbf{L}_c$.

$$\mathbf{L}_c = m_1\mathbf{r}'_1 \times \mathbf{v}'_1 + m_2\mathbf{r}'_2 \times \mathbf{v}'_2,$$

where $\mathbf{v}'_1 = \dot{\mathbf{r}}'_1$ and $\mathbf{v}'_2 = \dot{\mathbf{r}}'_2$. Using Eqs. (9.17$a$ and b) we have

$$\begin{aligned}
\mathbf{L}_c &= \frac{m_1 m_2}{m_1 + m_2}\mathbf{r} \times \mathbf{v}'_1 - \frac{m_1 m_2}{m_1 + m_2}\mathbf{r} \times \mathbf{v}'_2 \\
&= \mu\mathbf{r} \times (\mathbf{v}'_1 - \mathbf{v}'_2) \\
&= \mu\mathbf{r} \times \mathbf{v} \\
&= \mathbf{L}.
\end{aligned}$$

Similarly, the total energy E is the energy of m_1 and m_2 relative to their center of mass, E_c.

$$E_c = \tfrac{1}{2}m_1(\mathbf{v}'_1 \cdot \mathbf{v}'_1) + \tfrac{1}{2}m_2(\mathbf{v}'_2 \cdot \mathbf{v}'_2) + U(r).$$

From Eqs. (9.16a and b), we have $m_1\mathbf{v}'_1 = \mu\mathbf{v}$ and $m_2\mathbf{v}'_2 = -\mu\mathbf{v}$. Hence,

$$\begin{aligned}
E_c &= \tfrac{1}{2}\mu\mathbf{v} \cdot (\mathbf{v}'_1 - \mathbf{v}'_2) + U(r) \\
&= \tfrac{1}{2}\mu(\mathbf{v} \cdot \mathbf{v}) + U(r) \\
&= E.
\end{aligned}$$

9.5 The Energy Equation and Energy Diagrams

In Sec. 9.3 we found two equivalent ways of writing E, the total energy in the center of mass system. According to Eq. (9.8b),

$$E = \tfrac{1}{2}\mu v^2 + U(r),$$

and according to Eq. (9.11),

$$E = \tfrac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r).$$

We generally need to use both these forms in analyzing central force motion. The first form, $\frac{1}{2}\mu v^2 + U(r)$, is handy for evaluating E; all we need to know is the relative speed and position at some instant. However, $v^2 = \dot{r}^2 + (r\dot{\theta})^2$, and this dependence on two coordinates, r and θ, makes it difficult to visualize the motion. In contrast, the second form, $\frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$ depends on the single coordinate r. In fact, it is identical to the equation for the energy of a particle of mass μ constrained to move along a straight line with kinetic energy $\frac{1}{2}\mu\dot{r}^2$ and potential energy $U_{\text{eff}}(r)$. The coordinate θ is completely suppressed—the kinetic energy associated with the tangential motion, $\frac{1}{2}\mu(r\dot{\theta})^2$, is accounted for in the effective potential by the relations

$$\frac{1}{2}\mu(r\dot{\theta})^2 = \frac{l^2}{2\mu r^2}$$

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + U(r).$$

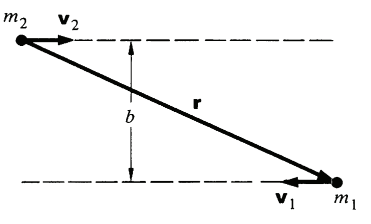
The equation

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

involves only the radial motion. Consequently, we can use the energy diagram technique developed in Chap. 4 to find the qualitative features of the radial motion.

To see how the method works, let's start by looking at a very simple system, two noninteracting particles.

Example 9.1 **Noninteracting Particles**



Two noninteracting particles m_1 and m_2 move toward each other with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Their paths are offset by distance b, as shown in the sketch. Let us investigate the equivalent one body description of this system.

The relative velocity is

$$\begin{aligned}\mathbf{v}_0 &= \dot{\mathbf{r}}\\ &= \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2\\ &= \mathbf{v}_1 - \mathbf{v}_2.\end{aligned}$$

$\mathbf{v}_0$ is constant since $\mathbf{v}_1$ and $\mathbf{v}_2$ are constant. The energy of the system relative to the center of mass is

$$E = \frac{1}{2}\mu v_0^2 + U(r) = \frac{1}{2}\mu v_0^2,$$

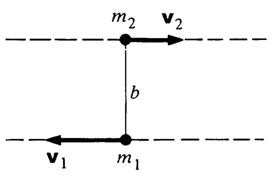
since $U(r) = 0$ for noninteracting particles.

In order to draw the energy diagram we need to find the effective potential

$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} + U(r) = \frac{l^2}{2\mu r^2}.$$

We could evaluate l by direct computation, but it is simpler to use the relation

$$E = \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2}$$
$$= \tfrac{1}{2}\mu v_0^2.$$



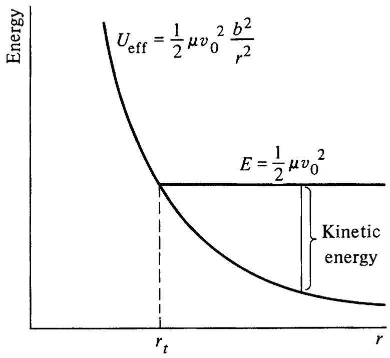
When m_1 and m_2 pass each other, $r = b$ and $\dot{r} = 0$. Hence

$$\frac{l^2}{2\mu b^2} = \tfrac{1}{2}\mu v_0^2,$$

$$l = \mu b v_0,$$

and

$$U_{\text{eff}} = \tfrac{1}{2}\mu v_0^2 \frac{b^2}{r^2}.$$



The energy diagram is shown in the sketch. The kinetic energy associated with radial motion is

$$K = \tfrac{1}{2}\mu\dot{r}^2$$
$$= E - U_{\text{eff}}.$$

K is never negative so that the motion is restricted to regions where $E - U_{\text{eff}} \geq 0$. Initially r is very large. As the particles approach, the kinetic energy decreases, vanishing at the *turning point* r_t, where the radial velocity is zero and the motion is purely tangential. At the turning point $E = U_{\text{eff}}(r_t)$, which gives

$$\tfrac{1}{2}\mu v_0^2 = \tfrac{1}{2}\mu v_0^2 \frac{b^2}{r_t^2}$$

or

$$r_t = b$$

as we expect, since r_t is the distance of closest approach of the particles. Once the turning point is passed, r increases and the particles separate. In our one dimensional picture, the particle μ "bounces off" the barrier of the effective potential.

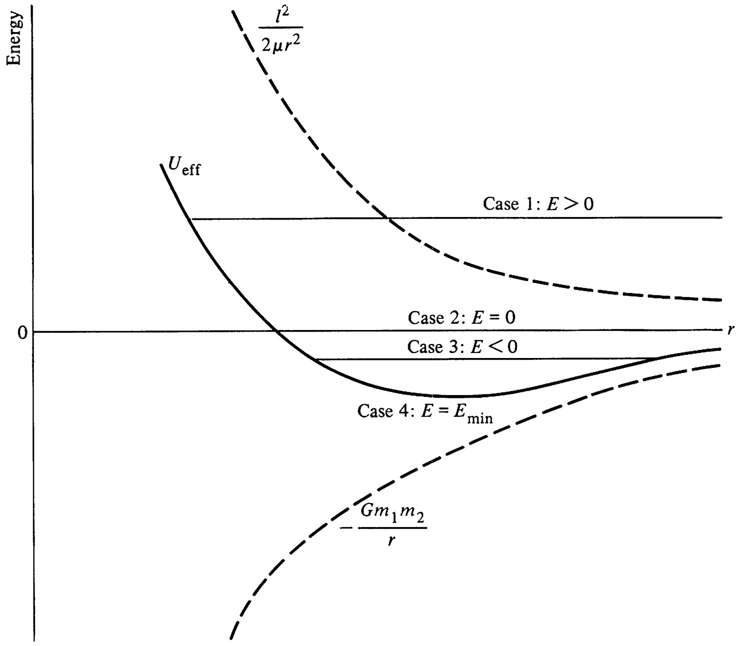
Now let us apply energy diagrams to the meatier problem of planetary motion. For the attractive gravitational force,

$$f(r) = -\frac{Gm_1m_2}{r^2}$$

$$U(r) = -\frac{Gm_1m_2}{r}.$$

(By the usual convention, we take $U(\infty) = 0$.) The effective potential energy is

$$U_{\text{eff}} = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2}.$$



If $l \neq 0$, the repulsive centrifugal potential $l^2/(2\mu r^2)$ dominates at small r, whereas the attractive gravitational potential $-Gm_1m_2/r$ dominates at large r. The drawing shows the energy diagram with various values of the total energy. The kinetic energy of radial motion is $K = E - U_{\text{eff}}$, and the motion is restricted to regions where $K \geq 0$. The nature of the motion is determined by the total energy. Here are the various possibilities:

1. $E > 0$: r is unbounded for large values but must exceed a certain minimum if $l \neq 0$. The particles are kept apart by the "centrifugal barrier."

2. $E = 0$: This is qualitatively similar to case 1 but on the boundary between unbounded and bounded motion.

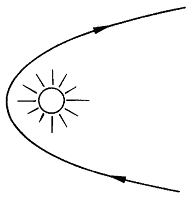
3. $E < 0$: The motion is bounded for both large and small r. The two particles form a bound system.

4. $E = E_{\min}$: r is restricted to one value. The particles stay a constant distance from one another.

In the next section we shall find that case 1 corresponds to motion in a hyperbola; case 2, to a parabola; case 3, to an ellipse; and case 4, to a circle.

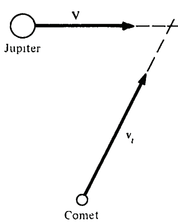
There is one other possibility, $l = 0$. In this case the particles move along a straight line on a collision course, since when l is zero there is no centrifugal barrier to hold them apart.

Example 9.2 **The Capture of Comets**



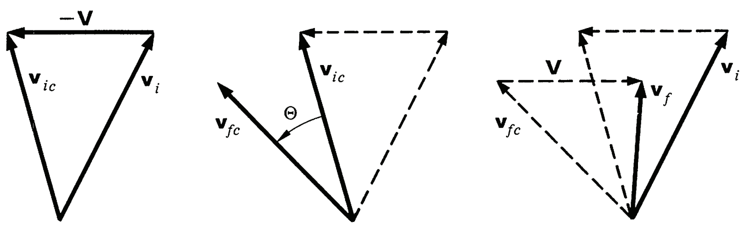
Suppose that a comet with $E > 0$ drifts into the solar system. From our discussion of the energy diagram for motion under a gravitational force, the comet will approach the sun and then swing away, never to return. In order for the comet to become a member of the solar system, its energy would have to be reduced to a negative value. However, the gravitational force is conservative and the comet's total energy cannot change.

The situation is quite different if more than two bodies are involved. For instance, if the comet is deflected by a massive planet like Jupiter, it can transfer energy to the planet and so become trapped in the solar system.



Suppose that a comet is heading outward from the sun toward the orbit of Jupiter, as shown in the sketch. Let the velocity of the comet before it starts to interact appreciably with Jupiter be $\mathbf{v}_i$, and let Jupiter's velocity be $\mathbf{V}$. For simplicity we shall assume that the orbits are not appreciably deflected by the sun during the time of interaction.

In the comet-Jupiter center of mass system Jupiter is essentially at rest, and the center of mass velocity of the comet is $\mathbf{v}_{ic} = \mathbf{v}_i - \mathbf{V}$, as shown in figure *a*.



(*a*) (*b*) (*c*)

In the center of mass system the path of the comet is deflected, but the final speed is equal to the initial speed v_{ic}. Hence, the interaction merely rotates $\mathbf{v}_{ic}$ through some angle Θ to a new direction $\mathbf{v}_{fc}$, as shown in Fig. *b*. The final velocity in the space fixed system is

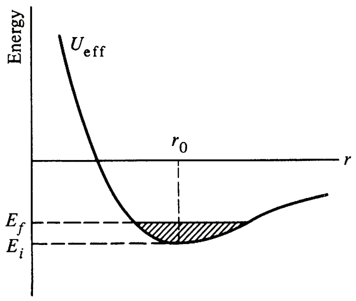
$$\mathbf{v}_f = \mathbf{v}_{fc} + \mathbf{V}.$$

Figure c shows $\mathbf{v}_f$ and, for comparison, $\mathbf{v}_i$. For the deflection shown, $v_f < v_i$, and the comet's energy has decreased. Conversely, if the deflection is in the opposite direction, interaction with Jupiter would increase the energy, possibly freeing a bound comet from the solar system. A large proportion of known comets have energies close to zero, so close that it is often difficult to determine from observations whether the orbit is elliptic ($E < 0$) or hyperbolic ($E > 0$). The interaction of a comet with Jupiter is therefore often sufficient to change the orbit from unbound to bound, or vice versa.

This mechanism for picking up energy from a planet can be used to accelerate an interplanetary spacecraft. By picking the orbit cleverly, the spacecraft can "hop" from planet to planet with a great saving in fuel.

The process we have described may seem to contradict the idea that the gravitational force is strictly conservative. Only gravity acts on the comet and yet its total energy can change. The reason is that the comet experiences a time-dependent gravitational force, and time-dependent forces are intrinsically nonconservative. Nevertheless, the total energy of the entire system is conserved, as we expect.

Example 9.3 Perturbed Circular Orbit



A satellite of mass m orbits the earth in a circle of radius r_0. One of its engines is fired briefly toward the center of the earth, changing the energy of the satellite but not its angular momentum. The problem is to find the new orbit.

The energy diagram shows the initial energy E_i and the final energy E_f. Note that firing the engine radially does not change the effective potential because l is not altered. Since the earth's mass M_e is much greater than m, the reduced mass is nearly m and the earth is effectively fixed.

If E_f is not much greater than E_i, the energy diagram shows that r never differs much from r_0. Rather than solve the planetary motion problem exactly, as we shall do in the next section, we instead approximate $U_{eff}(r)$ in the neighborhood of r_0 by a parabolic potential. As we know from our analysis of small oscillations of a particle about equilibrium, Sec. 4.10, the resulting radial motion of the satellite will be simple harmonic motion about r_0 to good accuracy.

The effective potential is, with $C \equiv GmM_e$,

$$U_{eff}(r) = -\frac{C}{r} + \frac{l^2}{2mr^2}.$$

The minimum of U_{eff} is at $r = r_0$. Since the slope is zero there, we have

$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = 0$$

$$= \frac{C}{r_0^2} - \frac{l^2}{mr_0^3},$$

which gives

$$l = \sqrt{mCr_0}. \qquad 1$$

(This result can also be found by applying Newton's second law to circular motion.) As we recall from Sec. 4.10, the frequency of oscillation of the system, which we shall denote by β, is

$$\beta = \sqrt{\frac{k}{m}},$$

where

$$k = \frac{d^2U_{\text{eff}}}{dr^2}\bigg|_{r_0}. \qquad 2$$

This is readily evaluated to yield

$$\beta = \sqrt{\frac{C}{mr_0^3}} = \frac{l}{mr_0^2}. \qquad 3$$

Hence, the radial position is given by

$$r = r_0 + A \sin \beta t. \qquad 4$$

We have omitted the term $B \cos \beta t$ in order to satisfy the initial condition $r(0) = r_0$. Although we could calculate the amplitude A in terms of E_f, we shall not bother with the algebra here except to note that $A \ll r_0$ for E_f nearly equal to E_i.

To find the new orbit, we must eliminate t and express r as a function of θ. For the circular orbit,

$$\dot{\theta} = \frac{l}{mr_0^2}, \qquad \text{or} \qquad 5$$

$$\theta = \left(\frac{l}{mr_0^2}\right) t. \qquad 6$$

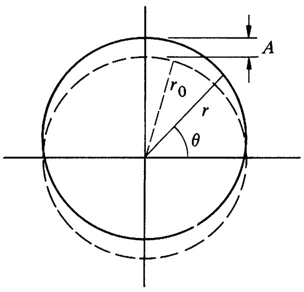
Equation (5) is accurate enough for our purposes, even though the radius oscillates slightly after the engine is fired; t occurs only in a small correction term to r in Eq. (4), and we are neglecting terms of order A and higher.

From Eqs. (1) and (5) we see that the frequency of rotation of the satellite around the earth is

$$\frac{l}{mr_0^2} = \frac{\sqrt{mCr_0}}{mr_0^2} = \sqrt{\frac{C}{mr_0^3}}$$

and

$$\theta = \frac{l}{mr_0^2} t = \beta t. \qquad 7$$



Surprisingly, the frequency of rotation is identical to the frequency of radial oscillation. If we substitute Eq. (7) in Eq. (4), we obtain

$$r = r_0 + A \sin \theta. \qquad 8$$

The new orbit is shown as the solid line in the sketch. The orbit looks almost circular, but it is no longer centered on the earth.

As we shall show in Sec. 9.6, the exact orbit for $E = E_f$ is an ellipse with the equation

$$r = \frac{r_0}{1 - (A/r_0) \sin \theta}.$$

If $A/r_0 \ll 1$,

$$\begin{aligned} r &= \frac{r_0}{1 - (A/r_0) \sin \theta} \\ &\approx r_0 \left(1 + \frac{A}{r_0} \sin \theta\right) \\ &= r_0 + A \sin \theta. \end{aligned}$$

To first order in A, Eq. (8) is the equation of an ellipse. However, the exact calculation is harder to derive (and to digest) than is the approximate result we found by using the energy diagram.

9.6 Planetary Motion

Let us now solve the main problem of the chapter—finding the orbit for the gravitational interaction

$$U(r) = -G\frac{Mm}{r} \equiv -\frac{C}{r},$$

where M is the mass of the sun and m is the mass of a planet. Alternatively, M could be the mass of a planet and m the mass of a satellite. Before proceeding with the calculation, it might be useful to consider whether or not this is a realistic description of the interaction of the sun and a planet. If both bodies were homogeneous spheres, they would interact like point particles as we saw in Note 2.1, and our formula would be exact. However, most of the members of the solar system are neither perfectly homogeneous nor perfectly spherical. For example, satellites around the moon are perturbed by mass concentrations ("mascons") in the moon, and the planet Mercury may be slightly perturbed by an equatorial bulge of the sun. Furthermore, the solar system is by no means a two body system. Each planet is attracted by all the other planets as well as by the sun.

Fortunately, none of these effects is particularly large. Most of the mass of the solar system is in the sun, so that the attraction of the planets for each other is quite feeble. The largest interaction is between Jupiter and Saturn. The effect of this perturbation is chiefly to change the speed of each planet, so that the law of equal areas no longer holds exactly. However, the perturbation never shifts Jupiter by more than a few minutes of arc from its expected position (one minute of arc is approximately equal to one-thirtieth the moon's diameter as seen from the earth). In practice, one first calculates planetary orbits neglecting the other planets and then calculates small corrections to the orbits due to their presence. Such a procedure is called a perturbation method. (The transuranic planets were actually discovered by their small perturbing effects on the orbits of the known outer planets.) Furthermore, if a body is not quite homogeneous or spherically symmetric, its gravitational field can be shown to have terms depending on $1/r^3$, $1/r^4$, etc., in addition to the main $1/r^2$ term. The coefficients depend on the size of the body compared with r; over the span of the solar system the higher order terms become negligible, although they may be important for a nearby satellite.

Returning to our idealized planetary motion problem $U(r) = -C/r$, we find that the equation for the orbit Eq. (9.16) becomes, using indefinite integrals,

$$\theta - \theta_0 = l \int \frac{dr}{r(2\mu Er^2 + 2\mu Cr - l^2)^{\frac{1}{2}}},$$

where θ_0 is a constant of integration. The integral over r is listed in tables of integrals. The result is

$$\theta - \theta_0 = \arcsin\left(\frac{\mu Cr - l^2}{r\sqrt{\mu^2C^2 + 2\mu El^2}}\right)$$

or

$$\mu Cr - l^2 = r\sqrt{\mu^2C^2 + 2\mu El^2}\sin(\theta - \theta_0).$$

Solving for r,

$$r = \frac{(l^2/\mu C)}{1 - \sqrt{1 + (2El^2/\mu C^2)}\sin(\theta - \theta_0)}. \qquad 9.18$$

The usual convention is to take $\theta_0 = -\pi/2$ and to introduce the parameters

$$r_0 \equiv \frac{l^2}{\mu C} \qquad 9.19$$

$$\epsilon \equiv \sqrt{1 + \frac{2El^2}{\mu C^2}}. \qquad 9.20$$

Physically, r_0 is the radius of the circular orbit corresponding to the given values of l, μ, and C. The dimensionless parameter ϵ, called the *eccentricity*, characterizes the shape of the orbit, as we shall see. With these replacements, Eq. (9.18) becomes

$$r = \frac{r_0}{1 - \epsilon \cos \theta}. \qquad 9.21$$

Equation (9.21) looks more familiar in cartesian coordinates $x = r \cos \theta$, $y = r \sin \theta$. Rewriting it in the form $r - \epsilon r \cos \theta = r_0$, we have

$$\sqrt{x^2 + y^2} - \epsilon x = r_0$$

or

$$(1 - \epsilon^2)x - 2r_0\epsilon x + y^2 = r_0^2. \qquad 9.22$$

Here are the possibilities:

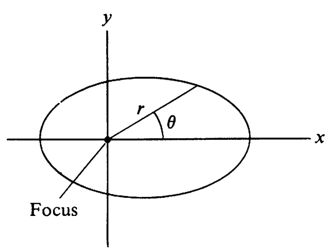
1. $\epsilon > 1$: The coefficients of x^2 and y^2 are unequal and opposite in sign; the equation has the form $y^2 - Ax^2 - Bx =$ constant, which is the equation of a *hyperbola.* From Eq. (9.20), $\epsilon > 1$ whenever $E > 0$.

2. $\epsilon = 1$: Eq. (9.22) becomes

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2}.$$

This is the equation of a *parabola*. $\epsilon = 1$ when $E = 0$.

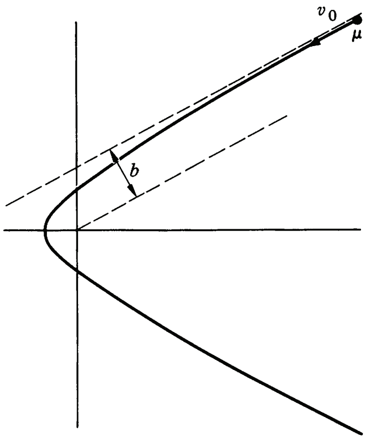
3. $0 \le \epsilon < 1$: The coefficients of x^2 and y^2 are unequal but of the same sign; the equation has the form $y^2 + Ax^2 - Bx =$ constant, which is the equation of an *ellipse*. The term linear in x means that the geometric center of the ellipse is not at the origin of coordinates. As proved in Note 9.1, one focus of the ellipse is at the origin. For $\epsilon < 1$, the allowed values of E are



$$-\frac{\mu C^2}{2l^2} \le E < 0.$$

When $E = -\mu C^2/2l^2$, $\epsilon = 0$ and the equation of the orbit becomes $x^2 + y^2 = r_0^2$; the ellipse degenerates to a *circle*.

Example 9.4 **Hyperbolic Orbits**



In order to use the orbit equation we must be able to express the orbit in terms of experimentally accessible parameters. For example, if the orbit is unbound, we might know the energy and the initial trajectory.

In this example we shall show how to relate some experimental parameters to the trajectory for the case of a hyperbolic orbit. The results could apply to the motion of a comet about the sun, or to the trajectory of a charged particle scattering off an atomic nucleus.

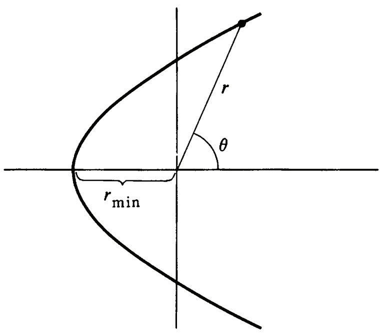
Let the speed of μ be v_0 when μ is far from the origin, and let the initial path pass the origin at distance b, as shown. b is commonly called the *impact parameter.* The angular momentum l and energy E are

$$l = \mu v_0 b$$
$$E = \tfrac{1}{2}\mu v_0^2.$$

For an inverse square force, $U(r) = -C/r$ and the equation of the orbit is

$$r = \frac{r_0}{1 - \epsilon\cos\theta},$$

where



$$r_0 = \frac{l^2}{\mu C} = \frac{\mu v_0^2 b^2}{C} = \frac{2Eb^2}{C}.$$

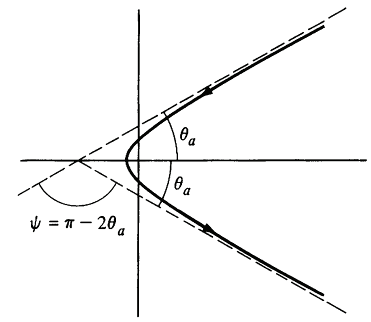
and

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu C^2}} = \sqrt{1 + \left(\frac{2Eb}{C}\right)^2}.$$

When $\theta = \pi$, $r = r_{\min}$,

$$r_{\min} = \frac{r_0}{1 + \epsilon} = \frac{2Eb^2/C}{1 + \sqrt{1 + (2Eb/C)^2}}.$$

For $E \rightarrow \infty$, $r_{\min} \rightarrow b$. Hence $0 < r_{\min} < b$.

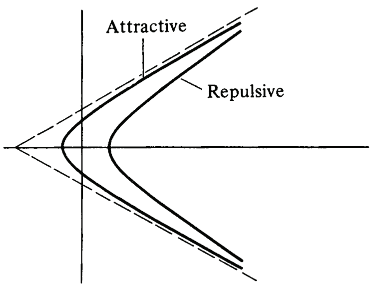


The angle of the asymptotes θ_a can be found from the orbit equation by letting $r \to \infty$. We find

$$\theta_a = \frac{1}{\ni}\cdot$$

on the interaction, μ is deflected through the angle $\psi = \pi - 2\theta_a$. The deflection angle ψ approaches 180° if $(2Eb/C)^2 \ll 1$.

Rutherford's classic experiment that established the nuclear model of the atom showed that fast alpha particles (doubly charged helium nuclei) interact with single atoms in thin gold foils according to the Coulomb potential $U(r) = -C'/r$. He found that the alpha particles followed hyperbolic orbits even when $r_{\min}$ was much less than the radius of the atom, proving that the charge of an atom must be concentrated in a small volume, the nucleus. Surprisingly, Rutherford was unable to determine whether the gold nuclei attracted ($C' > 0$) or repelled ($C' < 0$) alpha particles. The eccentricity, hence the scattering angle, depends on $(2Eb/C')^2$, making it impossible to determine the algebraic sign of the strength parameter C'.



Elliptical orbits ($E < 0$, $0 \leq \epsilon < 1$) are so important it is worth looking at their properties in more detail. From the orbit equation, Eq. (9.21),

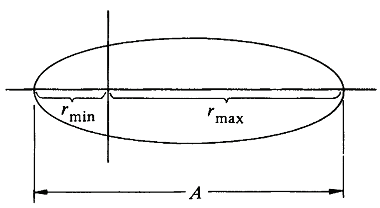
$$r = \frac{r_0}{1 - \epsilon \cos \theta},$$

The maximum value of r occurs at $\theta = 0$:

$$r_{\max} = \frac{r_0}{1 - \epsilon}\cdot \tag{9.23}$$

the minimum value of r occurs at $\theta = \pi$:

$$r_{\min} = \frac{r_0}{1 + \epsilon}\cdot \tag{9.24}$$

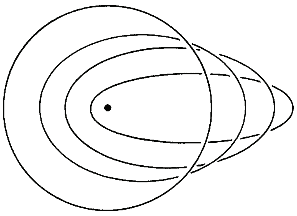


The length of the major axis is

$$\begin{aligned} A &= r_{\min} + r_{\max} \\ &= r_0\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) \\ &= \frac{2r_0}{1-\epsilon^2}\cdot \end{aligned} \tag{9.25}$$

Expressing r_0 and ϵ in terms of E, l, μ, C by Eq. (9.19) and (9.20) gives

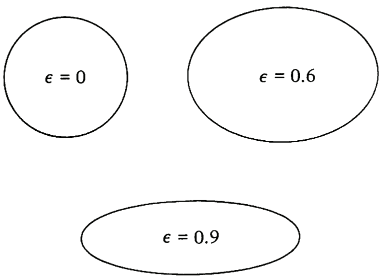
$$\begin{aligned} A &= \frac{2r_0}{1-\epsilon^2} \\ &= \frac{2l^2/(\mu C)}{1-[1+2El^2/(\mu C^2)]} \\ &= \frac{C}{(-E)}. \end{aligned} \qquad 9.26$$



The length of the major axis is independent of l; orbits with the same major axis have the same energy. For instance, all the orbits in the sketch correspond to the same value of E.

The ratio $r_{\text{max}}/r_{\text{min}}$ is

$$\begin{aligned} \frac{r_{\text{max}}}{r_{\text{min}}} &= \frac{r_0/(1-\epsilon)}{r_0/(1+\epsilon)} \\ &= \frac{1+\epsilon}{1-\epsilon}. \end{aligned}$$



When ϵ is near zero, $r_{\text{max}}/r_{\text{min}} \approx 1$ and the ellipse is nearly circular. When ϵ is near 1, the ellipse is very elongated. The shape of the ellipse is determined entirely by ϵ; r_0 only supplies the scale.

Table 9.1 gives the eccentricities of the orbits of the planets and Halley's comet. The table reveals why the Ptolemaic theory of circles moving on circles was reasonably successful in dealing with early observations. All the planetary orbits, except those of Mercury and Pluto, have eccentricities near zero and are nearly circular. Mercury is never far from the sun and is hard to observe, and Pluto was not discovered until 1930, so that neither of these

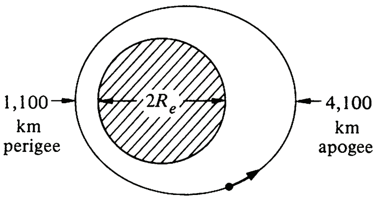
TABLE 9.1

PLANET	ECCENTRICITY
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.055
Uranus	0.051
Neptune	0.007
Pluto	0.252
Halley's Comet	0.967

planets was an impediment to the Ptolemaists. Mars has the most eccentric orbit of the easily observable planets, and its motion was a stumbling block to the Ptolemaic theory. Kepler discovered his laws of planetary motion by trying to fit his calculations to Brahe's accurate observations of Mars' orbit.

Note 9.1 derives the geometric properties of elliptical orbits. We turn now to some examples.

Example 9.5 **Satellite Orbit**



A satellite of mass $m = 2{,}000$ kg is in elliptic orbit about the earth. At perigee (closest approach to the earth) it has an altitude of 1,100 km and at apogee (farthest distance from the earth) its altitude is 4,100 km. What are the satellite's energy E and angular momentum l? How fast is it traveling at perigee and at apogee?

Since $m \ll M_e$, we can take $\mu \approx m$ and assume that the earth is fixed. The radius of the earth is $R_e = 6{,}400$ km, and the major axis of the orbit is therefore

$$A = [1{,}100 + 4{,}100 + 2(6{,}400)]\text{km}$$
$$= 1.8 \times 10^7 \text{ m}.$$

Knowing A, we can find E from Eq. (9.26):

$$A = \frac{C}{(-E)} \qquad \text{or} \qquad E = \frac{C}{A}.$$

$C = GmM_e = mgR_e^2$, since $g = GM_e/R_e^2$. Numerically,

$$C = (2 \times 10^3)(9.8)(6.4 \times 10^6)^2 = 8.0 \times 10^{17} \text{ J·m}.$$

$$E = -\frac{C}{A}$$
$$= -4.5 \times 10^{10} \text{ J}.$$

The initial energy of the satellite before launch was

$$E_i = -\frac{GmM_e}{R_e}$$
$$= -\frac{C}{R_e}$$
$$= -12.5 \times 10^{10} \text{ J}.$$

The energy needed to put the satellite into orbit, neglecting losses due to friction, is $E - E_i = 8 \times 10^{10}$ J.

We can find the angular momentum from the eccentricity. Since

$$r_{\text{min}} = \frac{r_0}{1 + \epsilon} \qquad \text{and} \qquad r_{\text{max}} = \frac{r_0}{1 - \epsilon},$$

we have

$$(1 + \epsilon)r_{\text{min}} = (1 - \epsilon)r_{\text{max}}$$

and

$$\begin{aligned}\epsilon &= \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}} \\ &= \frac{r_{\text{max}} - r_{\text{min}}}{A} \\ &= \frac{3 \times 10^3}{1.8 \times 10^4} \\ &= \frac{1}{6}.\end{aligned}$$

From the definition of ϵ, Eq. (9.20),

$$\epsilon^2 = 1 + \frac{2El^2}{mC^2}$$

which yields

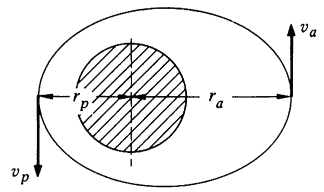
$$l = 1.2 \times 10^{14}\ \text{kg}\cdot\text{m}^2/\text{s}.$$

We can find the speed v of the satellite at any r from the energy equation

$$E = \frac{1}{2} mv^2 - \frac{C}{r}.$$

At perigee, $r = (1{,}100 + 6{,}400)$ km $= 7.5 \times 10^6$ m, and the speed at perigee is

$$v_p = 7{,}900 \text{ m/s}.$$



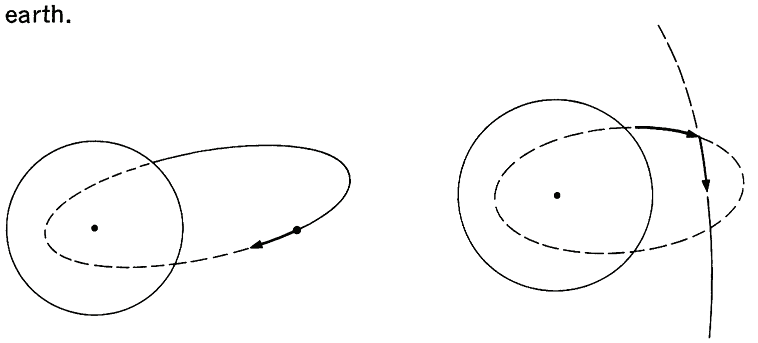
To find the speed at apogee, v_a, most simply, note that at apogee and perigee the velocity of the satellite is purely tangential. Hence, by conservation of angular momentum,

$$\mu v_p r_p = \mu v_a r_a,$$

and we find that

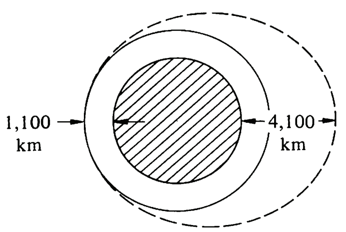
$$\begin{aligned}v_a &= \frac{v_p r_p}{r_a} \\ &= 5{,}600 \text{ m/s}.\end{aligned}$$

Suppose that a body is projected from the surface of the earth with initial velocity v_0. If v_0 is less than the escape velocity, 1.12×10^4 m/s, the total energy of the body is negative, and it travels in an elliptic orbit with one focus at the center of earth. As the drawing on the left shows, the body inevitably returns to earth.



In order to put a spacecraft into orbit around the earth, the magnitude and direction of its velocity must be altered at a point where the old and new orbits intersect. Orbit transfer maneuvers are frequently needed in astronautics. For example, on an Apollo moon flight the vehicle is first put into near earth orbit and is then transferred to a trajectory toward the moon. The next example illustrates the physical principles of orbit transfer.

Example 9.6 **Satellite Maneuver**



One of the commonest orbit maneuvers is the transfer between an elliptical and a circular orbit. This maneuver is used to inject spacecrafts into high orbits around the earth, or to put a planetary exploration satellite into a low orbit for surface inspection.

Suppose, for instance, that we want to transfer the satellite of Example 9.5 into a circular orbit at perigee, as shown in the sketch. Let E and l be the initial energy and angular momentum of the satellite and let E', l' be the parameters for the new orbit.

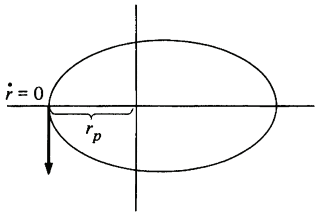
We start our analysis by finding E, l, E', l'. For simplicity, we shall assume that the amount of fuel burned by the satellite's rockets at transfer is negligible compared with the satellite's mass $m = 2{,}000$ kg.

From Eq. (9.26), $E = -C/A$. Since $A/r_p = 18 \times 10^6/(7.5 \times 10^6) = \frac{12}{5}$, we have

$$E = -\frac{5}{12}\frac{C}{r_p}. \qquad 1$$

r_p is the radius at perigee, hence the radius of the desired circular orbit.

An easy way to find l is to use the one dimensional energy equation, Eq. (9.9):



$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{C}{r}. \qquad 2$$

At perigee, $\dot{r} = 0$ and $r = r_p$, and we find

$$l^2 = \tfrac{7}{6}mCr_p. \qquad 3$$

For the circular orbit, the major axis is $2r_p$ and therefore

$$E' = -\frac{C}{2r_p}. \qquad 4$$

$\dot{r} = 0$ for the circular orbit, and from the one dimensional energy equation,

$$E' = \frac{l'^2}{2mr_p{}^2} - \frac{C}{r_p},$$

which yields

$$l'^2 = mCr_p. \qquad 5$$

How can we switch from E, l to E', l'? Since $E' < E$ and $l' < l$, we want to apply a braking thrust in order to reduce both the energy and the angular momentum. Thrust in the radial direction at perigee changes the energy but not the angular momentum, whereas tangential thrust changes both parameters. The old and new orbits are tangential where they intersect, and we might suspect that tangential thrust alone would be sufficient. We now show that this is correct.

At perigee, **v** is purely tangential, and tangential thrust changes the speed from v to v'. From the energy equation,

$$E = \frac{1}{2}mv^2 - \frac{C}{r},$$

and at perigee

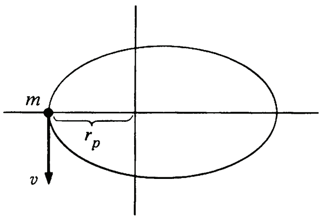
$$v^2 = \frac{2}{m}\left(E + \frac{C}{r_p}\right)$$
$$= \frac{7}{6}\frac{C}{mr_p},$$

using Eq. (1). Similarly,

$$v'^2 = \frac{2}{m}\left(E' + \frac{C}{r_p}\right)$$
$$= \frac{C}{mr_p},$$

using Eq. (4).

We now check to see if the angular momentum has its required value. At perigee, **v** is perpendicular to **r** and



$$l = mr_pv$$
$$= mr_p\sqrt{\frac{7}{6}\frac{C}{mr_p}}$$
$$= \sqrt{\frac{7}{6}mr_pC},$$

as we have already found, Eq. (3). Similarly,

$$l' = mr_pv'$$
$$= mr_p\sqrt{\frac{C}{mr_p}}$$
$$= \sqrt{mr_pC},$$

which is the required value according to Eq. (5).

The maneuver can be executed by applying a braking thrust tangential to the orbit at perigee to reduce the speed of the satellite from $v = \sqrt{7C/(6mr_p)} = 7{,}900$ m/s to $v' = \sqrt{C/(mr_p)} = 7{,}300$ m/s.

Practical orbit maneuvers are generally planned to economize on the fuel. According to our discussion of rockets in Sec. 3.5, if the mass of the spacecraft changes from M_i to $M_i - \Delta M$ during the rocket burn, its velocity changes by

$$\Delta\mathbf{v} = -\mathbf{u}\ln\left(\frac{M_i}{M_i - \Delta M}\right).$$

Therefore, the smaller the change in speed required by a maneuver, the more economical of fuel it is.

The maneuver described in this example reaches the maximum efficiency. At transfer,

$$E - E' = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv'^2$$
$$= \tfrac{1}{2}mv^2 - \tfrac{1}{2}m(\mathbf{v} - \Delta\mathbf{v})^2$$
$$\approx m\mathbf{v}\cdot\Delta\mathbf{v}.$$

$|\mathbf{v}|$ is greatest at perigee, and since $\Delta\mathbf{v}$ is parallel to $\mathbf{v}$, $|\Delta\mathbf{v}|$ is least there to obtain the needed value of $E - E'$.

9.7 Kepler's Laws

Johannes Kepler was the assistant of the sixteenth century Danish astronomer Tycho Brahe. They had a remarkable combination of talents. Brahe made planetary measurements of unprecedented accuracy, and Kepler had the mathematical genius and fortitude to show that Brahe's data could be fitted into three simple empirical laws. The task was formidable. It took Kepler 18 years of laborious calculation to obtain the following three laws:

1. Each planet moves in an ellipse with the sun at one focus.
2. The radius vector from the sun to a planet sweeps out equal areas in equal times.
3. The period of revolution T of a planet about the sun is related to the major axis of the ellipse A by

$$T^2 = kA^3,$$

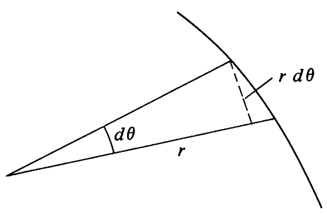
where k is the same for all the planets.

Kepler's first law follows from the results of the last section; elliptic orbits are characteristic of the inverse square law force. The second law is a general feature of central force motion as we demonstrated in Example 6.3.

Kepler's third law is easily proved by the following trick: We start with the definition of angular momentum, Eq. (9.8a),

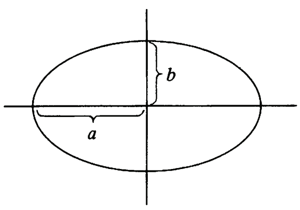
$$l = \mu r^2 \frac{d\theta}{dt},$$

which can be written



$$\frac{l}{2\mu} dt = \tfrac{1}{2} r^2 \, d\theta. \qquad 9.27$$

But $\frac{1}{2} r^2 \, d\theta$ is a differential element of area in polar coordinates. Over one complete period, the whole area of the ellipse is swept out, and integration of Eq. (9.27) yields



$$\frac{l}{2\mu} T = \text{area of ellipse} = \pi ab, \qquad 9.28$$

where $a = A/2$ is the semimajor axis and b is the semiminor axis. From Eq. (9.26),

$$a = \frac{C}{(-2E)},$$

and from Note 9.1,

$$b = \frac{l}{\sqrt{-2\mu E}},$$

Equation (9.28) becomes

$$
\begin{aligned}
T^2 &= \frac{4\mu^2}{l^2}\pi^2 a^2 b^2 \\
&= \frac{\pi^2 \mu C^2}{(-2E^3)} \\
&= \frac{\pi^2 \mu}{2C} A^3,
\end{aligned}
\qquad 9.29
$$

using $A = C/(-E)$. Since $C = GMm$ and $\mu = Mm/(M + m)$, we obtain finally

$$T^2 = \frac{\pi^2}{2(M + m)G} A^3. \qquad 9.30$$

This result shows that Kepler's third law is not exact; T^2/A^3 depends slightly on the planet's mass. However, even for Jupiter, the largest planet, m/M is only 1/1,000, so that Kepler's third law holds to good accuracy in the solar system.

Kepler's laws also apply to the motion of satellites around a planet. In Table 9.2 we show how his third law, the law of periods, holds for a number of artificial earth satellites. The ratio A^3/T^2 is constant to a fraction of a percent, although the periods vary by nearly a factor of 100. A more refined check would take into account the nonspherical shape of the earth and perturbations due to the moon.

TABLE 9.2*

SATELLITE	ϵ	A, km	T, min	A^3/T^2
Cosmos 358	0.002	13,823	95.2	2.91×10^8
Explorer 17	0.047	13,928	96.39	2.91×10^8
Cosmos 374	0.104	15,446	112.3	2.92×10^8
Cosmos 382	0.260	18,117	143	2.91×10^8
ATS 2	0.455	24,123	219.7	$2\ 91 \times 10^8$
15th Molniya I	0.738	52,537	706	2.91×10^8
Ers 13	0.887	117,390	2,352	2.92×10^8
Ogo 3	0.901	135,270	2,917	2.91×10^8
Explorer 34	0.940	224,150	6,225	2.91×10^8
Explorer 28	0.952	273,740	8,400	2.91×10^8

* Data taken from the data catalogs of the National Space Science Data Center and the World Data Center A. The catalogs give satellite altitudes relative to the surface of the earth; we assumed the diameter of the earth to be 12,757 km in calculating *A*.

Example 9.7 The Law of Periods

Here is a more general way of deriving the law of periods. Starting from Eq. (9.13) we have, with $U(r) = -C/r$,

$$\int_{t_a}^{t_b} dt = \mu \int_{r_a}^{r_b} \frac{r\,dr}{(2\mu E r^2 + 2\mu C r - l^2)^{\frac{1}{2}}}.$$

The integral is listed in standard tables. For the case of interest, $E < 0$, we find

$$t_b - t_a = \frac{\sqrt{2\mu E r^2 + 2\mu C r - l^2}}{2E}\Bigg|_{r_a}^{r_b} - \left(\frac{\mu C}{2E}\right)\frac{1}{\sqrt{-2\mu E}}\,\text{arcsin}\left(\frac{-2\mu E r - \mu C}{\sqrt{\mu^2 C^2 + 2\mu E l^2}}\right)\Bigg|_{r_a}^{r_b}$$

Fortunately this result can be greatly simplified. For a complete period, $t_b - t_a = T$, and $r_b = r_a$. The first term on the right hand side vanishes, and in the second term, the arcsine changes by 2π. The result is

$$T = \frac{\pi\mu C}{(-E)}\frac{1}{\sqrt{-2\mu E}}$$

or

$$T^2 = \frac{\pi^2\mu C^2}{(-2E^3)} = \frac{\pi^2\mu}{2C}A^3,$$

as we found earlier, Eq. (9.29).

Note 9.1 Properties of the Ellipse

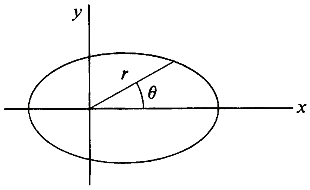
The equation of any conic section is, in polar coordinates,

$$r = \frac{r_0}{1 - \epsilon\cos\theta}. \qquad 1$$

Converting to cartesian coordinates $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, Eq. (1) becomes

$$(1 - \epsilon^2)x^2 - 2r_0 x + y^2 = r_0^2. \qquad 2$$

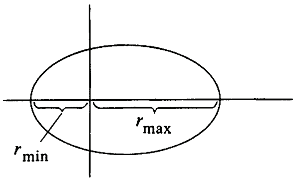
The ellipse corresponds to the case $0 \leq \epsilon < 1$. The ellipse described by Eqs. (1) and (2) is symmetrical about the x axis, but its center does not lie at the origin.



We can use Eq. (1) to determine the important dimensions of the ellipse. The maximum value of r, which occurs at $\theta = 0$, is

$$r_{\max} = \frac{r_0}{1 - \epsilon}.$$

The minimum value of r, which occurs at $\theta = \pi$, is



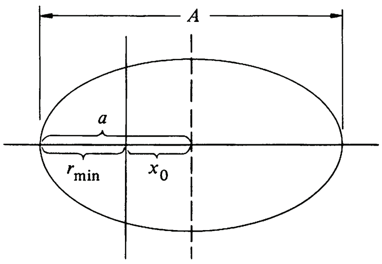
$$r_{\min} = \frac{r_0}{1 + \epsilon}.$$

The major axis is

$$\begin{aligned} A &= r_{\max} + r_{\min} \\ &= r_0\left(\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon}\right) \\ &= \frac{2r_0}{1 - \epsilon^2}. \end{aligned} \qquad 3$$

The semimajor axis is

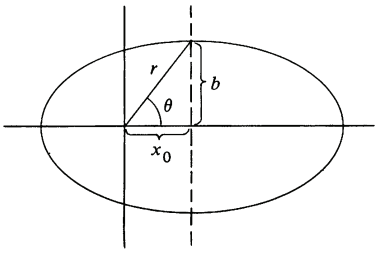
$$\begin{aligned} a &= \frac{A}{2} \\ &= \frac{r_0}{1 - \epsilon^2}. \end{aligned}$$



The distance from the origin to the center of the ellipse is

$$\begin{aligned} x_0 &= a - r_{\min} \\ &= r_0\left(\frac{1}{1-\epsilon^2} - \frac{1}{1+\epsilon}\right) \\ &= \frac{r_0\epsilon}{1 - \epsilon^2}. \end{aligned} \qquad 4$$

We see that the eccentricity is equal to the ratio x_0/a.



To find the length of the semiminor axis $b = \sqrt{r^2 - x_0^2}$, note that the tip of the semiminor axis has angular coordinates given by $\cos\theta = x_0/r$. We have

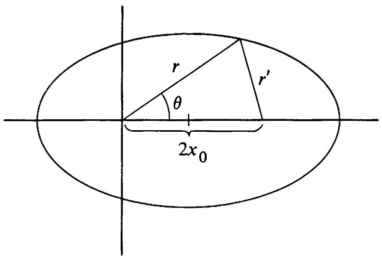
$$\begin{aligned} r &= \frac{r_0}{1 - \epsilon\cos\theta} \\ &= \frac{r_0}{1 - \epsilon x_0/r} \end{aligned}$$

or

$$r = r_0 + \epsilon x_0 = r_0\left(1 + \frac{\epsilon^2}{1 - \epsilon^2}\right)$$
$$= \frac{r_0}{1 - \epsilon^2}.$$

Hence,

$$b = \sqrt{r^2 - x_0^2} = \left(\frac{r_0}{1 - \epsilon^2}\right)\sqrt{1 - \epsilon^2}$$
$$= \frac{r_0}{\sqrt{1 - \epsilon^2}}.$$



Finally, we shall prove that the origin lies at a focus of the ellipse. According to the definition of an ellipse, the sum of the distances from the foci to a point on the ellipse is a constant. Hence, for the ellipse shown in the sketch we need to prove $r + r' =$ constant. By the law of cosines,

$$r'^2 = r^2 + 4x_0^2 - 4rx_0 \cos\theta. \qquad 5$$

From Eq. (1) we find that

$$r\cos\theta = \frac{r - r_0}{\epsilon}.$$

Equation (5) becomes

$$r'^2 = r^2 - \frac{4x_0}{\epsilon}r + 4x_0^2 + \frac{4r_0x_0}{\epsilon}.$$

Using the relation $x_0 = r_0\epsilon/(1 - \epsilon^2)$ from Eq. (4) gives

$$r'^2 = r^2 - \left(\frac{4r_0}{1 - \epsilon^2}\right)r + \frac{4r_0^2\epsilon^2}{(1 - \epsilon^2)^2} + \frac{4r_0^2}{(1 - \epsilon^2)}$$
$$= r^2 - \left(\frac{4r_0}{1 - \epsilon^2}\right)r + \frac{4r_0^2}{(1 - \epsilon^2)^2}.$$

The right hand side is a perfect square.

$$r' = \pm\left(r - \frac{2r_0}{1 - \epsilon^2}\right)$$
$$= \pm(r - A).$$

Since $A > r$, we must choose the negative sign to keep $r' > 0$. Therefore,

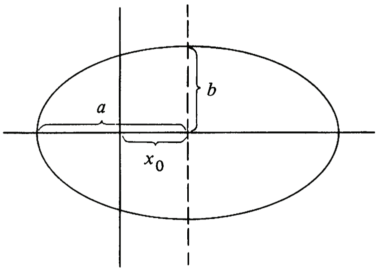
$$r' + r = A$$
$$= \text{constant}.$$

To conclude, we list a few of our results in terms of E, l, μ, C for the inverse square force problem $U(r) = -C/r$. When using these formulas, E must be taken to be a negative number. From Eqs. (9.19) and (9.20),

$$r_0 = \frac{l^2}{\mu C}$$

and

$$\epsilon = \sqrt{1 + 2El^2/(\mu C^2)}.$$



Hence,

$$\text{semimajor axis } a = \frac{r_0}{1-\epsilon^2} = \frac{C}{-2E}$$

$$\text{semiminor axis } b = \frac{r_0}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{-2\mu E}}$$

$$x_0 = \frac{r_0\epsilon}{1-\epsilon^2} = \left(\frac{C}{-2E}\right)\sqrt{1+\frac{2El^2}{\mu C^2}}.$$

Problems

9.1 Obtain Eqs. (9.7a and b) by differentiating Eqs. (9.8a and b) with respect to time.

9.2 A particle of mass 50 g moves under an attractive central force of magnitude $4r^3$ dynes. The angular momentum is equal to 1,000 g·cm^2/s.

a. Find the effective potential energy.

b. Indicate on a sketch of the effective potential the total energy for circular motion.

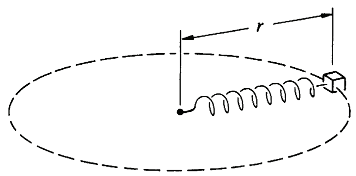
c. The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0.

Ans. (c) $r_0 \approx 2.8$ cm

9.3 A particle moves in a circle under the influence of an inverse cube law force. Show that the particle can also move with uniform radial velocity, either in or out. (This is an example of unstable motion. Any slight perturbation to the circular orbit will start the particle moving radially, and it will continue to do so.) Find θ as a function of r for motion with uniform radial velocity.

9.4 For what values of n are circular orbits stable with the potential energy $U(r) = -A/r^n$, where $A > 0$?

9.5 A 2-kg mass on a frictionless table is attached to one end of a massless spring. The other end of the spring is held by a frictionless pivot. The spring produces a force of magnitude $3r$ newtons on the mass, where



r is the distance in meters from the pivot to the mass. The mass moves in a circle and has a total energy of 12 J.

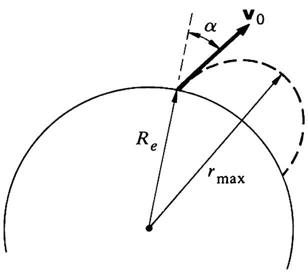
a. Find the radius of the orbit and the velocity of the mass.

b. The mass is struck by a sudden sharp blow, giving it instantaneous velocity of 1 m/s radially outward. Show the state of the system before and after the blow on a sketch of the energy diagram.

c. For the new orbit, find the maximum and minimum values of r.

9.6 A particle of mass m moves under an attractive central force Kr^4 with angular momentum l. For what energy will the motion be circular, and what is the radius of the circle? Find the frequency of radial oscillations if the particle is given a small radial impulse.

9.7 A rocket is in elliptic orbit around the earth. To put it into an escape orbit, its engine is fired briefly, changing the rocket's velocity by $\Delta\mathbf{V}$. Where in the orbit, and in what direction, should the firing occur to attain escape with a minimum value of $\Delta\mathbf{V}$?



9.8 A projectile of mass m is fired from the surface of the earth at an angle α from the vertical. The initial speed v_0 is equal to $\sqrt{GM_e/R_e}$. How high does the projectile rise? Neglect air resistance and the earth's rotation. (*Hint*: It is probably easier to apply the conservation laws directly instead of using the orbit equations.)

Ans. clue. If $\alpha = 60°$, then $r_{\text{max}} = 3R_e/2$

9.9 Halley's comet is in an elliptic orbit about the sun. The eccentricity of the orbit is 0.967 and the period is 76 years. The mass of the sun is 2×10^{30} kg, and $G = 6.67 \times 10^{-11}$ N·m²/kg².

a. Using these data, determine the distance of Halley's comet from the sun at perihelion and at aphelion.

b. What is the speed of Halley's comet when it is closest to the sun?

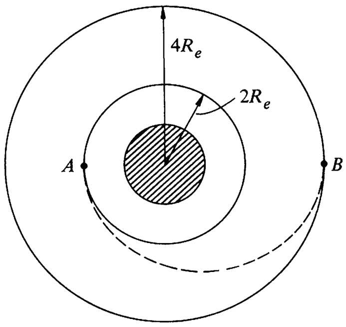
9.10 *a.* A satellite of mass m is in circular orbit about the earth. The radius of the orbit is r_0 and the mass of the earth is M_e. Find the total mechanical energy of the satellite.

b. Now suppose that the satellite moves in the extreme upper atmosphere of the earth where it is retarded by a constant feeble friction force f. The satellite will slowly spiral toward the earth. Since the friction force is weak, the change in radius will be very slow. We can therefore assume that at any instant the satellite is effectively in a circular orbit of average radius r. Find the approximate change in radius per revolution of the satellite, Δr.

c. Find the approximate change in kinetic energy of the satellite per revolution, ΔK.

Ans. (c) $\Delta K = +2\pi r f$ (note the sign!)

9.11 Before landing men on the moon, the Apollo 11 space vehicle was put into orbit about the moon. The mass of the vehicle was 9,979 kg and the period of the orbit was 119 min. The maximum and minimum distances from the center of the moon were 1,861 km and 1,838 km. Assuming the moon to be a uniform spherical body, what is the mass of the moon according to these data? $G = 6.67 \times 10^{-11}$ N·m²/kg².



9.12 A space vehicle is in circular orbit about the earth. The mass of the vehicle is 3,000 kg and the radius of the orbit is $2R_e = 12{,}800$ km. It is desired to transfer the vehicle to a circular orbit of radius $4R_e$.

a. What is the minimum energy expenditure required for the transfer?

b. An efficient way to accomplish the transfer is to use a semielliptical orbit (known as a Hohmann transfer orbit), as shown. What velocity changes are required at the points of intersection, A and B?