5 SOME MATHEMATICAL ASPECTS OF FORCE AND ENERGY

5.1 Introduction

The last chapter introduced quite a few new physical concepts—work, potential energy, kinetic energy, the work-energy theorem, conservative and nonconservative forces, and the conservation of energy.

In this chapter there are no new physical ideas; this chapter is on mathematics. We are going to introduce several mathematical techniques which will help express the ideas of the last chapter in a more revealing manner. The rationale for this is partly that mathematical elegance can be a source of pleasure, but chiefly that the results developed here will be useful in other areas of physics, particularly in the study of electricity and magnetism. We shall find how to tell whether or not a force is conservative and how to relate the potential energy to the force.

A word of reassurance: Don't be alarmed if the mathematics looks formidable at first. Once you have a little practice with the new techniques, they will seem quite straightforward. In any case, you will probably see the same techniques presented from a different point of view in your study of calculus.

In this chapter we must deal with functions of several variables, such as a potential energy function which depends on x, y, and z. Our first task is to learn how to take derivatives and find differentials of such functions. If you are already familiar with partial differentiation the next section can be skipped. Otherwise, read on.

5.2 Partial Derivatives

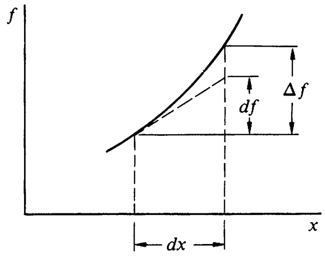
We start by reviewing briefly the concept of the differential of a function $f(x)$ which depends on the single variable x. (Differentials are discussed in greater detail in Note 1.1.)

Consider the value of $f(x)$ at any point x. Let dx be an increment in x, known as the differential of x, which can be any size we please. The differential df of f is defined to be

$$df \equiv \left(\frac{df}{dx}\right) dx.$$

Note that (df/dx) stands for the derivative

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$



The actual change in f is $\Delta f = f(x + dx) - f(x)$. Δf differs from df, as the sketch indicates, but if the limit $dx \to 0$ is to be taken, the difference can be neglected,[1] and we can use df and Δf interchangeably.

Now let us consider a function $f(x,y)$ which depends on two variables x and y. For instance, f could be the area of a rectangle of length x and width y. If we keep the variable y fixed and let the variable x change by dx, the differential of f in this case is

$$df = \left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x,y) - f(x,y)}{\Delta x}\right] dx.$$

The quantity in the bracket looks like a derivative. However, f depends on two variables and since we are differentiating with respect to only one variable, the quantity in the bracket is called a *partial derivative*. The partial derivative is denoted by $\partial f/\partial x$. (Calculus texts sometimes use f_x, but we shall avoid this notation to prevent confusion with vector components.) $\partial f/\partial x$ is read "the partial derivative of f with respect to x" or "the partial of f with respect to x." If we want to indicate that the partial derivative is to be evaluated at some particular point x_0, y_0, we can write

$$\frac{\partial f(x_0,y_0)}{\partial x} \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}.$$

The procedure for evaluating partial derivatives is straightforward; in evaluating $\partial f/\partial x$, for example, all variables but x are treated as constants.

Example 5.1 Partial Derivatives

Let

$$f = x^2 \sin y.$$

Then

$$\frac{\partial f}{\partial x} = 2x \sin y,$$

$$\frac{\partial f}{\partial y} = x^2 \cos y.$$

[1] Specifically, $(\Delta f - df)$ is of order $(dx)^2$, so that $\lim_{\Delta x \to 0} [(\Delta f - df)/\Delta x] = 0$.

We can generalize the procedure to any number of variables. For instance, let

$$f = y + e^{xz}.$$

Then

$$\frac{\partial f}{\partial x} = ze^{xz},$$

$$\frac{\partial f}{\partial y} = 1,$$

$$\frac{\partial f}{\partial z} = xe^{xz}.$$

Let us consider what happens to $f(x,y)$ if x and y both vary. Let x change by dx and y change by dy. The change in f is

$$\Delta f = f(x + dx,\ y + dy) - f(x,y).$$

The right hand side can be written as follows:

$$f(x + dx,\ y + dy) - f(x,y) = [f(x + dx,\ y + dy) - f(x,\ y + dy)] + [f(x,\ y + dy) - f(x,y)].$$

The first term on the right is the change in f due to dx; this is given approximately by

$$(\Delta f)_{\text{due to } x} \approx \frac{\partial f(x,\ y + dy)}{\partial x}\,\Delta x.$$

The second term on the right is

$$(\Delta f)_{\text{due to } y} \approx \frac{\partial f(x,y)}{\partial y}\,\Delta y.$$

The total change is

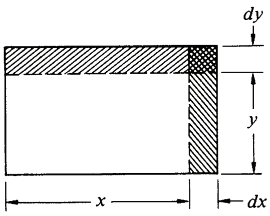
$$\Delta f \approx \frac{\partial f(x,\ y + dy)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy.$$

We define the differential of f to be

$$df \equiv \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy. \qquad 5.1$$

If we take the limit $dx \to 0$, $dy \to 0$, Δf approaches df. In applications where we are going to take the limit, we can use Δf and df interchangeably. Furthermore, even if we do not take the limit, the differential gives a good approximation to the actual value of the change in f if dx and dy are small, as the following example illustrates.

Example 5.2 **Applications of the Partial Derivative**



A. Suppose that f is the area of a rectangle of length x and width y. Then $f = xy$. The change in area if x increases by dx and y increases by dy is

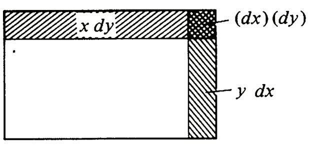
$$\begin{aligned}\Delta f &= f(x + dx, y + dy) - f(x,y)\\ &= (x + dx)(y + dy) - xy\\ &= y\,dx + x\,dy + (dx)(dy).\end{aligned}$$

The differential of f is

$$\begin{aligned}df &= \frac{\partial(xy)}{\partial x}dy + \frac{\partial(xy)}{\partial y}dy\\ &= y\,dx + x\,dy.\end{aligned}$$

We see that

$$\Delta f - df = (dx)(dy).$$



$(dx)(dy)$ is the area of the small rectangle in the figure. As $dx \to 0$ and $dy \to 0$, the area $(dx)(dy)$ becomes negligible compared with the area of the strips $x\,dy$ and $y\,dx$, and we can use the differential df as an accurate approximation to the actual change, Δf.

B. Consider the function

$$f(x,y) = y^3e^x.$$

At $x = 0$, $y = 1$ we have $f(0,1) = 1$. What is the value of $f(0.03,1.01)$? Approximating the change in f by df we have

$$\begin{aligned}\Delta f &\approx df\\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.\end{aligned}$$

The partial derivatives are easily evaluated.

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{0,1} &= \left.y^3e^x\right|_{0,1}\\ &= 1\\ \left.\frac{\partial f}{\partial y}\right|_{0,1} &= \left.3y^2e^x\right|_{0,1}\\ &= 3\end{aligned}$$

Taking $dx = 0.03$, $dy = 0.01$, we find

$$df = (1)(0.03) + 3(0.01)$$
$$= 0.06.$$

The actual value, to four significant figures, is

$$\Delta f = 0.0617.$$

5.3 How To Find the Force if You Know the Potential Energy

Our problem is this—suppose that we know the potential energy function $U(\mathbf{r})$; how do we find $\mathbf{F}(\mathbf{r})$? For one dimensional motion we already know the answer from Sec. 4.8: $F_x = -dU/dx$. It isn't difficult to generalize this result to three dimensions.

Our starting point is the definition of potential energy:

$$U_b - U_a = -\oint_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}. \qquad 5.2$$

Let us consider the change in potential energy when a particle acted on by $\mathbf{F}$ undergoes a displacement $\Delta\mathbf{r}$.

$$U(\mathbf{r} + \Delta\mathbf{r}) - U(\mathbf{r}) = -\oint_{\mathbf{r}}^{\mathbf{r}+\Delta\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \qquad 5.3$$

(We have labeled the dummy variable of integration by $\mathbf{r}'$ to avoid confusion with the end points of the line integral, $\mathbf{r}$ and $\mathbf{r} + \Delta\mathbf{r}$.) The left hand side of Eq. (5.3) is the difference in U at the two ends of the path. Let us call this ΔU. If $\Delta\mathbf{r}$ is so small that $\mathbf{F}$ does not vary appreciably over the path, the integral on the right is approximately $\mathbf{F} \cdot \Delta\mathbf{r}$. Therefore

$$\Delta U \approx -\mathbf{F} \cdot \Delta\mathbf{r}$$
$$= -(F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z). \qquad 5.4$$

We can obtain an alternative expression for ΔU by using the results of the last section. If we approximate ΔU by the differential of U, we have from Eq. (5.1)

$$\Delta U \approx \frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta y + \frac{\partial U}{\partial z}\Delta z. \qquad 5.5$$

Combining Eq. (5.4) and (5.5) yields

$$\frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta y + \frac{\partial U}{\partial z}\Delta z \approx -F_x\,\Delta x - F_y\,\Delta y - F_z\,\Delta z. \qquad 5.6$$

When we take the limit $(\Delta x,\Delta y,\Delta z) \to 0$, the approximation becomes exact. Since Δx, Δy, and Δz are independent, Eq. (5.6) remains valid even if we choose Δy and Δz to be zero. This requires that the coefficients of Δx on either side of the equation be equal. We conclude that

$$\begin{aligned}\frac{\partial U}{\partial x} &= -F_x \\ \frac{\partial U}{\partial y} &= -F_y \\ \frac{\partial U}{\partial z} &= -F_z.\end{aligned} \qquad 5.7$$

We have the answer to the problem set at the beginning of this section—how to find the force from the potential energy function. However, as we shall see in the next section, there is a much neater way of expressing Eq. (5.7).

5.4 The Gradient Operator

Equation (5.7) is really a vector equation. We can write it explicitly in vector form:

$$\begin{aligned}\mathbf{F} &= \hat{\mathbf{i}}F_x + \hat{\mathbf{j}}F_y + \hat{\mathbf{k}}F_z \\ &= -\hat{\mathbf{i}}\frac{\partial U}{\partial x} - \hat{\mathbf{j}}\frac{\partial U}{\partial y} - \hat{\mathbf{k}}\frac{\partial U}{\partial z}.\end{aligned} \qquad 5.8$$

A shorthand way to symbolize this result is

$$\mathbf{F} = -\nabla U, \qquad 5.9$$

where

$$\nabla U \equiv \hat{\mathbf{i}}\frac{\partial U}{\partial x} + \hat{\mathbf{j}}\frac{\partial U}{\partial y} + \hat{\mathbf{k}}\frac{\partial U}{\partial z}. \qquad 5.10$$

Equation (5.10) is a definition, so if the notation looks strange, it is not because you have missed something. Let's see what ∇U means.

∇U is a vector called the *gradient of* U or *grad* U. The symbol ∇ (called "del") can be written in vector form as follows:

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \qquad 5.11$$

Obviously ∇ is not really a vector; it is a *vector operator*. This means that when ∇ operates on a scalar function (the potential energy function in our case), it forms a vector.

The relation $\mathbf{F} = -\nabla U$ is a generalization of the one dimensional case. For example, suppose that U depends only on x. Then

$$\nabla U = \frac{\partial U(x)}{\partial x}\,\hat{\mathbf{i}}$$

and

$$F_x = -\frac{\partial U}{\partial x}.$$

However, for a function of a single variable the partial derivative is identical to the familiar total derivative. We have

$$F_x = -\frac{dU}{dx}.$$

Here are a few more examples.

Example 5.3 **Gravitational Attraction by a Particle**

If a particle of mass M is at the origin, the potential energy of mass m a distance r from the origin is

$$U(x,y,z) = -\frac{GMm}{r}.$$

Then

$$\begin{aligned}\mathbf{F} &= -\nabla U\\ &= +GMm\nabla\frac{1}{r}.\end{aligned}$$

Consider the x component of $\nabla(1/r)$. Since $r = \sqrt{x^2 + y^2 + z^2}$, we have

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} &= \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\\ &= -\frac{x}{r^3}.\end{aligned}$$

By symmetry the y and z terms are $-y/r^3$ and $-z/r^3$, respectively. Hence

$$\begin{aligned}\mathbf{F} &= GMm\left(\hat{\mathbf{i}}\frac{-x}{r^3} + \hat{\mathbf{j}}\frac{-y}{r^3} + \hat{\mathbf{k}}\frac{-z}{r^3}\right)\\ &= GMm\left[\frac{-\mathbf{r}}{r^3}\right]\\ &= -GMm\frac{\hat{\mathbf{r}}}{r^2}.\end{aligned}$$

We have recovered the familiar expression for the force of gravity between two particles.

Example 5.4 Uniform Gravitational Field

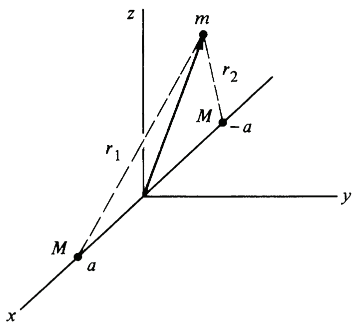
From the last chapter we know that the potential energy of mass m in a uniform gravitational field directed downward is

$$U(x,y,z) = mgz,$$

where z is the height above ground. The corresponding force is

$$\begin{aligned}\mathbf{F} &= -\nabla U \\ &= -mg\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)z \\ &= -mg\hat{\mathbf{k}}.\end{aligned}$$

Example 5.5 Gravitational Attraction by Two Point Masses



The previous examples were trivial, since the forces were obvious by inspection. Here is a more complicated case in which the energy method gives a helpful shortcut.

Two particles, each of mass M, lie on the x axis at $x = a$ and $x = -a$, respectively. Find the force on a particle of mass m located at $\mathbf{r}$.

We start by considering the potential energy of m due to the particle at $x = a$. The distance is $\sqrt{(x-a)^2 + y^2 + z^2}$, and the potential energy is $-GMm/\sqrt{(x-a)^2 + y^2 + z^2} = -GMm/r_1$. Similarly, the potential energy due to the mass at $x = -a$ is $-GMm/\sqrt{(x+a)^2 + y^2 + z^2} = -GMm/r_2$. The total potential energy is the sum of these terms. This illustrates a major advantage of working with energy rather than force. Energy is a scalar and is simply additive, whereas forces must be added vectorially.

We have $u = -GMm/r_1 - GMm/r_2$, or

$$U = -GMm\left\{\frac{1}{[(x-a)^2 + y^2 + z^2]^{\frac{1}{2}}} + \frac{1}{[(x+a)^2 + y^2 + z^2]^{\frac{1}{2}}}\right\}.$$

The force components are easily found by differentiation.

$$\begin{aligned}F_x(x,y,z) &= -\frac{\partial U}{\partial x} \\ &= -GMm\left\{\frac{(x-a)}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{(x+a)}{[(x+a)^2 + y^2 + z^2]^{\frac{3}{2}}}\right\} \\ &= -GMm\left(\frac{x-a}{r_1^3} + \frac{x+a}{r_2^3}\right)\end{aligned}$$

Similarly,

$$F_y(x,y,z) = -\frac{\partial U}{\partial y}$$
$$= -GMm\left(\frac{y}{r_1^3} + \frac{y}{r_2^3}\right)$$
$$F_z(x,y,z) = -\frac{\partial U}{\partial z}$$
$$= -GMm\left(\frac{z}{r_1^3} + \frac{z}{r_2^3}\right).$$

If m is far from the other two masses so that $|x| \gg a$, we have $r_1 \approx r$, $r_2 \approx r$. In this case

$$F_x \approx -\frac{2GMm}{r^2}\frac{x}{r}$$
$$F_y \approx -\frac{2GMm}{r^2}\frac{y}{r}$$
$$F_z \approx -\frac{2GMm}{r^2}\frac{z}{r}.$$

At large distances the force on m is like the force $(-2GMm/r^2)\hat{\mathbf{r}}$ that would be exerted by a single mass $2M$ located at the origin.

Perhaps these examples suggest something of the convenience of the energy method. Potential energy is much simpler to manipulate than force. If force is needed, we can obtain it from $\mathbf{F} = -\nabla U$. However, only conservative forces have potential energy functions associated with them. Nonconservative forces cannot be expressed as the gradient of a scalar function. Fortunately, most of the important forces of physics are conservative. In Sec. 5.6 we shall develop a simple means for telling whether a force is conservative or not.

We next turn to a discussion of the physical meaning of the gradient.

5.5 The Physical Meaning of the Gradient

Consider a particle moving under conservative forces with potential energy $U(x,y,z)$. As the particle moves from the point (x,y,z) to

$(x + dx, y + dy, z + dz)$, its potential energy changes by

$$U(x + dx, y + dy, z + dz) - U(x,y,z).$$

As explained in the last section, when we intend to take the limit $dx \to 0$, $dy \to 0$, $dz \to 0$, we can represent the change in U by the differential

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz.$$

The displacement is $d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$ and we can write

$$dU = \nabla U \cdot d\mathbf{r} \qquad 5.12$$

where ∇U, the gradient of U, is

$$\nabla U = \frac{\partial U}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\,\hat{\mathbf{k}}.$$

Equation (5.12) expresses the fundamental property of the gradient. The gradient allows us to find the change in a function induced by a change in its variables. In fact, Eq. (5.12) is actually the definition of gradient. Like a vector, the gradient operator is defined without reference to a particular coordinate system.

To develop physical insight into the meaning of ∇U, it is helpful to adopt a pictorial representation of potential energy. So let us make a brief digression.

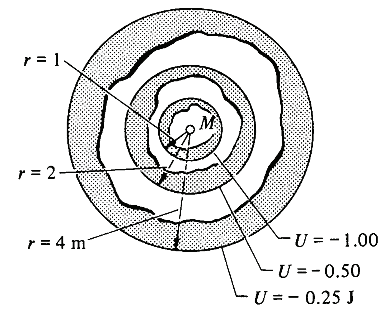
Constant Energy Surfaces and Contour Lines

The equation $U(x,y,z) = \text{constant} = C$ defines for each value of C a surface known as a *constant energy* surface. A particle constrained to move on such a surface has constant potential energy. For example, the gravitational potential energy of a particle m at distance $r = \sqrt{x^2 + y^2 + z^2}$ from particle M is $U = -GMm/r$. The surfaces of constant energy are given by

$$-\frac{GMm}{r} = C$$

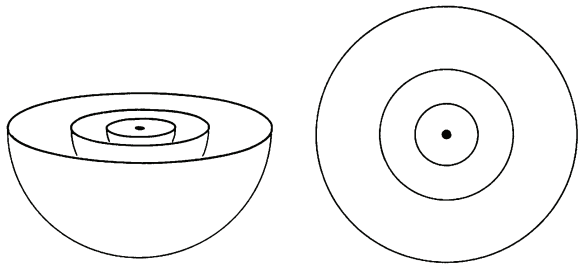
or

$$r = -\frac{GMm}{C}.$$

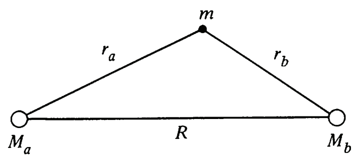


The constant energy surfaces are spheres centered on M, as shown in the drawing. (We have taken $GMm = 1$ N·m² for convenience.)

Constant energy surfaces are usually difficult to draw, and for this reason it is generally easier to visualize U by considering the lines of intersection of the constant energy surfaces with a plane. These lines are sometimes referred to as constant energy lines or, more simply, contour lines. For spherical energy surfaces the contour lines are circles. The next example discusses contour lines for a more complicated situation.



Example 5.6 **Energy Contours for a Binary Star System**



Consider a satellite of mass m in the gravitational field of a binary star system. The stars have masses M_a and M_b and are separated by distance R. The potential energy of the satellite is

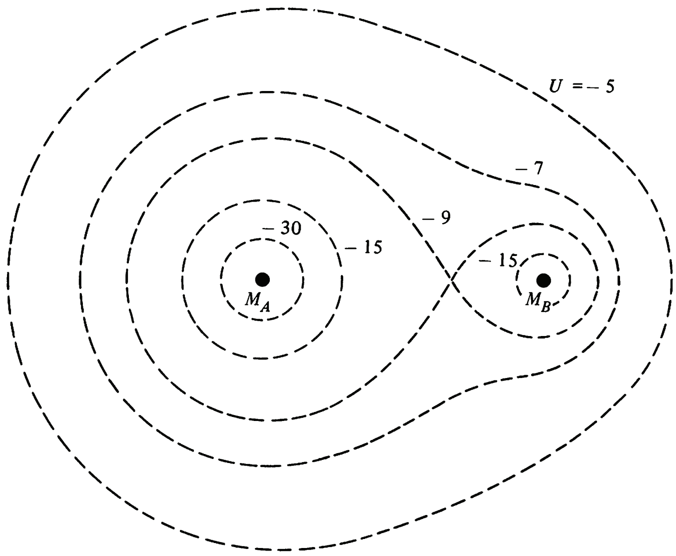
$$U = -\frac{GmM_a}{r_a} - \frac{GmM_b}{r_b},$$

where r_a and r_b are its distances from the two stars. Consider the contour lines in a plane through the axis of the stars. Near star a, where $r_a \ll r_b$, we have

$$U \approx -\frac{GmM_a}{r_a}.$$

Here the contour lines are effectively circles. Near star b, where $r_b \ll r_a$, the contour lines are also effectively circles.

In the intermediate region between the two stars the effects of both bodies are important. The contour lines in the drawing opposite were calculated numerically, with $GmM_b/R = 1$, and $M_b/M_a = \frac{1}{4}$.

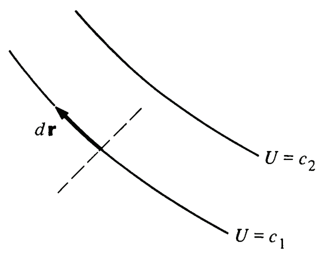


To see the relation between ∇U and contour lines, consider the change in U due to a displacement $d\mathbf{r}$ along a contour. In general

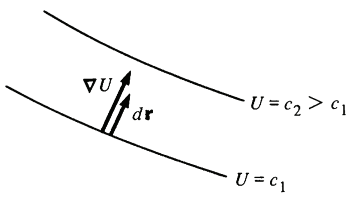
$$dU = \nabla U \cdot d\mathbf{r}.$$

However, on a contour line, U is constant and $dU = 0$. Hence

$$\nabla U \cdot d\mathbf{r} = 0 \qquad (d\mathbf{r} \text{ along contour line}).$$



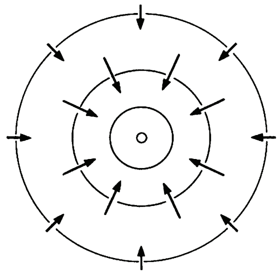
Since ∇U and $d\mathbf{r}$ are not zero, we see that the vector ∇U must be perpendicular to $d\mathbf{r}$. More generally, ∇U is perpendicular to any displacement $d\mathbf{r}$ on a constant energy surface. Hence, at every point in space, ∇U is perpendicular to the constant energy surface passing through that point.



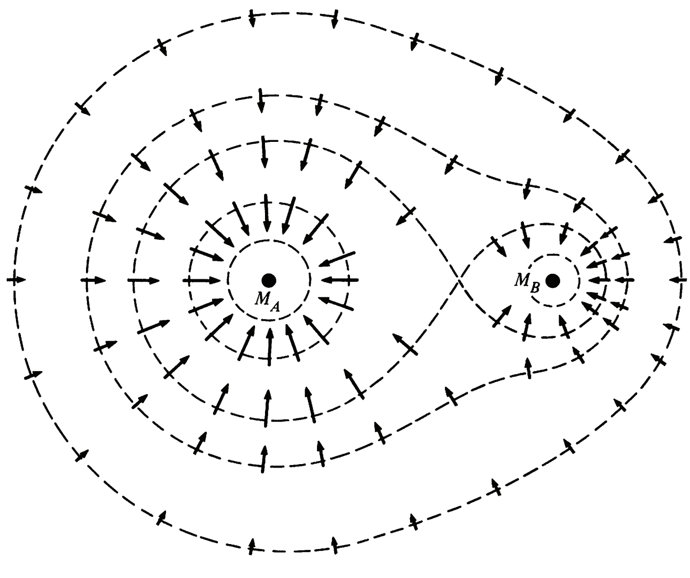
It is not hard to show that ∇U points from lower to higher potential energy. Consider a displacement $d\mathbf{r}$ pointing in the direction of increasing potential energy. For this displacement $dU > 0$, and since $dU = \nabla U \cdot d\mathbf{r} > 0$, we see that ∇U points from lower to higher potential energy. Hence the direction of ∇U is the direction in which U is increasing most rapidly.

Since $\nabla U = -\mathbf{F}$, we conclude that $\mathbf{F}$ is everywhere perpendicular to the constant energy surfaces and points from higher to lower potential energy.

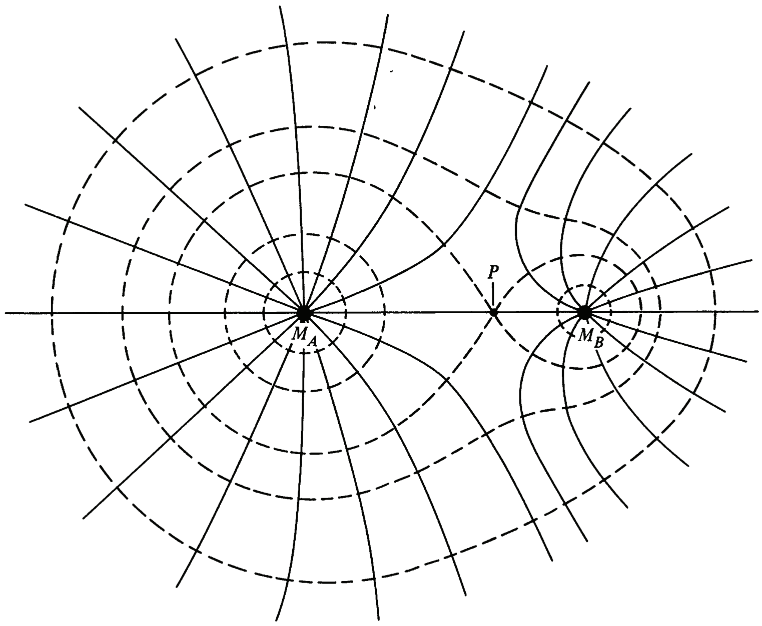
Given the contour lines, it is easy to sketch the force. For the gravitational interaction of a particle with a mass located at the origin, the contour lines are circles. The force points radially inward from higher to lower potential energy, as we expect.



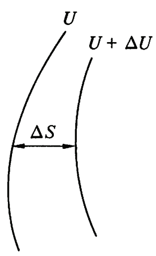
The drawing below shows the force at various points along the contour lines of the binary star system of Example 5.6. We can



extend the arrows to form a curve everywhere parallel to **F**. These lines show the direction of the force everywhere in space and provide a simple map of the force field. Note that the force lines are perpendicular to the energy contours everywhere. Point P, where



two energy contours intersect, presents a problem. How can the force point in two directions at once? The answer is that point P is the equilibrium point between the two stars where the force vanishes.



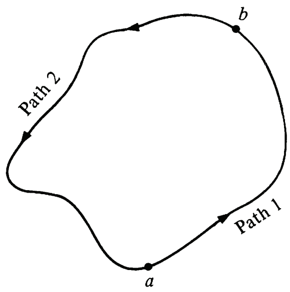
If two adjacent energy surfaces differ in energy by ΔU, then where the separation is ΔS,

$$|\nabla U| \approx \frac{\Delta U}{\Delta S}.$$

Hence, the closer the surfaces, the larger the gradient. More physically, the force is large where the potential energy is changing rapidly.

5.6 How to Find Out if a Force Is Conservative

Although we have seen numerous examples of conservative forces, we have no general test to tell us whether a given force $\mathbf{F(r)}$ is conservative. Let us now attack this problem.



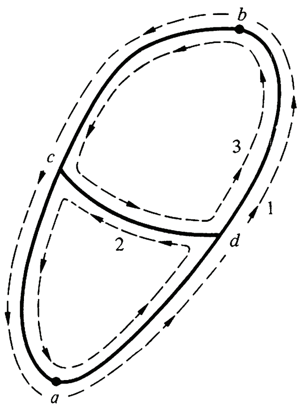
Our starting point is the observation that if **F(r)** is conservative, the work done on a particle by force **F** as it moves from a to b and back to a around a closed path is

$$\underset{\text{Path 1}}{\oint_a^b} \mathbf{F} \cdot d\mathbf{r} + \underset{\text{Path 2}}{\oint_b^a} \mathbf{F} \cdot d\mathbf{r} = (-U_b + U_a) + (-U_a + U_b) = 0.$$

Thus, the work done by a conservative force around a closed path must be zero. Symbolically,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0, \qquad 5.13$$

where the integral is a line integral taken around any closed path. (The symbol $\oint$ indicates that the path is closed.) Conversely, if a force **F** satisfies Eq. (5.13) for *all* paths (not just for a special path), the force must be conservative. Hence, Eq. (5.13) is a necessary and sufficient condition for a force to be conservative.

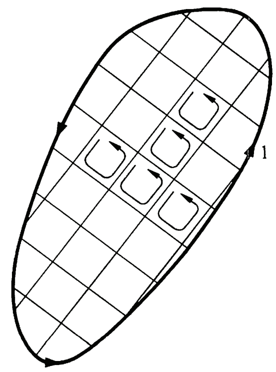


Although you may think that the problem is now more complicated than when we began, the fact is that we have taken a big step forward. However, in order to proceed we must further transform the problem.

Consider $\oint \mathbf{F} \cdot d\mathbf{r}$, where the integral is around loop 1. If we break the integral into two integrals, via the "shortcut" cd, we have

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} = \oint_2 \mathbf{F} \cdot d\mathbf{r} + \oint_3 \mathbf{F} \cdot d\mathbf{r}.$$

This identity follows because the contribution to $\oint_2 \mathbf{F} \cdot d\mathbf{r}$ from the line segment cd is exactly canceled by the contribution from the segment dc to $\oint_3 \mathbf{F} \cdot d\mathbf{r}$. Traversing the same line in two directions gives zero net contribution to the total work.

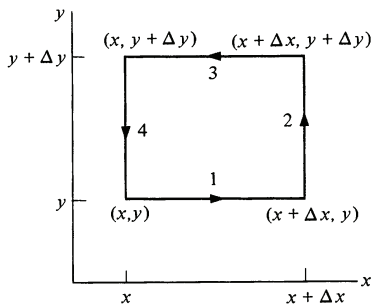


We can proceed to chop up the line integral into many small integrals around tiny loops, as shown in the sketch. When the work around each tiny loop is added, all the contributions from the interior paths cancel, and the total work is identical to the work done in traversing the original perimeter. Hence,

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} = \sum_i \oint_i \mathbf{F} \cdot d\mathbf{r} \qquad 5.14$$

where $\oint_i \mathbf{F} \cdot d\mathbf{r}$ is the work done in circling the ith tiny loop.

If you are wondering where this is leading, the answer is that by focusing our attention on one of the tiny paths we can convert



the original problem, which involves an integral over a large area, into a problem involving quantities at a single point in space. To do this, we must evaluate the line integral around one of the tiny loops. Let us consider a rectangular loop lying in the xy plane with sides of length Δx and Δy. The integral around the loop is

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint_1 \mathbf{F} \cdot d\mathbf{r} + \oint_2 \mathbf{F} \cdot d\mathbf{r} + \oint_3 \mathbf{F} \cdot d\mathbf{r} + \oint_4 \mathbf{F} \cdot d\mathbf{r}.$$

Integrals 1 and 3 both involve paths in the x direction, so let us consider them together. Integral 1 is

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} = \int_{x,y}^{x+\Delta x,\, y} F_x(x,y)\, dx. \qquad 5.15$$

If Δx is small,

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} \approx F_x(x,y)\, \Delta x.$$

Similarly, the integral along path 3 is

$$\oint_3 \mathbf{F} \cdot d\mathbf{r} \approx -F_x(x,\, y + \Delta y)\, \Delta x.$$

The integrals along paths 1 and 3 almost cancel. However, the small difference in y between the two paths is important. We have

$$\begin{aligned} \oint_1 \mathbf{F} \cdot d\mathbf{r} + \oint_3 \mathbf{F} \cdot d\mathbf{r} &\approx F_x(x,y)\, \Delta x - F_x(x,\, y + \Delta y)\, \Delta x \\ &= -[F_x(x,\, y + \Delta y) - F_x(x,y)]\, \Delta x. \qquad 5.16 \end{aligned}$$

You may be puzzled by the fact that we are allowing for the fact that y is different between the two paths but are ignoring the variation of x along each of the paths. The reason is simply that the variation in y has an effect in first order, whereas the variation in x does not, as you can verify for yourself.

We shall eventually take the limit $\Delta x \to 0$, $\Delta y \to 0$, and from the discussion of differentials in Sec. 5.2, we have

$$F_x(x,\, y + \Delta y) - F_x(x,y) = \frac{\partial F_x}{\partial y}\, \Delta y.$$

Hence Eq. (5.16) can be written

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} + \oint_3 \mathbf{F} \cdot d\mathbf{r} = -\frac{\partial F_x}{\partial y}\, \Delta x\, \Delta y.$$

Applying the same argument to paths 2 and 4 gives

$$\oint_2 \mathbf{F} \cdot d\mathbf{r} + \oint_4 \mathbf{F} \cdot d\mathbf{r} = \frac{\partial F_y}{\partial x} \Delta x \, \Delta y.$$

The line integral around the tiny rectangular loop in the xy plane is therefore

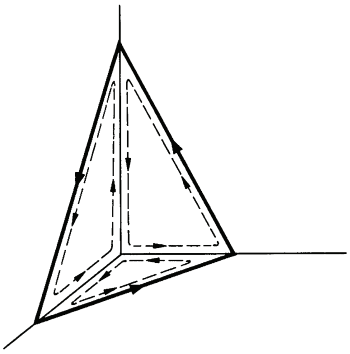
$$\oint_{xy \text{ plane}} \mathbf{F} \cdot d\mathbf{r} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \Delta x \, \Delta y. \qquad 5.17a$$

Although we shall not stop to prove it, this result holds for a small loop of any shape if $\Delta x \, \Delta y$ is replaced by the actual area ΔA.

The line integral around a tiny loop in the yz plane can be found by simply cycling the variables, $x \to y$, $y \to z$, $z \to x$. We find

$$\oint_{yz \text{ plane}} \mathbf{F} \cdot d\mathbf{r} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \Delta y \, \Delta z. \qquad 5.17b$$

Similarly, for a loop in the xz plane,

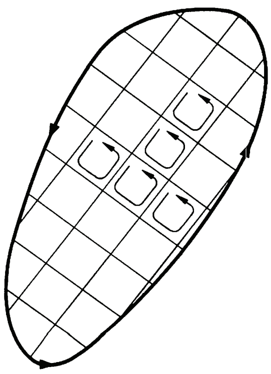


$$\oint_{xz \text{ plane}} \mathbf{F} \cdot d\mathbf{r} = \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \Delta x \, \Delta z. \qquad 5.17c$$

The line integral around a tiny loop in an arbitrary orientation can be decomposed into line integrals in the three coordinate planes, as the sketch suggests.

Accordingly, the line integral around any tiny loop will vanish provided

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 \qquad 5.18$$
$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0.$$



If Eq. (5.18) is satisfied everywhere, the line integral around any tiny loop vanishes and it follows that $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. Hence, a force satisfying Eq. (5.18) is conservative.

We have achieved our goal of finding a mathematical test for whether or not a given force is conservative. However, Eq. (5.18) is rather cumbersome as it stands. Fortunately, we can summarize it in simple vector notation. If we use the familiar rules of evaluating the cross product (Sec. 1.4) and treat the vector operator ∇ as if it were a vector, then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \qquad 5.19$$

$\nabla \times \mathbf{F}$ is called the *curl* of $\mathbf{F}$.

Example 5.7 The Curl of the Gravitational Force

We know that the gravitational force is conservative since it possesses a potential energy function. However, for purposes of illustration, let us prove that the force of gravity is conservative by showing that its curl is zero.

For the gravitational force between two particles we have

$$\mathbf{F} = \frac{A}{r^2}\hat{\mathbf{r}}$$

$$= A\frac{\mathbf{r}}{r^3} = A\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r^3}.$$

$$(\nabla \times \mathbf{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

$$= \frac{\partial}{\partial y}\left(\frac{Az}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{Ay}{r^3}\right).$$

The first term on the right hand side is

$$\frac{\partial}{\partial y} Az(x^2 + y^2 + z^2)^{-\frac{3}{2}} = Az(-\tfrac{3}{2})(x^2 + y^2 + z^2)^{-\frac{5}{2}}(2y)$$

$$= -3A\frac{zy}{r^5}.$$

Similarly,

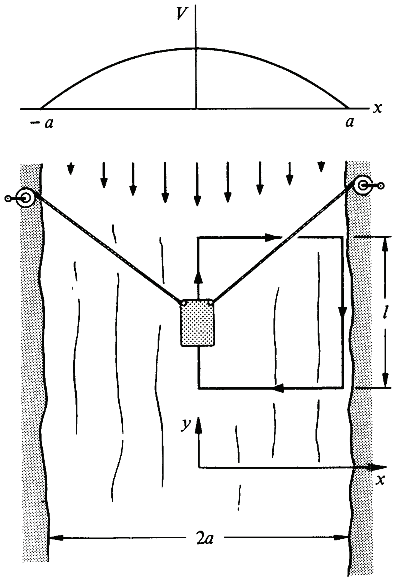
$$\frac{\partial}{\partial z}\frac{Ay}{r^3} = -3A\frac{yz}{r^5}.$$

Hence,

$$(\nabla \times \mathbf{F})_x = -3A\frac{zy}{r^5} + 3A\frac{yz}{r^5} = 0.$$

By cycling the coordinates, we see that the other components of $\nabla \times \mathbf{F}$ are also zero. Hence $\nabla \times \mathbf{F} = 0$ and the gravitational force is conservative.

Example 5.8 A Nonconservative Force



Here is an example of a nonconservative force: consider a river with a current whose velocity **V** is maximum at the center and drops to zero at either bank.

$$\mathbf{V} = -V_0\left(1 - \frac{x^2}{a^2}\right)\hat{\mathbf{j}}$$

The width of the river is $2a$, and the coordinates are shown in the sketch.

Suppose that a barge in the stream is hauled around the path shown, by winches on the banks. The barge is pulled slowly and we shall assume that the force exerted on it by the current is

$$\mathbf{F}_{\text{river}} = b\mathbf{V},$$

where b is a constant. The barge is effectively in equilibrium, so that the force exerted by the winches is

$$\begin{aligned}\mathbf{F} &= -\mathbf{F}_{\text{river}} = -b\mathbf{V}\\ &= bV_0\left(1 - \frac{x^2}{a^2}\right)\hat{\mathbf{j}}.\end{aligned}$$

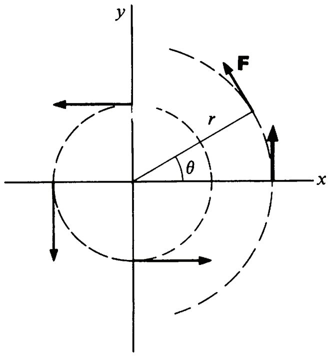
Let us evaluate $\nabla \times \mathbf{F}$ to determine whether or not the force is conservative. We have

$$\begin{aligned}(\nabla \times \mathbf{F})_x &= \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\\ &= 0\\ (\nabla \times \mathbf{F})_y &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\\ &= 0\\ (\nabla \times \mathbf{F})_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\\ &= \frac{\partial}{\partial x} bV_0\left(1 - \frac{x^2}{a^2}\right)\\ &= -\frac{2bV_0}{a^2}x.\end{aligned}$$

Since the curl does not vanish, the force is nonconservative and the winches must do work to pull the barge around the closed path. The work done going upstream is $F(x = 0)l$, and the work done going downstream is $-F(x = a)l$. (In this idealized problem no work is needed to move the barge cross stream.) Since $F(x) = bV_0(1 - x^2/a^2)$, the total work done by the winches is

$$\begin{aligned}W &= bV_0l - bV_0l\left(1 - \frac{a^2}{a^2}\right)\\ &= bV_0l.\end{aligned}$$

Example 5.9 A Most Unusual Force Field



The field described in this example has some very surprising properties.

Consider a particle moving in the xy plane under the force

$$\mathbf{F}(r) = \frac{A}{r}\hat{\boldsymbol{\theta}},$$

where A is a constant. The force decreases as $1/r$, and is directed tangentially about the origin, as shown.

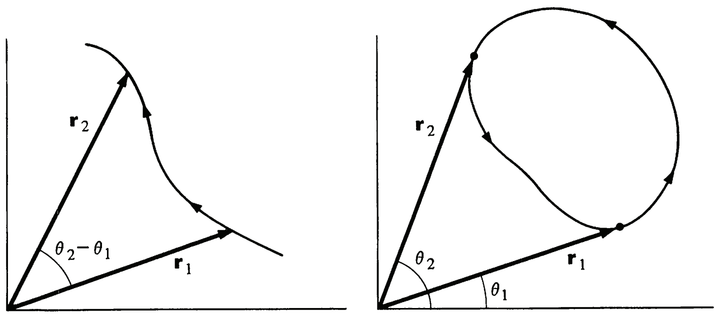
The work done as the particle travels through $d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$ is

$$\begin{aligned} dW &= \mathbf{F}\cdot d\mathbf{r} \\ &= \frac{A}{r}\,r\,d\theta \\ &= A\,d\theta. \end{aligned}$$

Surprisingly, the work does not depend on r, but only on the angle subtended.

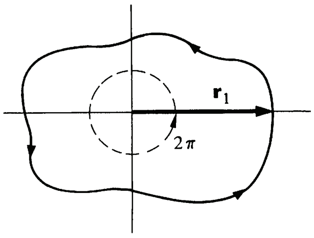
Offhand, $\mathbf{F}$ may seem to be conservative, since the work done in going from $\mathbf{r}_1$ to $\mathbf{r}_2$ in the drawing below, left, appears to be independent of path:

$$\begin{aligned} W &= \oint_{\mathbf{r}_1}^{\mathbf{r}_2} A\,d\theta \\ &= A(\theta_2 - \theta_1). \end{aligned}$$



For instance, for the closed path shown above right,

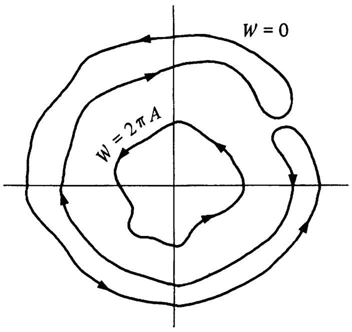
$$\begin{aligned} W &= \oint_{\mathbf{r}_1}^{\mathbf{r}_2} A\,d\theta + \oint_{\mathbf{r}_2}^{\mathbf{r}_1} A\,d\theta \\ &= A(\theta_2 - \theta_1) + A(\theta_1 - \theta_2) \\ &= 0, \end{aligned}$$



as we expect for a conservative force.

However, consider the work done along a closed path which encloses the origin as in the drawing at the left. Since $\theta_1 = 0$ and $\theta_2 = 2\pi$, the work $W = 2\pi A$. Evidently, $\mathbf{F}$ is not conservative.

Every time the particle makes a complete trip around the origin, the force does work $2\pi A$, but for a closed path that does not encircle the origin, $W = 0$. The force appears conservative provided that the path does not enclose the origin.



If you evaluate $\nabla \times \mathbf{F}$, you will find that it is zero everywhere except at the origin, where it has a singularity. It is this singularity which gives the force such peculiar properties. For the line integral of a force to vanish around a closed path, the curl must be zero everywhere inside the path. In this example, $\nabla \times \mathbf{F}$ is zero everywhere except at the origin.

If a force is conservative, it is always possible to find a potential energy function U such that $\mathbf{F} = -\nabla U$. The following example shows how this is done.

Example 5.10 Construction of the Potential Energy Function

In this example we shall find the potential energy function associated with the force

$$\mathbf{F} = A(x^2\hat{\mathbf{i}} + y\hat{\mathbf{j}}). \qquad 1$$

The first thing is to ascertain that $\nabla \times \mathbf{F} = 0$, for otherwise U does not exist. Since you can easily verify this for yourself, we proceed to determine U. U must obey

$$-\frac{\partial U}{\partial x} = F_x \qquad 2$$

$$= Ax^2$$

and

$$-\frac{\partial U}{\partial y} = F_y \qquad 3$$

$$= Ay.$$

We can integrate Eq. (2) to obtain

$$U(x,y) = -\frac{A}{3}x^3 + f(y). \qquad 4$$

Equation (4) needs some explanation. If U depended only on x, then integrating Eq. (2) would yield $U(x) = (-A/3)x^3 + C$, where C is a constant. However, U also depends on y. As far as partial differentiation with respect to x is concerned, $f(y)$ is a constant, since $\partial f(y)/\partial x = 0$.

Equation (4) is the most general solution of Eq. (2), and we can proceed to find the solution to Eq. (3). By substituting Eq. (4) into Eq. (3), we obtain

$$-\frac{\partial}{\partial y}\left[-\frac{A}{3}x^3 + f(y)\right] = Ay$$

or

$$-\frac{\partial f(y)}{\partial y} = -\frac{df(y)}{dy}$$
$$= Ay.$$

This can be integrated to give

$$f(y) = -\frac{A}{2}y^2 + C,$$

where C is a constant. [Since $f(y)$ is a function of the single variable y, the constant of integration cannot involve x.]

The potential energy is

$$U = -\frac{A}{3}x^3 - \frac{A}{2}y^2 + C.$$

Suppose that we try to apply this method to a nonconservative force. For instance, consider

$$\mathbf{F} = A(xy\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}).$$

The curl of **F** is not zero. Nevertheless, we can attempt to solve the equations

$$-\frac{\partial U}{\partial x} = F_x$$
$$= Axy \qquad 5$$
$$-\frac{\partial U}{\partial y} = F_y$$
$$= Ay^2. \qquad 6$$

The general solution of Eq. (5) is

$$U = -\frac{A}{2}x^2y + f(y).$$

If we substitute this into Eq. (6), we have

$$\frac{A}{2}x^2 - \frac{\partial f(y)}{\partial y} = Ay^2$$

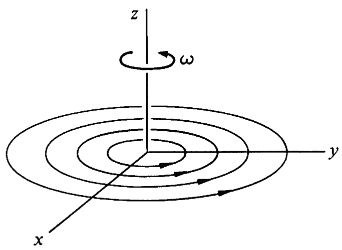
or

$$\frac{\partial f(y)}{\partial y} = -\frac{A}{2}x^2 - Ay^2.$$

But $f(y)$ cannot depend on x, so that this equation has no solution. Hence, it is impossible to construct a potential energy function for this force.

In closing this section, let's take a brief look at the physical meaning of the curl.

Example 5.11 How the Curl Got Its Name

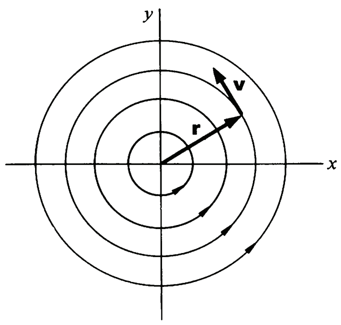


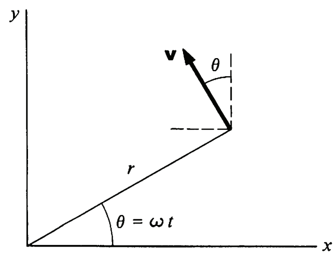
The curl was invented to help describe the properties of moving fluids. To see how the curl is connected with "curliness" or rotation, consider an idealized whirlpool turning with constant angular velocity ω about the z axis. The velocity of the fluid at **r** is

$$\mathbf{v} = r\omega\hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the tangential direction. In cartesian coordinates,

$$\begin{aligned}\mathbf{v} &= r\omega(-\sin\omega t\,\hat{\mathbf{i}} + \cos\omega t\,\hat{\mathbf{j}})\\ &= r\omega\left(-\frac{y}{r}\hat{\mathbf{i}} + \frac{x}{r}\hat{\mathbf{j}}\right)\\ &= -\omega y\hat{\mathbf{i}} + \omega x\hat{\mathbf{j}}.\end{aligned}$$





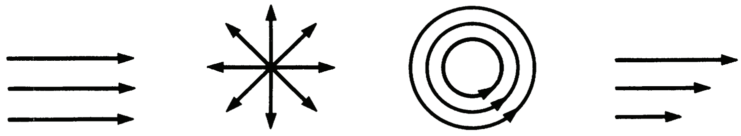
The curl of **v** is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(\omega x) + \frac{\partial}{\partial y}(\omega y)\right]$$

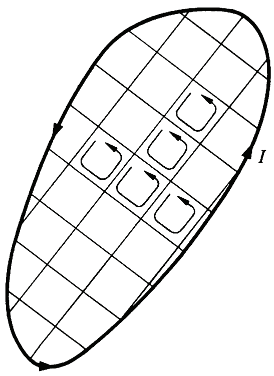
$$= 2\omega\hat{\mathbf{k}}.$$

If a paddle wheel is placed in the liquid, it will start to rotate. The rotation will be a maximum when the axis of the wheel points along the z axis parallel to $\nabla \times \mathbf{v}$. In Europe, curl is often called "rot" (for rotation). A vector field with zero curl gives no impression of rotation, as the sketches illustrate.



5.7 Stokes' Theorem

In Sec. 5.6 we stopped short of proving a remarkable result, known as Stokes' theorem, which relates the line integral of a vector field around a closed path to an integral over an area bounded by the path. Although Stokes' theorem is indispensible to the study of electricity and magnetism, we shall have little further use for it in our study of mechanics. Nevertheless, we have already developed most of the ideas involved in its proof, and only a brief additional discussion is needed.



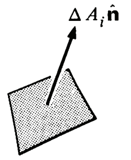
As we discussed earlier, the line integral of **F** around a closed path I can be written as the sum of the line integrals around each tiny loop.

$$\oint_{\mathrm{I}} \mathbf{F} \cdot d\mathbf{r} = \sum_i \oint_i \mathbf{F} \cdot d\mathbf{r}$$

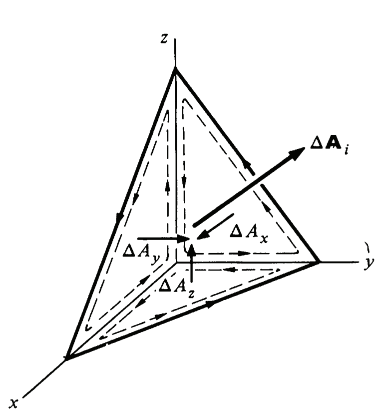
This result holds whether **F** is conservative or not; we shall not assume that **F** is conservative in this proof. Stokes' theorem contains no physics—it is a purely mathematical result.

Our starting point is Eq. (5.17). For a tiny rectangular loop in the xy plane,

$$\oint_i \mathbf{F} \cdot d\mathbf{r} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)_i (\Delta x \, \Delta y)_i.$$



As we have pointed out, the result is independent of the shape of the loop provided that we replace $(\Delta x \, \Delta y)_i$ by the loop's area ΔA_i. We can write the area element as a vector $\Delta \mathbf{A}_i = \Delta A_i \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is normal to the plane of the loop. (Example 1.4 discusses the use of vectors to represent areas.) For a loop in the xy plane, $\Delta \mathbf{A} = \Delta A_z \hat{\mathbf{k}}$ and we have



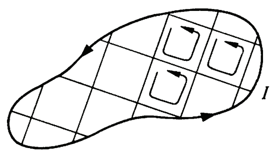
$$\oint_i \mathbf{F} \cdot d\mathbf{r} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)_i (\Delta A_z)_i$$
$$= [(\nabla \times \mathbf{F})_z \, \Delta A_z]_i. \qquad 5.20$$

If the tiny loop is at an arbitrary orientation, it is plausible that

$$\oint_i \mathbf{F} \cdot d\mathbf{r} = [(\text{curl } \mathbf{F})_x \, \Delta A_x + (\text{curl } \mathbf{F})_y \, \Delta A_y + (\text{curl } \mathbf{F})_z \, \Delta A_z]_i$$
$$= [\text{curl } \mathbf{F} \cdot \Delta \mathbf{A}]_i.$$

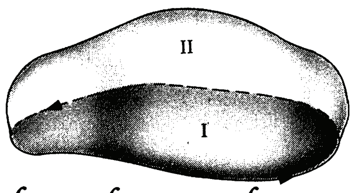
The line integral of $\mathbf{F}$ around path I is therefore

$$\oint \mathbf{F} \cdot d\mathbf{r} = \sum_i \oint_i \mathbf{F} \cdot d\mathbf{r}$$
$$= \sum_i (\text{curl } \mathbf{F} \cdot \Delta \mathbf{A})_i. \qquad 5.21$$



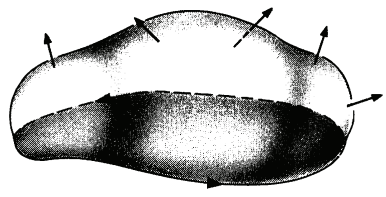
In words, the line integral is equal to the result of taking the scalar product of each vector area element with the curl of $\mathbf{F}$ at that element and summing over all elements bounded by the curve. In the limit $\Delta \mathbf{A}_i \rightarrow 0$, the number of area elements approaches infinity and the sum in Eq. (5.21) becomes an integral. We then have Stokes' theorem

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int \text{curl } \mathbf{F} \cdot d\mathbf{A}. \qquad 5.22$$



$\oint \mathbf{F} \cdot d\mathbf{r} = \int_{\substack{\text{area} \\ \text{I}}} \text{curl } \mathbf{F} \cdot d\mathbf{A} = \int_{\substack{\text{area} \\ \text{II}}} \text{curl } \mathbf{F} \cdot d\mathbf{A}$

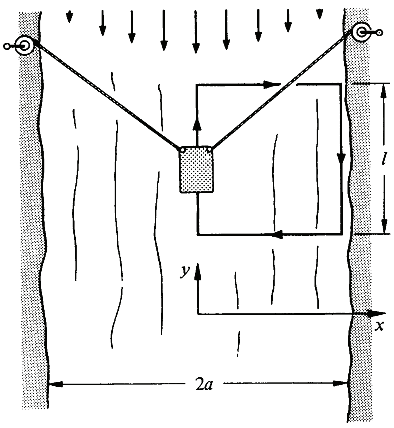
Two important remarks should be made about Stokes' theorem, Eq. (5.22). First, the area of integration on the right hand side can be *any* area bounded by the closed path. Second, there is an apparent ambiguity to the direction of $d\mathbf{A}$, since the normal can be out from either side of the area element. However, Eq. (5.17) was deduced using a counterclockwise circulation about the loop, and in defining the vector associated with the area element, we automatically set up the convention that the direction of $d\mathbf{A}$ is given by the right hand rule. If the circulation is counterclockwise as seen from above, the correct direction of $d\mathbf{A}$ is the one that tends to point "up."

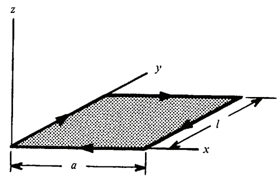


Example 5.12 **Using Stokes' Theorem**

In Example 5.8 we discussed a barge being towed against the current. We found the work done in going around the path in the sketch by evaluating the line integral $\oint \mathbf{F} \cdot d\mathbf{r} = W$. In this example we shall find the work by using Stokes' theorem

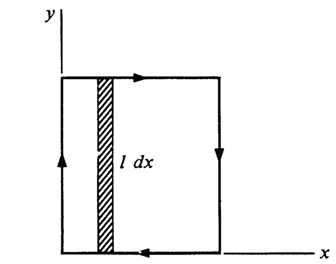
$W = \int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}.$





It is natural to integrate over the surface in the xy plane, as shown in the drawing above right. Since the direction of circulation is clockwise, $d\mathbf{A} = -\,dA\,\hat{\mathbf{k}}$, and we have $W = -\int (\nabla \times \mathbf{F})_z\,dA$.
From Example 5.8, the force is

$$\mathbf{F} = bV_0\left(1 - \frac{x^2}{a^2}\right)\hat{\mathbf{j}}$$



and

$$(\nabla \times \mathbf{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$
$$= -\frac{2bV_0x}{a^2}.$$

Since the integrand does not involve y, it is convenient to take $dA = l\,dx$. Then

$$W = \int_0^a \frac{2bV_0l}{a^2} x\,dx$$
$$= \frac{2bV_0l}{a^2}\left(\frac{a^2}{2}\right)$$
$$= bV_0l,$$

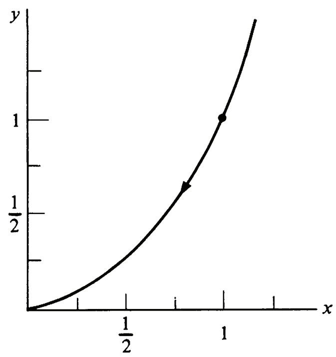
as we found previously by evaluating the line integral.

Problems 5.1 Find the forces for the following potential energies.

a. $U = Ax^2 + By^2 + Cz^2$

b. $U = A\ \ln(x^2 + y^2 + z^2)$ $\quad$ ($\ln = \log_e$)

c. $U = A \cos \theta/r^2$ $\quad$ (plane polar coordinates)



5.2 A particle of mass m moves in a horizontal plane along the parabola $y = x^2$. At $t = 0$ it is at the point (1,1) moving in the direction shown with speed v_0. Aside from the force of constraint holding it to the path, it is acted upon by the following external forces:

A radial force $\quad \mathbf{F}_a = -Ar^3\hat{\mathbf{r}}$
A force given by $\quad \mathbf{F}_b = B(y^2\hat{\mathbf{i}} - x^2\hat{\mathbf{j}})$

where A and B are constants.

a. Are the forces conservative?

b. What is the speed v_f of the particle when it arrives at the origin?
Ans. $v_f = (v_0^2 + A/2m + 3B/5m)^{\frac{1}{2}}$

5.3 Decide whether the following forces are conservative.

a. $\mathbf{F} = \mathbf{F}_0 \sin at$, where $\mathbf{F}_0$ is a constant vector.

b. $F = A\theta\hat{\mathbf{r}}$, A = constant and $0 \le \theta < 2\pi$. ($\mathbf{F}$ is limited to the xy plane.)

c. A force which depends on the velocity of a particle but which is always perpendicular to the velocity.

5.4 Determine whether each of the following forces is conservative. Find the potential energy function if it exists. A, α, β are constants.

a. $\mathbf{F} = A(3\hat{\imath} + z\hat{\jmath} + y\hat{k})$

b. $\mathbf{F} = Axyz(\hat{\imath} + \hat{\jmath} + \hat{k})$

c. $F_x = 3Ax^2y^5e^{\alpha z}$, $F_y = 5Ax^3y^4e^{\alpha z}$, $F_z = \alpha Ax^3y^5e^{\alpha z}$

d. $F_x = A \sin(\alpha y) \cos(\beta z)$, $F_y = -Ax\alpha \cos(\alpha y) \cos(\beta z)$, and $F_z = Ax \sin(\alpha y) \sin(\beta z)$

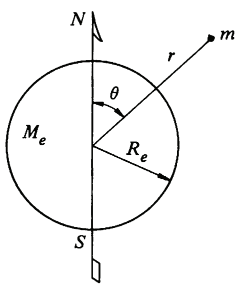
5.5 The potential energy function for a particular two dimensional force field is given by $U = Cxe^{-y}$, where C is a constant.

a. Sketch the constant energy lines.

b. Show that if a point is displaced by a short distance dx along a constant energy line, then its total displacement must be $d\mathbf{r} = dx(\hat{\imath} + \hat{\jmath}/x)$.

c. Using the result of b, show explicitly that ∇U is perpendicular to the constant energy line.

5.6 If $\mathbf{A}(\mathbf{r})$ is a vector function of $\mathbf{r}$ which everywhere satisfies $\nabla \times \mathbf{A} = 0$, show that $\mathbf{A}$ can be expressed by $\mathbf{A}(\mathbf{r}) = \nabla\phi(\mathbf{r})$, where $\phi(\mathbf{r})$ is some scalar function. (*Hint*: The result follows directly from physical arguments.)

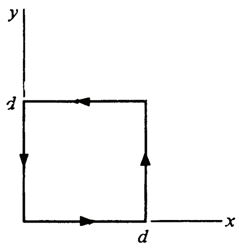


5.7 When the flattening of the earth at the poles is taken into account, it is found that the gravitational potential energy of a mass m a distance r from the center of the earth is approximately

$$U = -\frac{GM_em}{r}\left[1 - 5.4 \times 10^{-4}\left(\frac{R_e}{r}\right)^2 (3\cos^2\theta - 1)\right],$$

where θ is measured from the pole.

Show that there is a small tangential gravitational force on m except above the poles or the equator. Find the ratio of this force to GM_em/r^2 for $\theta = 45°$ and $r = R_e$.

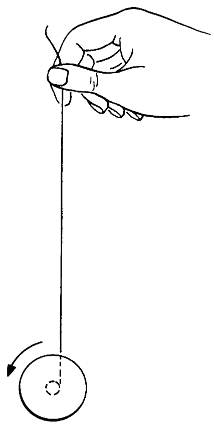


5.8 How much work is done around the path that is shown by the force $\mathbf{F} = A(y^2\hat{\imath} + 2x^2\hat{\jmath})$, where A is a constant and x and y are in meters? Find the answer by evaluating the line integral, and also by using Stokes' theorem.

Ans. $W = Ad^3$

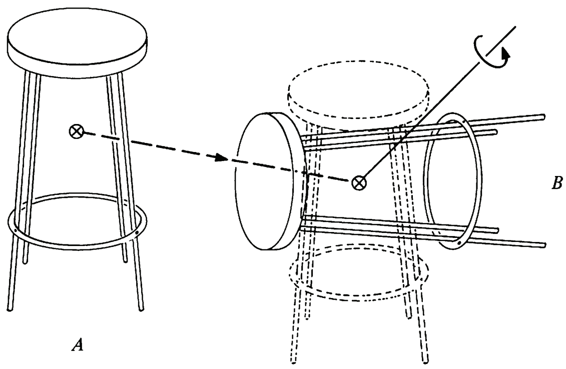
6 ANGULAR MOMENTUM AND FIXED AXIS ROTATION

6.1 Introduction



Our development of the principles of mechanics in the past five chapters is lacking in one important respect: we have not developed techniques to handle the rotational motion of solid bodies. For example, consider the common Yo-Yo running up and down its string as the spool winds and unwinds. In principle we already know how to analyze the motion: each particle of the Yo-Yo moves according to Newton's laws. Unfortunately, analyzing rotational problems on a particle-by-particle basis is an impossible task. What we need is a simple method for treating the rotational motion of an extended body as a whole. The goal of this chapter is to develop such a method. In attacking the problem of translational motion, we needed the concepts of force, linear momentum, and center of mass; in this chapter we shall develop for rotational motion the analogous concepts of torque, angular momentum, and moment of inertia.

Our aim, of course, is more ambitious than merely to understand Yo-Yos; our aim is to find a way of analyzing the general motion of a rigid body under any combination of applied forces. Fortunately this problem can be divided into two simpler problems —finding the center of mass motion, a problem we have already solved, and finding the rotational motion about the center of mass, the task at hand. The justification for this is a theorem of rigid body motion which asserts that any displacement of a rigid body can be decomposed into two independent motions: a translation of the center of mass and a rotation about the center



To bring the body from position A to some new position B, first translate it so that the center of mass coincides with the new center of mass, and then rotate it around the appropriate axis through the center of mass until the body is in the desired position.

of mass. A few minutes spent playing with a rigid body such as a book or a chair should convince you that the theorem is plausible. Note that the theorem does not say that this is the *only* way to represent a general displacement—merely that it is one possible way of doing so. The general proof of this theorem[1] is presented in Note 6.1 at the end of the chapter. However, detailed attention to a formal proof is not necessary at this point. What is important is being able to visualize any displacement as the combination of a single translation and a single rotation.

Leaving aside extended bodies for a time, we start in the best tradition of physics by considering the simplest possible system—a particle. Since a particle has no size, its orientation in space is of no consequence, and we need concern ourselves only with translational motion. In spite of this, particle motion is useful for introducing the concepts of angular momentum and torque. We shall then move to progressively more complex systems, culminating, in Chap. 7, with a treatment of the general motion of a rigid body.

6.2 Angular Momentum of a Particle

Here is the formal definition of the *angular momentum* **L** of a particle which has momentum **p** and position vector **r** with respect to a given coordinate system.

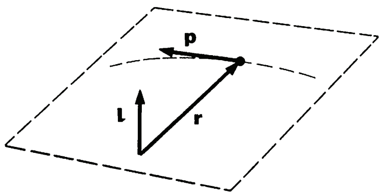
$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad 6.1$$

The unit of angular momentum is $kg{\cdot}m^2/s$ in the SI system or $g{\cdot}cm^2/s$ in cgs. There are no special names for these units.

Angular momentum is our first physical quantity to involve the cross product. (See Secs. 1.2 and 1.4 if you need to review the cross product.) Because angular momentum is so different from anything we have yet encountered, we shall discuss it in great detail at first.

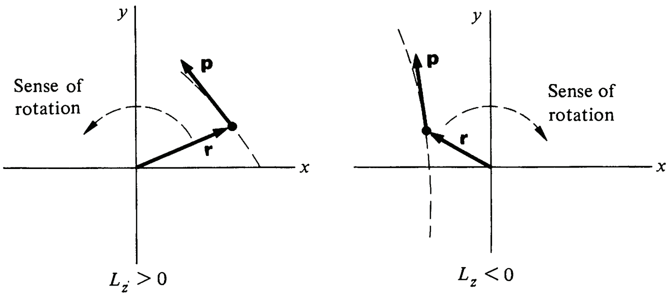
Possibly the strangest aspect of angular momentum is its direction. The vectors **r** and **p** determine a plane (sometimes known as the plane of motion), and by the properties of the cross product, **L** is perpendicular to this plane. There is nothing particularly "natural" about the definition of angular momentum. However, **L** obeys a very simple dynamical equation, as we shall see, and therein lies its usefulness.

[1] Euler proved that the general displacement of a rigid body with one point fixed is a rotation about some axis; the theorem quoted in the text, called Chasle's theorem, follows directly from this.



The diagram at left shows the trajectory and instantaneous position and momentum of a particle. $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{p}$, and points in the direction dictated by the right hand rule for vector multiplication. Although $\mathbf{L}$ has been drawn through the origin, this location has no significance. Only the direction and magnitude of $\mathbf{L}$ are important.

If $\mathbf{r}$ and $\mathbf{p}$ lie in the xy plane, then $\mathbf{L}$ is in the z direction. $\mathbf{L}$ is in the positive z direction if the "sense of rotation" of the point about the origin is counterclockwise, and in the negative z direction if the sense of rotation is clockwise. Note that the sense of rotation is well defined even if the trajectory is a straight line. The only exception is when the trajectory aims at the origin, in which case $\mathbf{r}$ and $\mathbf{p}$ are along the same line so that $\mathbf{L}$ is 0 anyway.



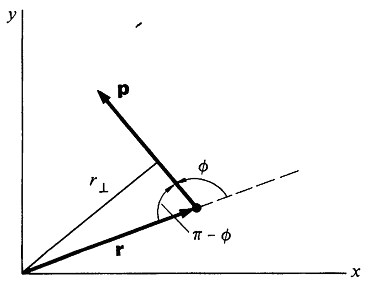
There are various methods for visualizing and calculating angular momentum. Here are three ways to calculate the angular momentum of a particle moving in the xy plane.

Method 1

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$= rp \sin \phi \hat{\mathbf{k}}$$

or

$$L_z = rp \sin \phi.$$



For motion in the xy plane, $\mathbf{L}$ lies in the z direction. Its magnitude has a simple geometrical interpretation: the line $r_\perp$ has length $r_\perp = r \sin(\pi - \phi) = r \sin \phi$. Therefore,

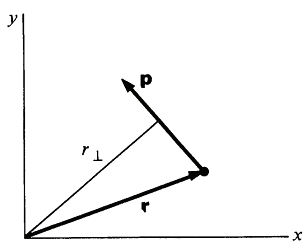
$$L_z = r_\perp p,$$

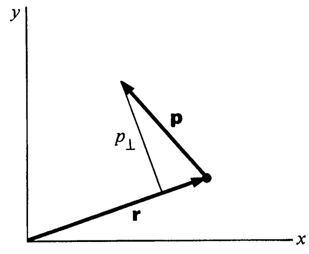
where $r_\perp$ is the perpendicular distance between the origin and the line of **p**. This result illustrates that angular momentum is proportional to the distance from the origin to the line of motion.

As the sketches show, an alternative way of writing L_z is

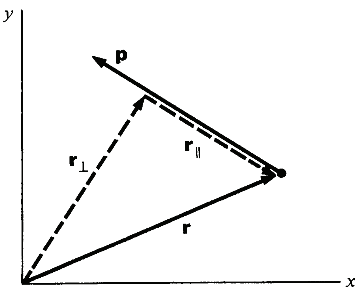
$$L_z = rp_\perp,$$

where $p_\perp$ is the component of **p** perpendicular to **r**.





Method 2



Resolve **r** into two vectors $\mathbf{r}_\perp$ and $\mathbf{r}_\parallel$,

$$\mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_\parallel,$$

such that $\mathbf{r}_\perp$ is perpendicular to **p**, and $\mathbf{r}_\parallel$ is parallel to **p**. Then

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times \mathbf{p} = (\mathbf{r}_\perp + \mathbf{r}_\parallel) \times \mathbf{p} \\ &= (\mathbf{r}_\perp \times \mathbf{p}) + (\mathbf{r}_\parallel \times \mathbf{p}) \\ &= \mathbf{r}_\perp \times \mathbf{p},\end{aligned}$$

since $\mathbf{r}_\parallel \times \mathbf{p} = 0$. (Parallel vectors have zero cross product.) Evaluating the cross product $\mathbf{r}_\perp \times \mathbf{p}$ is trivial because the vectors are perpendicular by construction. We have

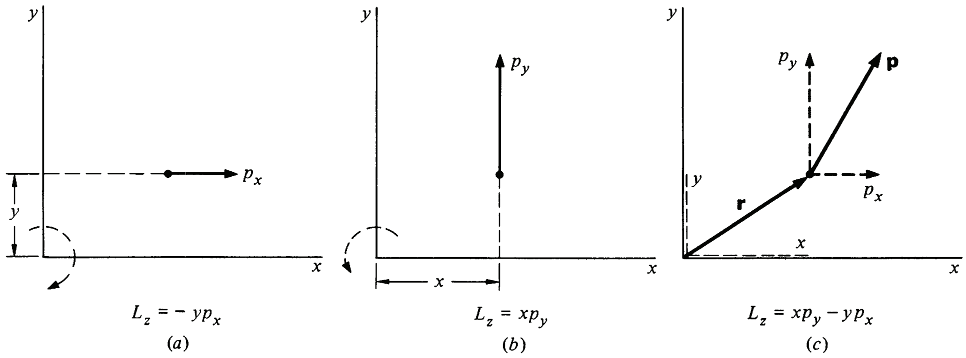
$$L_z = |\mathbf{r}_\perp|\,|\mathbf{p}|$$

as before. By a similar argument,

$$L_z = |\mathbf{r}|\,|\mathbf{p}_\perp|.$$

Method 3

Consider motion in the xy plane, first in the x direction and then in the y direction, as in drawings a and b on the next page.



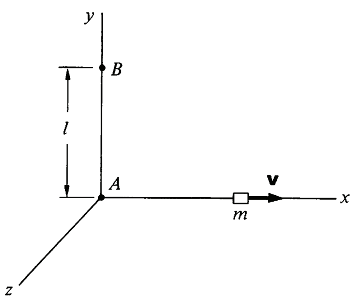
The most general case involves both these motions simultaneously, as drawings above show.

Hence $L_z = xp_y - yp_x$, as you can verify by inspection or by evaluating the cross product as follows. Using $\mathbf{r} = (x,y,0)$ and $\mathbf{p} = (p_x,p_y,0)$, we have

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times \mathbf{p}\\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix}\\ &= (xp_y - yp_x)\hat{\mathbf{k}}.\end{aligned}$$

We have limited our illustrations to motion in the xy plane where the angular momentum lies entirely along the z axis. There is, however, no difficulty applying any of these methods to the general case where $\mathbf{L}$ has components along all three axes.

Example 6.1 Angular Momentum of a Sliding Block



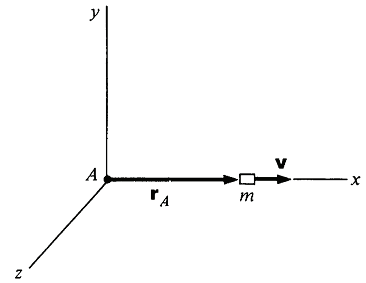
Consider a block of mass m and negligible dimensions sliding freely in the x direction with velocity $\mathbf{v} = v\hat{\mathbf{i}}$, as shown in the sketch. What is its angular momentum $\mathbf{L}_A$ about origin A and its angular momentum $\mathbf{L}_B$ about the origin B?

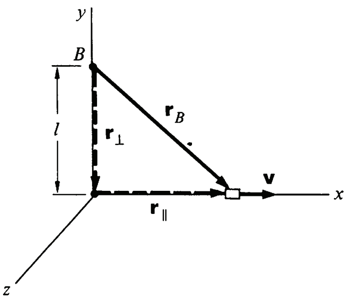
As shown in the drawing on the top of page 237, the vector from origin A to the block is

$$\mathbf{r}_A = x\hat{\mathbf{i}}.$$

Since $\mathbf{r}_A$ is parallel to $\mathbf{v}$, their cross product is zero and

$$\begin{aligned}\mathbf{L}_A &= m\mathbf{r}_A \times \mathbf{v}\\ &= 0.\end{aligned}$$





Taking origin B, we can resolve the position vector $\mathbf{r}_B$ into a component $\mathbf{r}_\parallel$ parallel to $\mathbf{v}$ and a component $\mathbf{r}_\perp$ perpendicular to $\mathbf{v}$. Since $\mathbf{r}_\parallel \times \mathbf{v} = 0$, only $\mathbf{r}_\perp$ gives a contribution to $\mathbf{L}_B$. We have $|\mathbf{r}_\perp \times \mathbf{v}| = lv$ and

$$\begin{aligned}\mathbf{L}_B &= m\mathbf{r}_B \times \mathbf{v}\\ &= mlv\hat{\mathbf{k}}.\end{aligned}$$

$\mathbf{L}_B$ lies in the positive z direction because the sense of rotation is counterclockwise about the z axis.

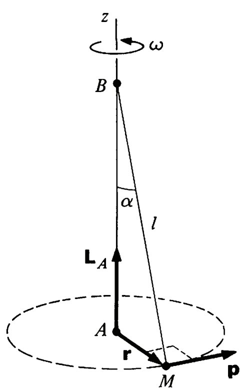
To calculate $\mathbf{L}_B$ formally we can write $\mathbf{r}_B = x\hat{\mathbf{i}} - l\hat{\mathbf{j}}$ and evaluate $\mathbf{r}_B \times \mathbf{v}$ using our determinantal form.

$$\begin{aligned}\mathbf{L}_B &= m\mathbf{r}_B \times \mathbf{v}\\ &= m\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ x & -l & 0\\ v & 0 & 0\end{vmatrix}\\ &= mlv\hat{\mathbf{k}}\end{aligned}$$

as before.

The following example shows in a striking way how $\mathbf{L}$ depends on our choice of origin.

Example 6.2 Angular Momentum of the Conical Pendulum



Let us return to the conical pendulum, which we encountered in Example 2.8, to illustrate some features of angular momentum. Assume that the pendulum is in steady circular motion with constant angular velocity ω.

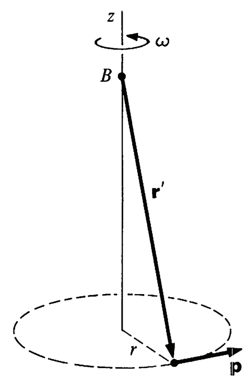
We begin by evaluating $\mathbf{L}_A$, the angular momentum about origin A. From the sketch we see that $\mathbf{L}_A$ lies in the positive z direction. It has magnitude $|\mathbf{r}_\perp|\,|\mathbf{p}| = |\mathbf{r}|\,|\mathbf{p}| = rp$, where r is the radius of the circular motion. Since

$$\begin{aligned}|\mathbf{p}| &= Mv\\ &= Mr\omega,\end{aligned}$$

we have

$$\mathbf{L}_A = Mr^2\omega\mathbf{k}.$$

Note that $\mathbf{L}_A$ is constant, both in magnitude and direction.

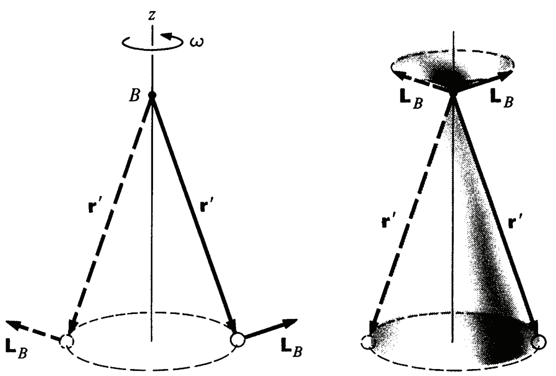


Now let us evaluate the angular momentum about the origin B located at the pivot. The magnitude of $\mathbf{L}_B$ is

$$
\begin{aligned}
|\mathbf{L}_B| &= |\mathbf{r}' \times \mathbf{p}| \\
&= |\mathbf{r}'|\,|\mathbf{p}| = l|\mathbf{p}| \\
&= Mlr\omega,
\end{aligned}
$$

where $|\mathbf{r}'| = l$, the length of the string. It is apparent that the magnitude of $\mathbf{L}$ depends on the origin we choose.

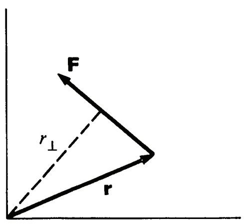
Unlike $\mathbf{L}_A$, the direction of $\mathbf{L}_B$ is not constant. $\mathbf{L}_B$ is perpendicular to both $\mathbf{r}'$ and $\mathbf{p}$, and the sketches below show $\mathbf{L}_B$ at different times. Two sketches are given to emphasize that only the magnitude and direction of $\mathbf{L}$ are important, not the position at which we choose to draw it. The magnitude of $\mathbf{L}_B$ is constant, but its *direction* is obviously not constant; as the bob swings around, $\mathbf{L}_B$ sweeps out the shaded cone shown in the sketch at the right. The z component of $\mathbf{L}_B$ is constant, but the horizontal component travels around the circle with the bob. We shall see the dynamical consequences of this in Example 6.6.



6.3 Torque

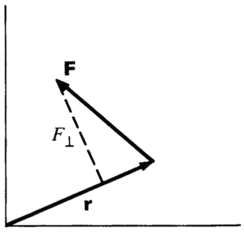
To continue our development of rotational motion we must introduce a new quantity *torque* $\boldsymbol{\tau}$. The torque due to force $\mathbf{F}$ which acts on a particle at position $\mathbf{r}$ is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \qquad 6.2$$



In the last section we discussed several ways of evaluating angular momentum, $\mathbf{r} \times \mathbf{p}$. The mathematical methods we developed for calculating the cross product can also be applied to torque $\mathbf{r} \times \mathbf{F}$. For example, we have

$$|\boldsymbol{\tau}| = |\mathbf{r}_\perp|\,|\mathbf{F}|$$

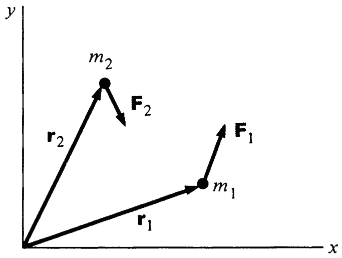


or

$$|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}_\perp|$$

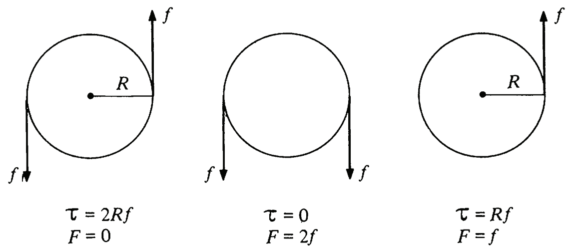
or, formally,

$$\boldsymbol{\tau} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}.$$



We can also associate a "sense of rotation" using **r** and **F**. Assume in the sketch that all the vectors are in the xy plane. The torque on m_1 due to $\mathbf{F}_1$ is along the positive z axis (out of the paper) and the torque on m_2 due to $\mathbf{F}_2$ is along the negative z axis (into the paper).

It is important to realize that torque and force are entirely different quantities. For one thing, torque depends on the origin we choose but force does not. For another, we see from the definition $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ that $\boldsymbol{\tau}$ and $\mathbf{F}$ are always mutually perpendicular. There can be a torque on a system with zero net force, and there can be force with zero net torque. In general, there will be both torque and force. These three cases are illustrated in the sketches below. (The torques are evaluated about the centers of the disks.)



Torque is important because it is intimately related to the rate of change of angular momentum:

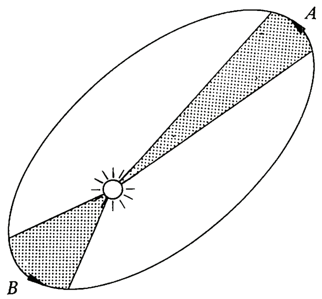
$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$
$$= \left(\frac{d\mathbf{r}}{dt} \times \mathbf{p}\right) + \left(\mathbf{r} \times \frac{d\mathbf{p}}{dt}\right).$$

But $(d\mathbf{r}/dt) \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = 0$, since the cross product of two parallel vectors is zero. Also, $d\mathbf{p}/dt = \mathbf{F}$, by Newton's second law. Hence, the second term is $\mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}$, and we have

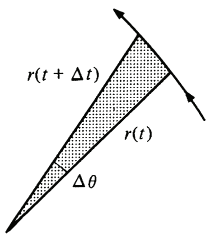
$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \qquad 6.3$$

Equation (6.3) shows that if the torque is zero, $\mathbf{L}$ = constant and the angular momentum is conserved. As you may already realize from our work with linear momentum and energy, conservation laws are powerful tools. However, because we have considered only the angular momentum of a single particle, the conservation law for angular momentum has not been presented in much generality. In fact, Eq. (6.3) follows directly from Newton's second law—only when we talk about extended systems does angular momentum assume its proper role as a new physical concept. Nevertheless, even in its present context, considerations of angular momentum lead to some surprising simplifications, as the next two examples show.

Example 6.3 Central Force Motion and the Law of Equal Areas

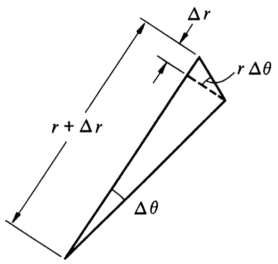


In 1609 Kepler announced his second law of planetary motion, the law of equal areas: that is, the area swept out by the radius vector from the sun to a planet in a given time is the same for any location of the planet in its orbit. The sketch (not to scale) shows the areas swept out by the earth during a month at two different seasons. The shorter radius vector at B is compensated by the greater speed of the earth when it is nearer the sun. We shall now show that the law of equal areas follows directly from considerations of angular momentum, and that it holds not only for motion under the gravitational force but also for motion under any central force.



Consider a particle moving under a central force, $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$, where $f(r)$ has any dependence on r we care to choose. The torque on the particle about the origin is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = \mathbf{r} \times f(\mathbf{r})\hat{\mathbf{r}} = 0$. Hence, the angular momentum of the particle $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is constant both in magnitude and direction. An immediate consequence is that the motion is confined to a plane; otherwise the direction of $\mathbf{L}$ would change with time. We shall now prove that the rate at which area is swept out is constant, a result that leads directly to the law of equal areas.

Consider the position of the particle at t and $t + \Delta t$, when its polar coordinates are, respectively, (r,θ) and $(r + \Delta r,\ \theta + \Delta\theta)$. The area swept out is shown shaded in the drawing at left.



For small values of $\Delta\theta$, the area ΔA is approximately equal to the area of a triangle with base $r + \Delta r$ and altitude $r\,\Delta\theta$, as shown.

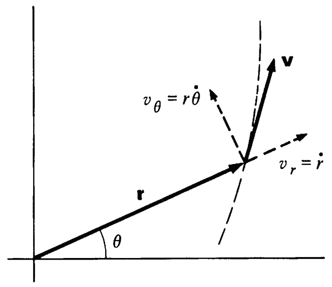
$$\Delta A \approx \tfrac{1}{2}(r + \Delta r)(r\,\Delta\theta)$$
$$= \tfrac{1}{2}r^2\,\Delta\theta + \tfrac{1}{2}r\,\Delta r\,\Delta\theta$$

The rate at which area is swept out is

$$\frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t} + r\frac{\Delta\theta\,\Delta r}{\Delta t}\right)$$
$$= \frac{1}{2}r^2\frac{d\theta}{dt}.$$

(The small triangle with sides $r\,\Delta\theta$ and Δr makes no contribution in the limit.)

In polar coordinates the velocity of the particle is $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$. Its angular momentum is



$$\mathbf{L} = (\mathbf{r}\times m\mathbf{v}) = r\hat{\mathbf{r}}\times m(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}\hat{\mathbf{k}}.$$

(Note that $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$). Hence,

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$$
$$= \frac{L_z}{2m}.$$

Since L_z is constant for any central force, it follows that dA/dt is constant also.

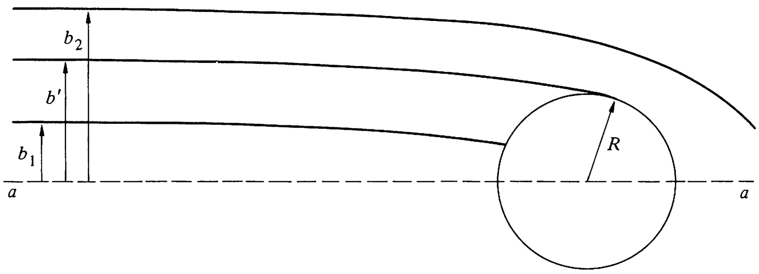
Here is another way to prove the law of equal areas. For a central force, $F_\theta = 0$, so that $a_\theta = 0$. It follows that $ra_\theta = 0$, but $ra_\theta = r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = (d/dt)(r^2\dot{\theta}) = 2(d/dt)(dA/dt)$. Hence, dA/dt = constant.

Example 6.4 Capture Cross Section of a Planet

This example concerns the problem of aiming an unpowered spacecraft to hit a far-off planet. If you have ever looked at a planet through a telescope, you know that it appears to have the shape of a disk. The area of the disk is πR^2, where R is the planet's radius. If gravity played no role, we would have to aim the spacecraft to head for this area in order to assure a hit. However, the situation is more favorable than this because of the gravitational attraction of the spacecraft by the planet. Gravity tends to deflect the spacecraft toward the planet, so that some trajectories which are aimed outside the planetary disk nevertheless end in a hit. Consequently, the effective area for a hit A_e is greater than the geometrical area $A_g = \pi R^2$. Our problem is to find A_e.

We shall neglect effects of the sun and other planets here, although they would obviously have to be taken into account for a real space mission.

One approach to the problem would be to work out the full solution for the orbit of the spacecraft in the gravitational field of the planet. This involves a lengthy calculation which is not really necessary; by using conservation of energy and angular momentum, we can find the answer in a few short steps.



The sketch shows several possible trajectories of the spacecraft. The distance between the launch point and the target planet is assumed to be extremely large compared with R, so that the different trajectories are effectively parallel before the gravitational force of the planet becomes important. The line aa is parallel to the initial trajectories and passes through the center of the planet. The distance b between the initial trajectory and line aa is called the *impact parameter* of the trajectory. The largest value of b for which the trajectory hits the planet is indicated by b' in the sketch. The area through which the trajectory must pass to assure a hit is $A_e = \pi(b')^2$. (If there were no attraction, the trajectories would be straight lines. In this case, $b' = R$ and $A_e = \pi R^2 = A_g$.)

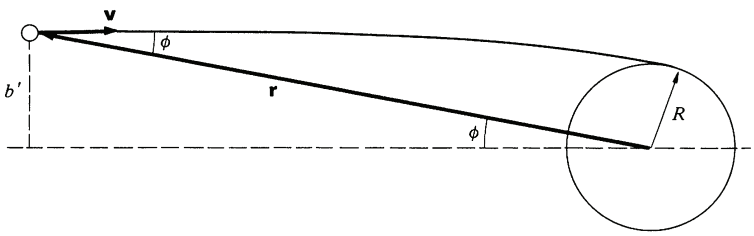
To find b', we note that both the energy and angular momentum of the spacecraft are conserved. (Linear momentum of the spacecraft is not conserved. Do you see why?)

The kinetic energy is $\frac{1}{2}mv^2$, and the potential energy is $-mMG/r$. The total energy $E = K + V$ is

$$E = \frac{1}{2}mv^2 - mMG\frac{1}{r}.$$

The angular momentum about the center of the planet is

$$L = -mrv \sin \phi.$$



Initially, $r \to \infty$, $v = v_0$, and $r \sin \phi = b'$. Hence,

$$L = -mb'v_0,$$

$$E = \frac{1}{2} mv_0^2.$$

The point of collision occurs at the distance of closest approach of the orbit, $r = R$; otherwise the trajectory would not "just graze" the planet. At the distance of closest approach, **r** and **v** are perpendicular. If $v(R)$ is the speed at this point,

$$L = -mRv(R)$$

$$E = \frac{1}{2} mv(R)^2 - \frac{mMG}{R}.$$

Since L and E are conserved, their values at $r = R$ must be the same as their values at $r = \infty$. Hence

$$-mb'v_0 = -mRv(R) \qquad 1$$

$$\frac{1}{2} mv_0^2 = \frac{1}{2} mv(R)^2 - \frac{mMG}{R}. \qquad 2$$

Equation (1) gives $v(R) = v_0b'/R$, and by substituting this in Eq. (2) we obtain

$$(b')^2 = R^2\left(1 + \frac{mMG/R}{mv_0^2/2}\right).$$

The effective area is

$$A_e = \pi(b')^2$$

$$= \pi R^2\left(1 + \frac{mMG/R}{mv_0^2/2}\right).$$

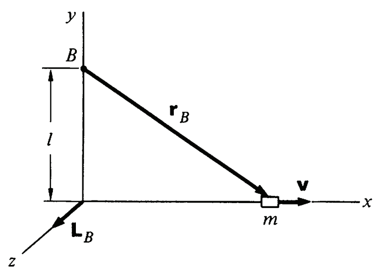
As we expect, the effective area is greater than the geometrical area. Since $mMG/R = -U(R)$, and $mv_0^2/2 = E$, we have

$$A_e = A_g\left(1 - \frac{U(R)}{E}\right).$$

If we "turn off" gravity, $U(R) \to 0$ and $A_e \to A_g$, as we require. Furthermore, as $E \to 0$, $A_e \to \infty$, which means that it is impossible to miss the planet, provided that you start from rest. For $E = 0$, the spacecraft inevitably falls into the planet.

If there is a torque on a system the angular momentum must change according to $\boldsymbol{\tau} = d\mathbf{L}/dt$, as the following examples illustrate.

Example 6.5 Torque on a Sliding Block

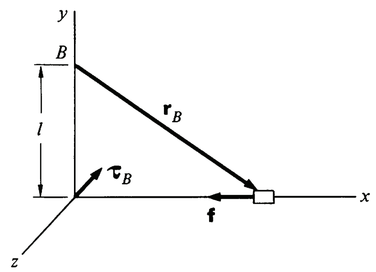


For a simple illustration of the relation $\boldsymbol{\tau} = d\mathbf{L}/dt$, consider a small block of mass m sliding in the x direction with velocity $\mathbf{v} = v\hat{\mathbf{i}}$. The angular momentum of the block about origin B is

$$\mathbf{L}_B = m\mathbf{r}_B \times \mathbf{v} \quad 1$$
$$= mlv\hat{\mathbf{k}},$$

as we discussed in Example 6.1. If the block is sliding freely, $\mathbf{v}$ does not change, and $\mathbf{L}_B$ is therefore constant, as we expect, since there is no torque acting on the block.

Suppose now that the block slows down because of a friction force $\mathbf{f} = -f\hat{\mathbf{i}}$. The torque on the block about origin B is

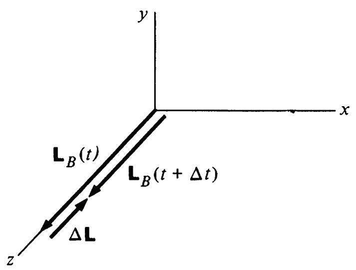


$$\boldsymbol{\tau}_B = \mathbf{r}_B \times \mathbf{f}$$
$$= -lf\hat{\mathbf{k}}. \quad 2$$

We see from Eq. (1) that as the block slows, $\mathbf{L}_B$ remains along the positive z direction but its magnitude decreases. Therefore, the change $\Delta\mathbf{L}_B$ in $\mathbf{L}_B$ points in the negative z direction, as shown in the lower sketch. The direction of $\Delta\mathbf{L}_B$ is the same as the direction of $\boldsymbol{\tau}_B$. Since $\boldsymbol{\tau} = d\mathbf{L}/dt$ in general, the vectors $\boldsymbol{\tau}$ and $\Delta\mathbf{L}$ are always parallel.

From Eq. (1),

$$\Delta\mathbf{L}_B = ml\,\Delta v\,\hat{\mathbf{k}}, \quad 3$$



where $\Delta v < 0$. Dividing Eq. (3) by Δt and taking the limit $\Delta t \to 0$, we have

$$\frac{d\mathbf{L}_B}{dt} = ml\frac{dv}{dt}\hat{\mathbf{k}}. \quad 4$$

By Newton's second law, $m\,dv/dt = -f$ and Eq. (4) becomes

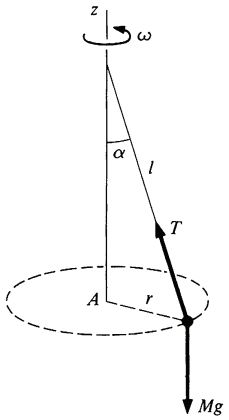
$$\frac{d\mathbf{L}_B}{dt} = -lf\hat{\mathbf{k}}$$
$$= \boldsymbol{\tau}_B,$$

as we expect.

It is important to keep in mind that since τ and $\mathbf{L}$ depend on the choice of origin, the same origin must be used for both when applying the relation $\tau = d\mathbf{L}/dt$, as we were careful to do in this problem.

The angular momentum of the block in this example changed only in magnitude and not in direction, since τ and $\mathbf{L}$ happened to be along the same line. In the next example we return to the conical pendulum to study a case in which the angular momentum is constant in magnitude but changes direction due to an applied torque.

Example 6.6 Torque on the Conical Pendulum



In Example 6.2, we calculated the angular momentum of a conical pendulum about two different origins. Now we shall complete the analysis by showing that the relation $\tau = d\mathbf{L}/dt$ is satisfied.

The sketch illustrates the forces on the bob. T is the tension in the string. For uniform circular motion there is no vertical acceleration, and consequently

$$T\cos\alpha - Mg = 0. \qquad 1$$

The total force $\mathbf{F}$ on the bob is radially inward: $\mathbf{F} = -T\sin\alpha\hat{\mathbf{r}}$. The torque on M about A is

$$\begin{aligned}\tau_A &= \mathbf{r}_A \times \mathbf{F}\\ &= 0,\end{aligned}$$

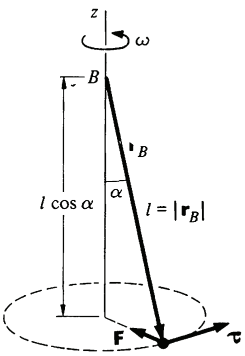
since $\mathbf{r}_A$ and $\mathbf{F}$ are both in the $\hat{\mathbf{r}}$ direction. Hence

$$\frac{d\mathbf{L}_A}{dt} = 0$$

and we have the result

$$\mathbf{L}_A = \text{constant}$$

as we already know from Example 6.2.



The problem looks entirely different if we take the origin at B. The torque τ_B is

$$\tau_B = \mathbf{r}_B \times \mathbf{F}.$$

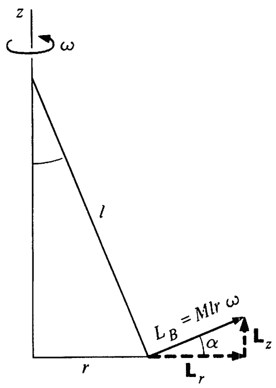
Hence

$$\begin{aligned}|\tau_B| &= l\cos\alpha F = l\cos\alpha\, T\sin\alpha\\ &= Mgl\sin\alpha,\end{aligned}$$

where we have used Eq. (1), $T\cos\alpha = Mg$. The direction of τ_B is tangential to the line of motion of M:

$$\tau_B = Mgl\sin\alpha\hat{\boldsymbol{\theta}}, \qquad 2$$

where $\hat{\boldsymbol{\theta}}$ is the unit tangential vector in the plane of motion.



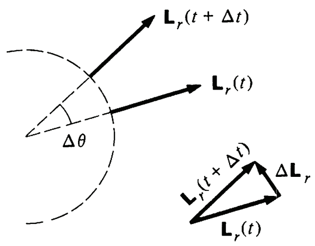
Our problem is to show that the relation

$$\boldsymbol{\tau}_B = \frac{d\mathbf{L}_B}{dt} \qquad 3$$

is satisfied. From Example 6.2, we know that $\mathbf{L}_B$ has constant magnitude $Mlr\omega$. As the diagram at left shows, $\mathbf{L}_B$ has a vertical component $L_z = Mlr\omega \sin\alpha$ and a horizontal radial component $L_r = Mlr\omega \cos\alpha$. Writing $\mathbf{L}_B = \mathbf{L}_z + \mathbf{L}_r$, we see that $\mathbf{L}_z$ is constant, as we expect, since $\boldsymbol{\tau}_B$ has no vertical component. $\mathbf{L}_r$ is not constant; it changes direction as the bob swings around. However, the magnitude of $\mathbf{L}_r$ is constant. We encountered such a situation in Sec. 1.8, where we showed that the only way a vector $\mathbf{A}$ of constant magnitude can change in time is to rotate, and that if its instantaneous rate of rotation is $d\theta/dt$, then $|d\mathbf{A}/dt| = A\, d\theta/dt$. We can employ this relation directly to obtain

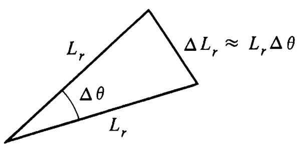
$$\left|\frac{d\mathbf{L}_r}{dt}\right| = L_r\omega.$$

However, since we shall invoke this result frequently, let us take a moment to rederive it geometrically.



The vector diagrams show $\mathbf{L}_r$ at some time t and at $t + \Delta t$. During the interval Δt, the bob swings through angle $\Delta\theta = \omega\, \Delta t$, and $\mathbf{L}_r$ rotates through the same angle. The magnitude of the vector difference $\Delta\mathbf{L}_r = \mathbf{L}_r(t + \Delta t) - \mathbf{L}_r(t)$ is given approximately by

$$|\Delta\mathbf{L}_r| \approx L_r\, \Delta\theta.$$



In the limit $\Delta t \to 0$, we have

$$\frac{dL_r}{dt} = L_r \frac{d\theta}{dt} = L_r\omega.$$

Since $L_r = Mlr\omega \cos\alpha$, we have

$$\frac{dL_r}{dt} = Mlr\omega^2 \cos\alpha.$$

$Mr\omega^2$ is the radial force, $T \sin\alpha$, and since $T\cos\alpha = Mg$, we have

$$\frac{dL_r}{dt} = Mgl \sin\alpha,$$

which agrees with the magnitude of $\boldsymbol{\tau}_B$ from Eq. (2). Furthermore, as the vector drawings indicate, $d\mathbf{L}_r/dt$ lies in the tangential direction, parallel to $\boldsymbol{\tau}_B$, as we expect.

Another way to calculate $d\mathbf{L}_B/dt$ is to write $\mathbf{L}_B$ in vector form and then differentiate:

$$\mathbf{L}_B = (Mlr\omega \sin \alpha)\hat{\mathbf{k}} + (Mlr\omega \cos \alpha)\hat{\mathbf{r}}.$$

$$\frac{d\mathbf{L}_B}{dt} = Mlr\omega \cos \alpha \frac{d\hat{\mathbf{r}}}{dt}$$

$$= Mlr\omega^2 \cos \alpha\hat{\boldsymbol{\theta}},$$

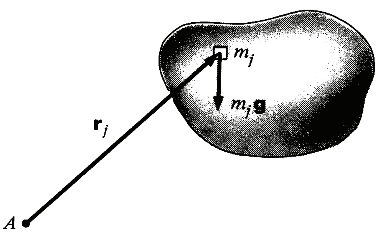
where we have used $d\hat{\mathbf{r}}/dt = \omega\hat{\boldsymbol{\theta}}$.

It is important to be able to visualize angular momentum as a vector which can rotate in space. This type of reasoning occurs often in analyzing the motion of rigid bodies; we shall find it particularly helpful in understanding gyroscope motion in Chap. 7.

Example 6.7 Torque due to Gravity

We often encounter systems in which there is a torque exerted by gravity. Examples include a pendulum, a child's top, and a falling chimney. In the usual case of a uniform gravitational field, the torque on a body about any point is $\mathbf{R} \times \mathbf{W}$, where $\mathbf{R}$ is a vector from the point to the center of mass and $\mathbf{W}$ is the weight. Here is the proof.

The problem is to find the torque on a body of mass M about origin A when the applied force is due to a uniform gravitational field $\mathbf{g}$. We can regard the body as a collection of particles. The torque $\boldsymbol{\tau}_j$ on the jth particle is



$$\boldsymbol{\tau}_j = \mathbf{r}_j \times m_j\mathbf{g},$$

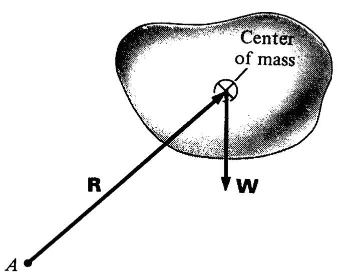
where $\mathbf{r}_j$ is the position vector of the jth particle from origin A, and m_j is its mass.

The total torque is

$$\boldsymbol{\tau} = \Sigma\boldsymbol{\tau}_j$$

$$= \Sigma\mathbf{r}_j \times m_j\mathbf{g}$$

$$= (\Sigma m_j\mathbf{r}_j) \times \mathbf{g}.$$



By definition of center of mass,

$$\Sigma m_j\mathbf{r}_j = M\mathbf{R},$$

where $\mathbf{R}$ is the position vector of the center of mass. Hence

$$\boldsymbol{\tau} = M\mathbf{R} \times \mathbf{g}$$

$$= \mathbf{R} \times M\mathbf{g}$$

$$= \mathbf{R} \times \mathbf{W}.$$

A corollary to this result is that in order to balance an object, the pivot point must be at the center of mass.

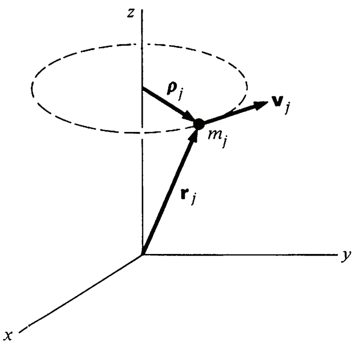
6.4 Angular Momentum and Fixed Axis Rotation

The most prominent application of angular momentum in classical mechanics is to the analysis of the motion of rigid bodies. The general case of rigid body motion involves free rotation about any axis—for instance, the motion of a baseball bat flung spinning and tumbling into the air. Analysis of the general case involves a number of mathematical complexities which we are going to postpone for a chapter, and in this chapter we restrict ourselves to a special, but important, case—rotation about a fixed axis. By fixed axis we mean that the *direction* of the axis of rotation is always along the same line; the axis itself may translate. For example, a car wheel attached to an axle undergoes fixed axis rotation as long as the car drives straight ahead. If the car turns, the wheel must rotate about a vertical axis while simultaneously spinning on the axle; the motion is no longer fixed axis rotation. If the wheel flies off the axle and wobbles down the road, the motion is definitely not rotation about a fixed axis.

We can choose the axis of rotation to be in the z direction, without loss of generality. The rotating object can be a wheel or a baseball bat, or anything we choose, the only restriction being that it is rigid—which is to say that its shape does not change as it rotates.

When a rigid body rotates about an axis, every particle in the body remains at a fixed distance from the axis. If we choose a coordinate system with its origin lying on the axis, then for each particle in the body, $|\mathbf{r}|$ = constant. The only way that $\mathbf{r}$ can change while $|\mathbf{r}|$ remains constant is for the velocity to be perpendicular to $\mathbf{r}$. Hence, for a body rotating about the z axis,

$$|\mathbf{v}_j| = |\dot{\mathbf{r}}_j| \qquad 6.4$$
$$= \omega\rho_j,$$



where ρ_j is the perpendicular distance from the axis of rotation to particle m_j of the rigid body and $\boldsymbol{\rho}_j$ is the corresponding vector. ω is the rate of rotation, or angular velocity. Since the axis of rotation lies in the z direction, we have $\rho_j = (x_j^2 + y_j^2)^{\frac{1}{2}}$. [In this chapter and the next we shall use the symbol ρ to denote perpendicular distance to the axis of rotation. Note that r stands for the distance to the origin: $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.]

The angular momentum of the jth particle of the body, $\mathbf{L}(j)$, is.

$$\mathbf{L}(j) = \mathbf{r}_j \times m_j\mathbf{v}_j.$$

In this chapter we are concerned only with L_z, the component of angular momentum along the axis of rotation. Since $\mathbf{v}_j$ lies in the xy plane,

$$L_z(j) = m_j v_j \times \text{(distance to } z \text{ axis)} = m_j v_j \rho_j.$$

Using Eq. (6.4), $v_j = \omega\rho_j$, we have

$$L_z(j) = m_j \rho_j^2 \omega.$$

The z component of the total angular momentum of the body L_z is the sum of the individual z components:

$$\begin{aligned} L_z &= \sum_j L_z(j) \\ &= \Sigma m_j \rho_j^2 \omega, \end{aligned} \qquad 6.5$$

where the sum is over all particles of the body. We have taken ω to be constant throughout the body; can you see why this must be so?

Equation (6.5) can be written as

$$L_z = I\omega, \qquad 6.6$$

where

$$I \equiv \sum_j m_j \rho_j^2. \qquad 6.7$$

I is a geometrical quantity called the *moment of inertia*. I depends on both the distribution of mass in the body and the location of the axis of rotation. (We shall give a more general definition for I in the next chapter when we talk about unrestricted rigid body motion.) For continuously distributed matter we can replace the sum over mass particles by an integral over differential mass elements. In this case

$$\sum_j m_j \rho_j^2 \rightarrow \int \rho^2 \, dm,$$

and

$$\begin{aligned} I &= \int \rho^2 \, dm \\ &= \int (x^2 + y^2) \, dm. \end{aligned}$$

To evaluate such an integral we generally replace the mass element dm by the product of the density (mass per unit volume) w at the position of dm and the volume dV occupied by dm:

$$dm = w \, dV.$$

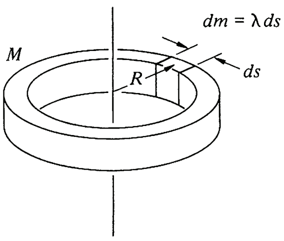
(Often ρ is used to denote density, but that would cause confusion here.) We can write

$$I = \int \rho^2 dm$$
$$= \int (x^2 + y^2) w \, dV.$$

For simple shapes with a high degree of symmetry, calculation of the moment of inertia is straightforward, as the following examples show.

Example 6.8 **Moments of Inertia of Some Simple Objects**

a. UNIFORM THIN HOOP OF MASS M AND RADIUS R, AXIS THROUGH THE CENTER AND PERPENDICULAR TO THE PLANE OF THE HOOP



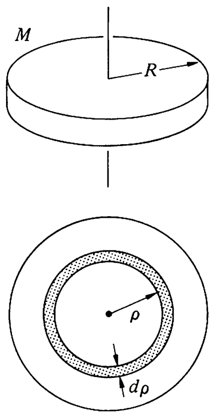
The moment of inertia about the axis is given by

$$I = \int \rho^2 \, dm.$$

Since the hoop is thin, $dm = \lambda ds$, where $\lambda = M/2\pi R$ is the mass per unit length of the hoop. All points on the hoop are distance R from the axis so that $\rho = R$, and we have

$$I = \int_0^{2\pi R} R^2 \lambda \, ds$$
$$= R^2 \left(\frac{M}{2\pi R}\right) s \Big|_0^{2\pi R}$$
$$= MR^2.$$

b. UNIFORM DISK OF MASS M, RADIUS R, AXIS THROUGH THE CENTER AND PERPENDICULAR TO THE PLANE OF THE DISK



We can subdivide the disk into a series of thin hoops with radius ρ width $d\rho$, and moment of inertia dI. Then $I = \int dI$.

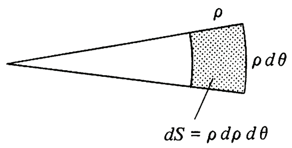
The area of one of the thin hoops is $dA = 2\pi\rho \, d\rho$, and its mass is

$$dm = M \frac{dA}{A} = \frac{M 2\pi\rho \, d\rho}{\pi R^2}$$
$$= \frac{2M\rho \, d\rho}{R^2}.$$
$$dI = \rho^2 \, dm = \frac{2M\rho^3 \, d\rho}{R^2}$$
$$I = \int_0^R \frac{2M\rho^3 \, d\rho}{R^2}$$
$$= \frac{1}{2} MR^2.$$

Let us also solve this problem by double integration to illustrate the most general approach.

$$I = \int \rho^2\, dm$$
$$= \int \rho^2 \sigma\, dS,$$

where σ is the mass per unit area. For the uniform disk, $\sigma = M/\pi R^2$, Polar coordinates are the obvious choice. In plane polar coordinates,



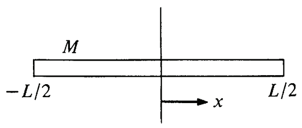
$$dS = \rho\, d\rho\, d\theta.$$

Then

$$I = \int \rho^2\, \sigma dS$$
$$= \left(\frac{M}{\pi R^2}\right) \int \rho^2\, dS$$
$$= \left(\frac{M}{\pi R^2}\right) \int_0^R \int_0^{2\pi} \rho^2 \rho\, d\rho\, d\theta$$
$$= \left(\frac{2M}{R^2}\right) \int_0^R \rho^3\, d\rho$$
$$= \tfrac{1}{2} MR^2,$$

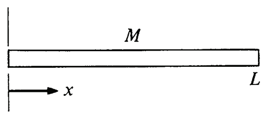
as before.

c. UNIFORM THIN STICK OF MASS M, LENGTH L, AXIS THROUGH THE MIDPOINT AND PERPENDICULAR TO THE STICK



$$I = \int_{-L/2}^{+L/2} x^2\, dm$$
$$= \frac{M}{L} \int_{-L/2}^{+L/2} x^2\, dx$$
$$= \frac{M}{L} \frac{1}{3} x^3 \Big|_{-L/2}^{+L/2}$$
$$= \tfrac{1}{12} ML^2$$

d. UNIFORM THIN STICK, AXIS AT ONE END AND PERPENDICULAR TO THE STICK



$$I = \frac{M}{L} \int_0^L x^2\, dx$$
$$= \tfrac{1}{3} ML^2.$$

e. UNIFORM SPHERE OF MASS M, RADIUS R, AXIS THROUGH CENTER

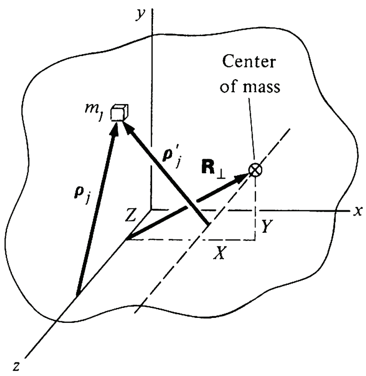
We quote this result without proof—perhaps you can derive it for yourself.

$$I = \tfrac{2}{5} MR^2.$$

Example 6.9 The Parallel Axis Theorem

This handy theorem tells us I, the moment of inertia about any axis, provided that we know I_0, the moment of inertia about a parallel axis through the center of mass. If the mass of the body is M and the distance between the axes is l, the theorem states that

$$I = I_0 + Ml^2.$$



To prove this, consider the moment of inertia of the body about an axis which we choose to have lie in the z direction. The vector from the z axis to particle j is

$$\boldsymbol{\rho}_j = x_j\hat{\mathbf{i}} + y_j\hat{\mathbf{k}},$$

and

$$I = \Sigma m_j \rho_j^2.$$

If the center of mass is at $\mathbf{R} = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}} + Z\hat{\mathbf{k}}$, the vector perpendicular from the z axis to the center of mass is

$$\mathbf{R}_\perp = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}.$$

If the vector from the axis through the center of mass to particle j is $\boldsymbol{\rho}'_j$, then the moment of inertia about the center of mass is

$$I_0 = \Sigma m_j \rho_j'^2.$$

From the diagram we see that

$$\boldsymbol{\rho}_j = \boldsymbol{\rho}'_j + \mathbf{R}_\perp,$$

so that

$$\begin{aligned} I &= \Sigma m_j \rho_j^2 \\ &= \Sigma m_j(\boldsymbol{\rho}'_j + \mathbf{R}_\perp)^2 \\ &= \Sigma m_j(\rho_j'^2 + 2\boldsymbol{\rho}'_j \cdot \mathbf{R}_\perp + R_\perp^2). \end{aligned}$$

The middle term vanishes, since

$$\begin{aligned} \Sigma m_j \boldsymbol{\rho}'_j &= \Sigma m_j(\boldsymbol{\rho}_j - \mathbf{R}_\perp) = M(\mathbf{R}_\perp - \mathbf{R}_\perp) \\ &= 0. \end{aligned}$$

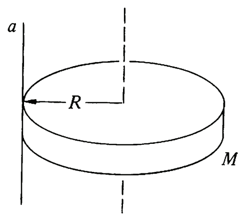
If we designate the magnitude of $\mathbf{R}_\perp$ by l, then

$$I = I_0 + Ml^2.$$

For example, in Example 6.8*c* we showed that the moment of inertia of a stick about its midpoint is $ML^2/12$. The moment of inertia about its end, which is $L/2$ away from the center of mass, is therefore

$$\begin{aligned} I_a &= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 \\ &= \frac{ML^2}{3}, \end{aligned}$$

which is the result we found in Example 6.8*d*.



Similarly, the moment of inertia of a disk about an axis at the rim, perpendicular to the plane of the disk, is

$$I_a = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}.$$

6.5 Dynamics of Pure Rotation about an Axis

In Chap. 3 we showed that the motion of a system of particles is simple to describe if we distinguish between external forces and internal forces acting on the particles. The internal forces cancel by Newton's third law, and the momentum changes only because of external forces. This leads to the law of conservation of momentum: the momentum of an isolated system is constant. In describing rotational motion we are tempted to follow the same procedure and to distinguish between external and internal *torques*. Unfortunately, there is no way to prove from Newton's laws that the internal torques add to zero. However, it is an experimental fact that they always do cancel, since the angular momentum of an isolated system has never been observed to change. We shall discuss this more fully in Sec. 7.5 and for the remainder of this chapter simply assume that only external torques change the angular momentum of a rigid body.

In this section we consider fixed axis rotation with no translation of the axis, as, for instance, the motion of a door on its hinges or the spinning of a fan blade. Motion like this, where there is an axis of rotation at rest, is called *pure rotation*. Pure rotation is important because it is simple and because it is frequently encountered.

Consider a body rotating with angular velocity ω about the z axis. From Eq. (6.6) the z component of angular momentum is

$$L_z = I\omega.$$

Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, where $\boldsymbol{\tau}$ is the external torque, we have

$$\begin{aligned}\tau_z &= \frac{d}{dt}(I\omega)\\ &= I\frac{d\omega}{dt}\\ &= I\alpha,\end{aligned}$$

where $\alpha = d\omega/dt$ is called the *angular acceleration*. In this chapter we are concerned with rotation only about the z axis, so we drop the subscript z and write

$$\tau = I\alpha. \qquad 6.8$$

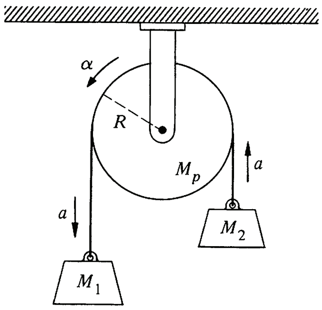
Equation (6.8) is reminiscent of $\mathbf{F} = m\mathbf{a}$, and in fact there is a close analogy between linear and rotational motion. We can develop this further by evaluating the kinetic energy of a body undergoing pure rotation:

$$\begin{aligned} K &= \Sigma \tfrac{1}{2} m_j v_j^2 \\ &= \Sigma \tfrac{1}{2} m_j \rho_j^2 \omega^2 \\ &= \tfrac{1}{2} I \omega^2, \end{aligned}$$

where we have used $v_j = \rho_j \omega$ and $I = \Sigma m_j \rho_j^2$.

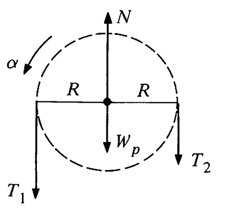
The method of handling problems involving rotation under applied torques is a straightforward extension of the familiar procedure for treating translational motion under applied forces, as the following example illustrates.

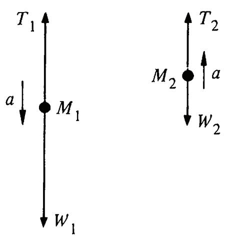
Example 6.10 **Atwood's Machine with a Massive Pulley**



The problem is to find the acceleration a for the arrangement shown in the sketch. The effect of the pulley is to be included.

Force diagrams for the three masses are shown below left. The points of application of the forces on the pulley are shown; this is necessary whenever we need to calculate torques. The pulley evidently undergoes pure rotation about its axle, so we take the axis of rotation to be the axle.





The equations of motion are

$$\left.\begin{aligned} W_1 - T_1 &= M_1 a \\ T_2 - W_2 &= M_2 a \end{aligned}\right\} \text{Masses}$$

$$\left.\begin{aligned} \tau = T_1 R - T_2 R &= I\alpha \\ N - T_1 - T_2 - W_p &= 0 \end{aligned}\right\} \text{Pulley}$$

Note that in the torque equation, α must be positive counterclockwise to correspond to our convention that torque out of the paper is positive. N is the force on the axle, and the last equation simply assures that the pulley does not fall. Since we don't need to know N, it does not contribute to the solution.

There is a constraint relating a and α, assuming that the rope does not slip. The velocity of the rope is the velocity of a point on the surface of the wheel, $v = \omega R$, from which it follows that

$$a = \alpha R.$$

We can now eliminate T_1, T_2, and α;

$$W_1 - W_2 - (T_1 - T_2) = (M_1 + M_2)a$$

$$T_1 - T_2 = \frac{I\alpha}{R} = \frac{Ia}{R^2}$$

$$W_1 - W_2 - \frac{Ia}{R^2} = (M_1 + M_2)a.$$

If the pulley is a simple disk, we have

$$I = \frac{M_pR^2}{2}$$

and it follows that

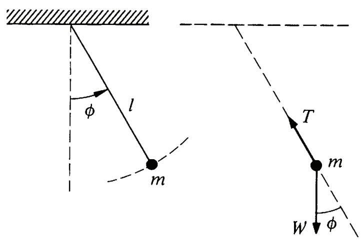
$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + M_p/2}.$$

The pulley increases the total inertial mass of the system, but in comparison with the hanging weights, the effective mass of the pulley is only one-half its real mass.

6.6 The Physical Pendulum

A mass hanging from a string is a *simple pendulum* if we assume that the mass has negligible size and the mass of the string is zero. We shall review its behavior as an introduction to the more realistic object, the *physical pendulum*, for which we do not need to make these assumptions.

The Simple Pendulum



At the left is a sketch of a simple pendulum and the force diagram. The tangential force is $-W \sin \phi$, and we obtain

$$ml\ddot{\phi} = -W \sin \phi.$$

(Incidentally, we get the same result by considering pure rotation about the point of suspension: $I = ml^2$, $\alpha = \ddot{\phi}$, and $\tau = -Wl \sin \phi$, so $ml^2\ddot{\phi} = -Wl \sin \phi$.) We can rewrite the equation of motion as

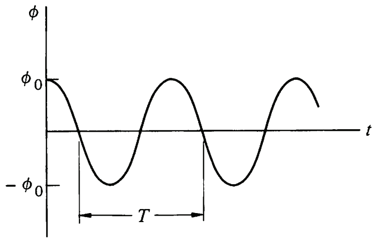
$$l\ddot{\phi} + g \sin \phi = 0.$$

This equation cannot be solved in terms of familiar functions. However, if the pendulum never swings far from the vertical, then $\phi \ll 1$, and we can use the approximation $\sin \phi \approx \phi$. Then

$$l\ddot{\phi} + g\phi = 0.$$

This is the equation for simple harmonic motion. (See Example 2.14.) The solution is $\phi = A \sin \omega t + B \cos \omega t$, where $\omega = \sqrt{g/l}$ and A and B are constants. If the pendulum starts from rest at angle ϕ_0, the solution is

$$\phi = \phi_0 \cos \omega t.$$



The motion is *periodic*, which means it occurs identically over and over again. The *period* T, the time between successive repetitions of the motion, is given by $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\sqrt{g/l}}$$

$$= 2\pi\sqrt{\frac{l}{g}}.$$

The maximum angle ϕ_0 is called the *amplitude* of the motion. The period is independent of the amplitude, which is why the pendulum is so well suited to regulating the rate of a clock. However, this feature of the motion is a consequence of the approximation $\sin\phi \approx \phi$. The exact solution, which is developed in Note 6.2 at the end of the chapter, shows that the period lengthens slightly with increasing amplitude. The following example illustrates the consequence of this.

Example 6.11 Grandfather's Clock

As shown in Eq. (7) of Note 6.2, for small amplitudes the period of a pendulum is given by

$$T = T_0(1 + \tfrac{1}{16}\phi_0^2 + \cdots). \qquad 1$$

where

$$T_0 = 2\pi\sqrt{\frac{l}{g}}.$$

For $\phi_0 \approx 0$ we have our previous result, $T = 2\pi\sqrt{l/g}$. The correction term, $\frac{1}{16}\phi_0^2$ is surprisingly small: Consider a grandfather's clock with $T_0 = 2$ s and $l \approx 1$ m. If the pendulum swings 4 cm to either side, then $\phi_0 = 4 \times 10^{-2}$ rad and the correction term is $\phi_0^2/16 = 10^{-4}$. This by itself is of no consequence, since the length of the pendulum can be adjusted to make the clock run at any desired rate. However, the amplitude may vary slightly due to friction and other effects. Suppose that the amplitude changes by an amount $d\phi$. Taking differentials of Eq. (1) gives

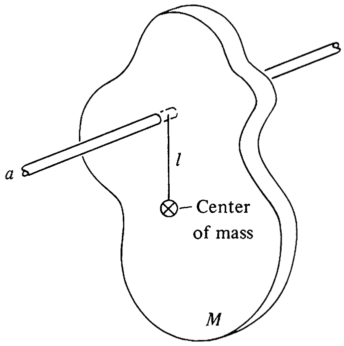
$$dT = \tfrac{1}{8}T_0\phi_0\,d\phi.$$

The fractional change in T is

$$\frac{dT}{T_0} = \frac{1}{8}\phi_0\,d\phi.$$

If the amplitude changes by 10 percent, then $d\phi = 0.1\phi_0 = 4 \times 10^{-3}$ rad, and $dT/T_0 = 2 \times 10^{-5}$, giving an error of about 2 seconds per day.

The Physical Pendulum

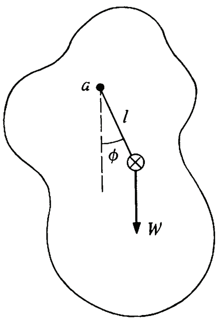


Now let us turn to the physical pendulum such as the one in the sketch. The swinging object can have any shape. Its mass is M, and its center of mass is at distance l from the pivot. One other quantity we need is the moment of inertia about the pivot, I_a. The motion is pure rotation about the pivot. Choosing the axis of rotation through the pivot, we find that the only torque is that due to gravity, and we have

$$-lW \sin \phi = I_a\ddot{\phi}.$$

Making the small angle approximation,

$$I_a\ddot{\phi} + Mlg\phi = 0.$$



This is again the equation of simple harmonic motion with the solution

$$\phi = A \cos \omega t + B \sin \omega t,$$

where $\omega = \sqrt{Mlg/I_a}$.

We can write this result in a simpler form if we introduce the *radius of gyration*. If the moment of inertia of an object about its center of mass is I_0, the radius of gyration k is defined as

$$k = \sqrt{\frac{I_0}{M}} \qquad \text{or} \qquad I_0 = Mk^2.$$

For instance, for a hoop of radius R, $k = R$; for a disk, $k = \sqrt{\frac{1}{2}}\,R$; and for a solid sphere, $k = \sqrt{\frac{2}{5}}\,R$.

By the parallel axis theorem we have

$$\begin{aligned} I_a &= I_0 + Ml^2 \\ &= M(k^2 + l^2), \end{aligned}$$

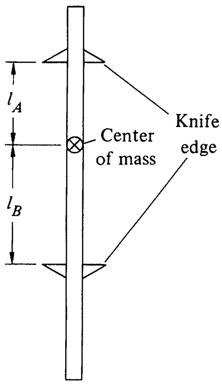
so that

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}.$$

The simple pendulum corresponds to $k = 0$, and in this case we obtain $\omega = \sqrt{g/l}$, as before.

Example 6.12 Kater's Pendulum

Between the sixteenth and twentieth centuries, the most accurate measurements of g were obtained from experiments with pendulums. The method is attractive because the only quantities needed are the period of the pendulum, which can be determined to great accuracy by counting many swings, and the pendulum's dimensions. For very precise measurements, the limiting feature turns out to be the precision with which the center of mass of the pendulum and its radius of gyration can be determined. A clever invention, named after the nineteenth century English physicist, surveyor, and inventor Henry Kater, overcomes this difficulty.



Kater's pendulum has two knife edges; the pendulum can be suspended from either. If the knife edges are distances l_A and l_B from the center of mass, then the period for small oscillations from each of these is, respectively,

$$T_A = 2\pi\left(\frac{k^2 + l_A{}^2}{gl_A}\right)^{\frac{1}{2}}$$

$$T_B = 2\pi\left(\frac{k^2 + l_B{}^2}{gl_B}\right)^{\frac{1}{2}}.$$

l_A or l_B is adjusted until the periods are identical: $T_A = T_B = T$. We can then eliminate T and solve for k^2:

$$k^2 = \frac{l_A l_B{}^2 - l_B l_A{}^2}{l_B - l_A}$$

$$= l_A l_B.$$

Then

$$T = 2\pi\left(\frac{l_A l_B + l_A{}^2}{gl_A}\right)^{\frac{1}{2}}$$

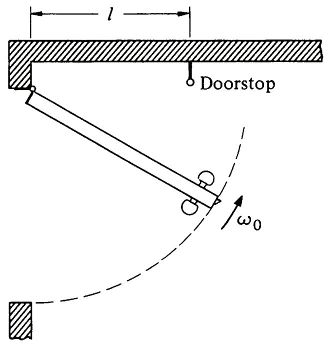
$$= 2\pi\left(\frac{l_A + l_B}{g}\right)^{\frac{1}{2}}$$

or

$$g = 4\pi^2\left(\frac{l_A + l_B}{T^2}\right)$$

The beauty of Kater's invention is that the only geometrical quantity needed is $l_A + l_B$, the distance between the knife edges, which can be measured to great accuracy. The position of the center of mass need not be known.

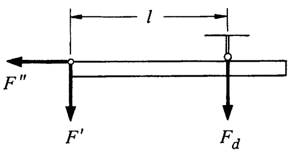
Example 6.13 **The Doorstop**



The banging of a door against its stop can tear loose the hinges. However, by the proper choice of l, the impact forces on the hinge can be made to vanish.

The forces on the door during impact are F_d, due to the stop, and F' and F'' due to the hinge. F'' is the small radial force which provides the centripetal acceleration of the swinging door. F' and F_d are the large impact forces which bring the door to rest when it bangs against the stop. The force on the hinges is equal and opposite to F' and F'' To minimize the stress on the hinges, we must make F' as small as possible.

To derive an expression for F', we shall consider in turn the angular momentum of the door about the hinges and the linear momentum of the center of mass.



Since $dL = \tau dt$, we have

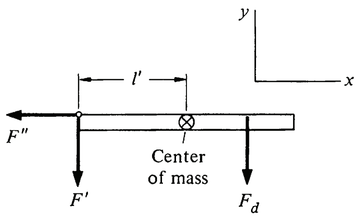
$$L_{\text{final}} - L_{\text{initial}} = \int_{t_i}^{t_f} \tau\, dt.$$

The initial angular momentum of the door is $I\omega_0$, where I is the moment of inertia about the hinges. Since the door comes to rest, $L_{\text{final}} = 0$. The torque on the door during the collision is $\tau = -lF_d$, and we obtain

$$I\omega_0 = l \int F_d\, dt, \qquad 1$$

where the integral is over the duration of the collision.

The center of mass motion obeys



$$\mathbf{P}_{\text{final}} - \mathbf{P}_{\text{initial}} = \int \mathbf{F}\, dt,$$

where $\mathbf{F}$ is the total force. The momentum in the y direction immediately before the collision is $MV_y = Ml'\omega_0$, where l' is the distance from the hinge to the center of mass of the door. $P_{\text{final}} = 0$, and the y component of $\mathbf{F}$ is $F_y = -(F' + F_d)$. Hence,

$$Ml'\omega_0 = \int (F' + F_d)\, dt. \qquad 2$$

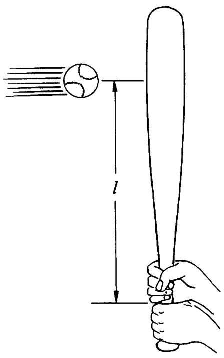
According to Eq. (1), $\int F_d\, dt = I\omega_0/l$, and substituting this in Eq. (2) gives

$$\int F'\, dt = \left(Ml' - \frac{I}{l}\right)\omega_0.$$

By choosing

$$l = \frac{I}{Ml'}, \qquad 3$$

the impact force is made zero. If the door is uniform, and of width w, then $I = Mw^2/3$ and $l' = w/2$. In this case $l = \frac{2}{3}w$.



Incidentally, the stop must be at the height of the center of mass rather than at floor level. Otherwise the impact forces will not be identic l on the two hinges and the door will tend to rotate about a horizontal axis, an effect we have not taken into account.

The distance l specified by Eq. (3) is called the *center of percussion.* In batting a baseball it is important to hit the ball at the bat's center of percussion to avoid a reaction on the batter's hands and a painful sting.

6.7 Motion Involving Both Translation and Rotation

Often translation and rotation occur simultaneously, as in the case of a rolling drum. There is no obvious axis as there was in Sec. 6.5 when we analyzed pure rotation, and the problem seems confusing until we recall the theorem in Sec. 6.1—that one possible way to describe a general motion is by a translation of the center of mass plus a rotation about the center of mass. By using center of mass coordinates we will find it a straightforward matter to obtain simple expressions for both the angular momentum and the torque and to find the dynamical equation connecting them.

As before, we shall consider only motion for which the axis of rotation remains parallel to the z axis. We shall show that L_z, the z component of the angular momentum of the body, can be written as the sum of two terms. L_z is the angular momentum $I_0\omega$ due to rotation of the body about its center of mass, plus the angular momentum $(\mathbf{R} \times M\mathbf{V})_z$ due to motion of the center of mass with respect to the origin of the inertial coordinate system:

$$L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z,$$

where $\mathbf{R}$ is the position vector of the center of mass and $\mathbf{V} = \dot{\mathbf{R}}$.

To find the angular momentum, we start by considering the body to be an aggregation of N particles with masses $m_j (j = 1, \ldots, N)$ and position vectors $\mathbf{r}_j$ with respect to an inertial coordinate system. The angular momentum of the body can be written

$$\mathbf{L} = \sum_{j=1}^{N} (\mathbf{r}_j \times m_j\dot{\mathbf{r}}_j). \qquad 6.9$$

The center of mass of the body has position vector $\mathbf{R}$:

$$\mathbf{R} = \frac{\Sigma m_j\mathbf{r}_j}{M}, \qquad 6.10$$

where M is the total mass. The center of mass coordinates $\mathbf{r}'_j$ can be introduced as we did in Sec. 3.3:

$$\mathbf{r}_j = \mathbf{R} + \mathbf{r}'_j.$$

Eliminating $\mathbf{r}_j$ from Eq. (6.9) gives

$$\begin{aligned}\mathbf{L} &= \Sigma(\mathbf{r}_j \times m_j\dot{\mathbf{r}}_j)\\ &= \Sigma(\mathbf{R} + \mathbf{r}'_j) \times m_j(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_j)\\ &= \mathbf{R} \times \Sigma m_j\dot{\mathbf{R}} + \Sigma m_j\mathbf{r}'_j \times \dot{\mathbf{R}} + \mathbf{R} \times \Sigma m_j\dot{\mathbf{r}}'_j + \Sigma m_j\mathbf{r}'_j \times \dot{\mathbf{r}}'_j.\end{aligned}$$

This expression looks cumbersome, but we can show that the middle two terms are identically zero and that the first and last terms have simple physical interpretations. Starting with the second term, we have

$$\begin{aligned}\Sigma m_j\mathbf{r}'_j &= \Sigma m_j(\mathbf{r}_j - \mathbf{R})\\ &= \Sigma m_j\mathbf{r}_j - M\mathbf{R}\\ &= 0.\end{aligned}$$

by Eq. (6.10). The third term is also zero; since $\Sigma m_j\mathbf{r}'_j$ is identically zero, its time derivative $\Sigma m_j\dot{\mathbf{r}}'_j = 0$ as well.

The first term is

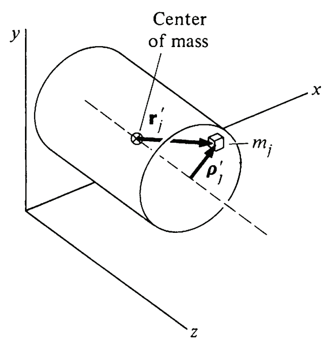
$$\begin{aligned}\mathbf{R} \times \Sigma m_j\dot{\mathbf{R}} &= \mathbf{R} \times M\dot{\mathbf{R}}\\ &= \mathbf{R} \times M\mathbf{V},\end{aligned}$$

where $\mathbf{V} \equiv \dot{\mathbf{R}}$ is the velocity of the center of mass with respect to the inertial system. The expression for $\mathbf{L}$ then becomes

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j. \qquad 6.11$$

The first term of Eq. (6.11) represents the angular momentum due to the center of mass motion. The second term represents angular momentum due to motion around the center of mass. The only way for the particles of a rigid body to move with respect to the center of mass is for the body as a whole to rotate. We shall evaluate the second term for an arbitrary axis of rotation in the next chapter. In this chapter, however, we are restricting ourselves to fixed axis rotation about the z axis. Taking the z component of Eq. (6.11) gives

$$L_z = (\mathbf{R} \times M\mathbf{V})_z + (\Sigma\mathbf{r}'_j \times m_j\dot{\mathbf{r}}'_j)_z. \qquad 6.12$$



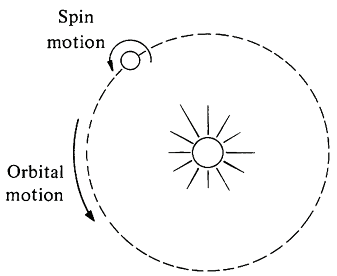
For rotation about the z axis, the second term $(\Sigma \mathbf{r}_j' \times m_j\dot{\mathbf{r}}_j')_z$ can be simplified by recognizing that we dealt with this kind of expression before, in Sec. 6.4. The body has angular velocity $\omega\hat{\mathbf{k}}$ about its center of mass, and since the origin of $\mathbf{r}_j'$ is the center of mass, the second term is identical in form to the case of pure rotation we treated in Sec. 6.4.

$$\begin{aligned}(\Sigma m_j\mathbf{r}_j' \times \dot{\mathbf{r}}_j')_z &= (\Sigma m_j\boldsymbol{\rho}_j' \times \dot{\boldsymbol{\rho}}_j')_z \\ &= \Sigma m_j\rho_j'^2\omega = I_0\omega,\end{aligned}$$

where $\boldsymbol{\rho}_j'$ is the vector to m_j perpendicular from an axis in the z direction through the center of mass. $I_0 = \Sigma m_j\rho_j'^2$ is the moment of inertia of the body about this axis.

Collecting our results, we have

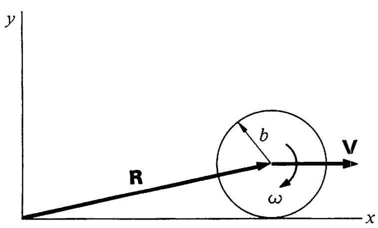
$$L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z. \qquad 6.13$$



We have proven the result stated at the beginning of this section. The angular momentum of a rigid object is the sum of the angular momentum about its center of mass and the angular momentum of the center of mass about the origin. These two terms are often referred to as the *spin* and *orbital* terms, respectively. The earth illustrates them nicely. The daily rotation of the earth about its axis gives rise to the earth's spin angular momentum, and its annual revolution about the sun gives rise to the earth's orbital angular momentum about the sun. An important feature of the spin angular momentum is that it is independent of the coordinate system. In this sense it is intrinsic to the body; no change in coordinate system can eliminate spin, whereas orbital angular momentum disappears if the origin is along the line of motion.

It should be kept in mind that Eq. (6.13) is valid even when the center of mass is accelerating, since **L** was calculated with respect to an inertial coordinate system.

Example 6.14 Angular Momentum of a Rolling Wheel



In this example we apply Eq. (6.13) to the calculation of the angular momentum of a uniform wheel of mass M and radius b which rolls uniformly and without slipping. The moment of inertia of the wheel about its center of mass is $I_0 = \frac{1}{2}Mb^2$ and its angular momentum about the center of mass is

$$\begin{aligned}L_0 &= -I_0\omega \\ &= -\tfrac{1}{2}Mb^2\omega.\end{aligned}$$

L_0 is parallel to the z axis. The minus sign indicates that L_0 is directed into the paper, in the negative z direction.

If we calculate the angular momentum of the center of mass of the wheel with respect to the origin, we have

$$(\mathbf{R} \times M\mathbf{V})_z = -MbV.$$

The total angular momentum about the origin is then

$$\begin{aligned} L_z &= -\tfrac{1}{2}Mb^2\omega - MbV \\ &= -\tfrac{1}{2}Mb^2\omega - Mb^2\omega \\ &= -\tfrac{3}{2}Mb^2\omega, \end{aligned}$$

where we have used the result $V = b\omega$, which holds for a wheel that rolls without slipping.

Torque also naturally divides itself into two components. The torque on a body is

$$\begin{aligned} \boldsymbol{\tau} &= \Sigma\mathbf{r}_j \times \mathbf{f}_j \\ &= \Sigma(\mathbf{r}_j' + \mathbf{R}) \times \mathbf{f}_j \\ &= \Sigma(\mathbf{r}_j' \times \mathbf{f}_j) + \mathbf{R} \times \mathbf{F}, \end{aligned} \qquad 6.14$$

where $\mathbf{F} = \Sigma\mathbf{f}_j$ is the total applied force. The first term in Eq. (6.14) is the torque about the center of mass due to the various external forces, and the second term is the torque due to the total external force acting at the center of mass. For fixed axis rotation $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, and Eq. (6.14) can be written

$$\tau_z = \tau_0 + (\mathbf{R} \times \mathbf{F})_z, \qquad 6.15$$

where τ_0 is the z component of the torque about the center of mass. But from Eq. (6.13) for L_z we have

$$\begin{aligned} \frac{dL_z}{dt} &= I_0\frac{d\omega}{dt} + \frac{d}{dt}(\mathbf{R} \times M\mathbf{V})_z \\ &= I_0\alpha + (\mathbf{R} \times M\mathbf{a})_z. \end{aligned} \qquad 6.16$$

Using $\tau_z = dL_z/dt$, Eq. (6.15) and (6.16) yield

$$\begin{aligned} \tau_0 + (\mathbf{R} \times \mathbf{F})_z &= I_0\alpha + (\mathbf{R} \times M\mathbf{a})_z \\ &= I_0\alpha + (\mathbf{R} \times \mathbf{F})_z, \end{aligned}$$

since $\mathbf{F} = M\mathbf{a}$. Hence,

$$\tau_0 = I_0\alpha. \qquad 6.17$$

According to Eq. (6.17), rotational motion about the center of mass depends only on the torque about the center of mass, independent of the translational motion. In other words, Eq. (6.17) is correct even if the axis is accelerating.

These relations will seem quite natural when we use them. Before doing so, we complete the development by examining the kinetic energy.

$$\begin{aligned} K &= \tfrac{1}{2}\Sigma m_j v_j^2 \\ &= \tfrac{1}{2}\Sigma m_j(\dot{\boldsymbol{\rho}}_j' + \mathbf{V})^2 \\ &= \tfrac{1}{2}\Sigma m_j \dot{\rho}_j'^2 + \Sigma m_j \dot{\boldsymbol{\rho}}_j' \cdot \mathbf{V} + \tfrac{1}{2}\Sigma m_j V^2 \\ &= \tfrac{1}{2}I_0\omega^2 + \tfrac{1}{2}MV^2 \end{aligned} \qquad 6.18$$

The first term corresponds to the kinetic energy of spin, while the last term arises from the orbital center of mass motion.

Here is a summary of these results.

TABLE 6.1
Summary of Dynamical Formulas for Fixed Axis Motion

a Pure rotation about an axis—no translation.

$L = I\omega$

$\tau = I\alpha$

$K = \tfrac{1}{2}I\omega^2$

b Rotation and translation (subscript 0 refers to center of mass)

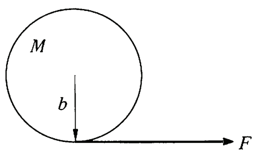
$L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z$

$\tau_z = \tau_0 + (\mathbf{R} \times \mathbf{F})_z$

$\tau_0 = I_0\alpha$

$K = \tfrac{1}{2}I_0\omega^2 + \tfrac{1}{2}MV^2$

Example 6.15 Disk on Ice



A disk of mass M and radius b is pulled with constant force F by a thin tape wound around its circumference. The disk slides on ice without friction. What is its motion?

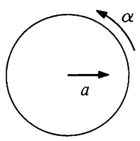
We shall solve the problem by two different methods.

METHOD 1
Analyzing the motion about the center of mass we have

$$\begin{aligned} \tau_0 &= bF \\ &= I_0\alpha \end{aligned}$$

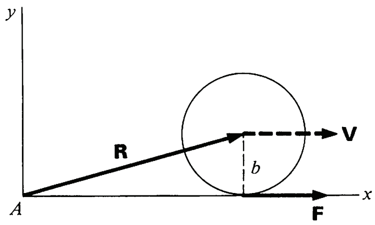
or

$$\alpha = \frac{bF}{I_0}.$$



The acceleration of the center of mass is

$$a = \frac{F}{M}.$$



METHOD 2

We choose a coordinate system whose origin A is along the line of **F**. The torque about A is, from Table 6.1*b*,

$$\tau_z = \tau_0 + (\mathbf{R} \times \mathbf{F})_z$$
$$= bF - bF = 0.$$

The torque is zero, as we expect, and angular momentum about the origin is conserved. The angular momentum about A is, from Table 6.1*b*,

$$L_z = I_0\omega + (\mathbf{R} \times M\mathbf{V})_z$$
$$= I_0\omega - bMV.$$

Since $dL_z/dt = 0$, we have

$$0 = I_0\alpha - bMa$$

or

$$\alpha = \frac{bMa}{I_0} = \frac{bF}{I_0},$$

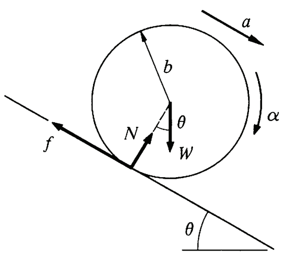
as before.

Example 6.16 Drum Rolling down a Plane

A uniform drum of radius b and mass M rolls without slipping down a plane inclined at angle θ. Find its acceleration along the plane. The moment of inertia of the drum about its axis is $I_0 = Mb^2/2$.

METHOD 1

The forces acting on the drum are shown in the diagram. f is the force of friction. The translation of the center of mass along the plane is given by



$$W \sin\theta - f = Ma$$

and the rotation about the center of mass by

$$bf = I_0\alpha.$$

For rolling without slipping, we also have

$$a = b\alpha.$$

If we eliminate f, we obtain

$$W \sin\theta - I_0 \frac{\alpha}{b} = Ma.$$

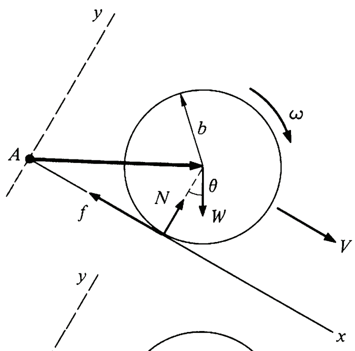
Using $I_0 = Mb^2/2$, and $\alpha = a/b$, we obtain

$$Mg \sin\theta - \frac{Ma}{2} = Ma,$$

or

$$a = \tfrac{2}{3} g \sin\theta.$$

METHOD 2

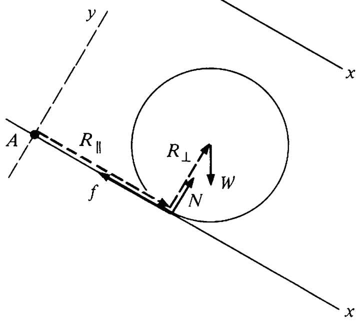


Choose a coordinate system whose origin A is on the plane. The torque about A is

$$\begin{aligned}\tau_s &= \tau_0 + (\mathbf{R} \times \mathbf{F})_z \\ &= -R_\perp f + R_\perp(f - W \sin\theta) + R_\parallel(N - W\cos\theta) \\ &= -bW \sin\theta,\end{aligned}$$

since $R_\perp = b$ and $W \cos\theta = N$. The angular momentum about A is

$$\begin{aligned}L_z &= -I_0\omega + (\mathbf{R} \times M\mathbf{V})_z \\ &= -\tfrac{1}{2}Mb^2\omega - Mb^2\omega \\ &= -\tfrac{3}{2}Mb^2\omega,\end{aligned}$$



where $(\mathbf{R} \times M\mathbf{V})_z = -Mb^2\omega$, as in Example 6.14. Since $\tau_z = dL_z/dt$, we have

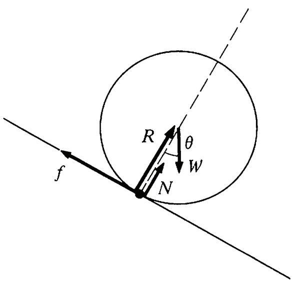
$$bW \sin\theta = \frac{3}{2} Mb^2\alpha,$$

or

$$\alpha = \frac{2}{3}\frac{W}{Mb} \sin\theta = \frac{2}{3}\frac{g \sin\theta}{b}.$$

For rolling without slipping, $a = b\alpha$ and

$$a = \tfrac{2}{3} g \sin\theta.$$



Note that the analysis would have been even more direct if we had chosen the origin at the point of contact. In this case we can calculate τ_z directly from

$$\tau_z = \Sigma(\mathbf{r}_j \times \mathbf{f}_j)_z.$$

Since $\mathbf{f}$ and $\mathbf{N}$ act at the origin, the torque is due only to W, and

$$\tau_z = -bW \sin\theta$$

as we obtained above. With this origin, however, the unknown forces $\mathbf{f}$ and $\mathbf{N}$ do not appear.

The Work-energy Theorem

In Chap. 4 we derived the work-energy theorem for a particle

$$K_b - K_a = W_{ba}$$

where

$$W_{ba} = \oint_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}.$$

We can generalize this for a rigid body and show that the work-energy theorem divides naturally into two parts, one dealing with translational energy and one dealing with rotational energy.

To derive the translational part, we start with the equation of motion for the center of mass.

$$\mathbf{F} = M \frac{d^2\mathbf{R}}{dt^2}$$
$$= M \frac{d\mathbf{V}}{dt}.$$

The work done when the center of mass is displaced by $d\mathbf{R} = \mathbf{V}\,dt$ is

$$\mathbf{F} \cdot d\mathbf{R} = M \frac{d\mathbf{V}}{dt} \cdot \mathbf{V}\,dt$$
$$= d(\tfrac{1}{2}MV^2).$$

Integrating, we obtain

$$\oint_{\mathbf{R}_a}^{\mathbf{R}_b} \mathbf{F} \cdot d\mathbf{R} = \tfrac{1}{2}MV_b^2 - \tfrac{1}{2}MV_a^2. \qquad 6.19$$

Now let us evaluate the work associated with the rotational kinetic energy. The equation of motion for fixed axis rotation about the center of mass is

$$\tau_0 = I_0\alpha$$
$$= I_0 \frac{d\omega}{dt}.$$

Rotational kinetic energy has the form $\frac{1}{2}I_0\omega^2$, which suggests that we multiply the equation of motion by $d\theta = \omega\,dt$:

$$\tau_0\,d\theta = I_0 \frac{d\omega}{dt}\,\omega\,dt$$
$$= d(\tfrac{1}{2}I_0\omega^2).$$

Integrating, we find that

$$\int_{\theta_a}^{\theta_b} \tau_0 \, d\theta = \tfrac{1}{2} I_0 \omega_b^2 - \tfrac{1}{2} I_0 \omega_a^2. \qquad 6.20$$

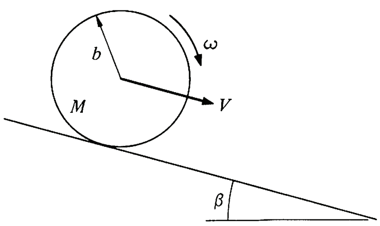
The integral on the left evidently represents the work done by the applied torque.

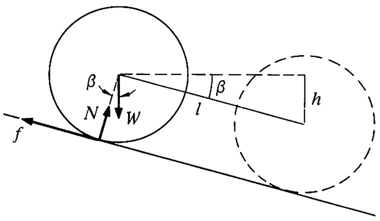
The general work-energy theorem for a rigid body is therefore

$$K_b - K_a = W_{ba},$$

where $K = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2$ and W_{ba} is the total work done on the body as it moves from position a to position b. We see from Eqs. (6.19) and (6.20) that the work-energy theorem is composed of two independent theorems, one for translation and one for rotation. In many problems these theorems can be applied separately, as the following example shows.

Example 6.17 Drum Rolling down a Plane: Energy Method





Consider once again a uniform drum of radius b, mass M, and moment of inertia $I_0 = Mb^2/2$ on a plane of angle β. If the drum starts from rest and rolls without slipping, find the speed of its center of mass, V, after it has descended a height h.

The forces on the drum are shown in the sketch. The energy equation for the translational motion is

$$\int_a^b \mathbf{F} \cdot d\mathbf{r} = \tfrac{1}{2}MV_b^2 - \tfrac{1}{2}MV_a^2$$

or

$$(W \sin\beta - f)l = \tfrac{1}{2}MV^2, \qquad 1$$

where $l = h/\sin\beta$ is the displacement of the center of mass as the drum descends height h.

The energy equation for the rotational motion is

$$\int_{\theta_a}^{\theta_b} \tau \, d\theta = \tfrac{1}{2}I_0\omega_b^2 - \tfrac{1}{2}I_0\omega_a^2$$

or

$$fb\theta = \tfrac{1}{2}I_0\omega^2,$$

where θ is the rotation angle about the center of mass. For rolling without slipping, $b\theta = l$. Hence,

$$fl = \tfrac{1}{2}I_0\omega^2. \qquad 2$$

We also have $\omega = V/b$, so that

$$fl = \frac{1}{2}\frac{I_0V^2}{b^2}.$$

Using this in Eq. (1) to eliminate f gives

$$\begin{aligned} Wh &= \frac{1}{2}\left(\frac{I_0}{b^2} + M\right)V^2 \\ &= \frac{1}{2}\left(\frac{M}{2} + M\right)V^2 \\ &= \tfrac{3}{4}MV^2 \end{aligned}$$

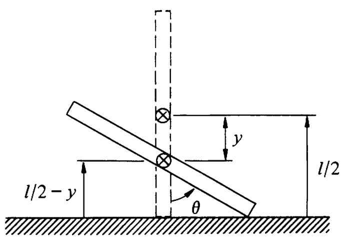
or

$$V = \sqrt{\frac{4gh}{3}}.$$

An interesting point in this example is that the friction force is not dissipative. From Eq. (1), friction decreases the translational energy by an amount fl. However, from Eq. (2), the torque exerted by friction increases the rotational energy by the same amount. In this motion, friction simply transforms mechanical energy from one mode to another. If slipping occurs, this is no longer the case and some of the mechanical energy is dissipated as heat.

We conclude this section with an example involving constraints which is easily handled by energy methods.

Example 6.18 **The Falling Stick**



A stick of length l and mass M, initially upright on a frictionless table, starts falling. The problem is to find the speed of the center of mass as a function of position.

The key lies in realizing that since there are no horizontal forces, the center of mass must fall straight down. Since we must find velocity as a function of position, it is natural to apply energy methods.

The sketch shows the stick after it has rotated through angle θ and the center of mass has fallen distance y. The initial energy is

$$\begin{aligned} E &= K_0 + U_0 \\ &= \frac{Mgl}{2}. \end{aligned}$$

The kinetic energy at a later time is

$$K = \tfrac{1}{2}I_0\dot{\theta}^2 + \tfrac{1}{2}M\dot{y}^2$$

and the corresponding potential energy is

$$U = Mg\left(\frac{l}{2} - y\right).$$

Since there are no dissipative forces, mechanical energy is conserved and $K + U = K_0 + U_0 = Mgl/2$. Hence,

$$\tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}I_0\dot{\theta}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}.$$

We can eliminate $\dot{\theta}$ by turning to the constraint equation. From the sketch we see that

$$y = \frac{l}{2}(1 - \cos\theta).$$

Hence,

$$\dot{y} = \frac{l}{2}\sin\theta\,\dot{\theta}$$

and

$$\dot{\theta} = \frac{2}{l\sin\theta}\dot{y}.$$

Since $I_0 = M(l^2/12)$, we obtain

$$\tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}M\frac{l^2}{12}\left(\frac{2}{l\sin\theta}\right)^2\dot{y}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}$$

or

$$\dot{y}^2 = \frac{2gy}{[1 + 1/(3\sin^2\theta)]},$$

$$\dot{y} = \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}.$$

6.8 The Bohr Atom

We conclude this chapter with an historical account of the Bohr theory of the hydrogen atom. Although this material represents an interesting application of the principles we have encountered, it is not essential to our development of classical mechanics.

The Bohr theory of the hydrogen atom is the major link between classical physics and quantum mechanics. We present here a brief outline of the Bohr theory as an exciting example of the application of concepts we have studied, particularly energy and angular momentum. Our description is similar, though not identical, to Bohr's original paper which he published in 1913 at the age of 26. Although this brief account cannot deal adequately with the background to the Bohr theory, it may give some of the flavor of one of the great chapters in physics.

The development of optical spectroscopy in the nineteenth century made available a great deal of experimental data on the structure of atoms. The light from atoms excited by an electric discharge is radiated only at certain discrete wavelengths characteristic of the element involved, and the last half of the nineteenth century saw tremendous effort in the measurement and interpretation of these line spectra. The wavelength measurements represented a notable experimental achievement; some were made to an accuracy of better than a part in a million. Interpretation, on the other hand, was a dismal failure; aside from certain empirical rules which gave no insight into the underlying physical laws, there was no progress.

The most celebrated empirical formula was discovered in 1886 by the Swiss high school art teacher Joseph Balmer. He found that the wavelengths of the optical spectrum of atomic hydrogen are given within experimental accuracy by the formula

$$\frac{1}{\lambda} = Ry\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \qquad n = 3, 4, 5, \ldots,$$

where λ is the wavelength of a particular spectral line, and Ry is a constant, named the *Rydberg constant* after the Swedish spectroscopist who modified Balmer's formula to apply to certain other spectra. Numerically, $Ry = 109{,}700\ \text{cm}^{-1}$. (In this section we shall follow the tradition of atomic physics by using cgs units.)

Not only did Balmer's formula account for the known lines of hydrogen, $n = 3$ through $n = 6$, it predicted other lines, $n = 7$, $8, \ldots$, which were quickly found. Furthermore, Balmer suggested that there might be other lines given by

$$\lambda = Ry\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \qquad m = 3, 4, 5, \ldots \qquad n = m + 1, m + 2, \ldots \qquad 6.21$$

and these, too, were found. (Balmer overlooked the series with $m = 1$, lying in the ultraviolet, which was found in 1916.)

Undoubtedly the Balmer formula contained the key to the structure of hydrogen. Yet no one was able to create a model for an atom which could radiate such a spectrum.

Bohr was familiar with the Balmer formula. He was also familiar with ideas of atomic structure current at the time, ideas based on the experimental researches of J. J. Thomson and Ernest Rutherford. Thomson, working in the Cavendish physical laboratory at Cambridge University, surmised the existence of electrons in 1897. This first indication of the divisibility of the atom stimulated further work, and in 1911 Ernest Rutherford's[1] alpha scattering experiments at the University of Manchester showed that atoms have a charged core which contains most of the mass. Each atom has an integral number of electrons and an equal number of positive charges on the massive core.

A further development in physics which played an essential role in Bohr's theory was Einstein's theory of the photoelectric effect. In 1905, the same year that he published the special theory of relativity, Einstein proposed that the energy transmitted by light consists of discrete "packages," or quanta. The quantum of light is called a *photon*, and Einstein asserted that the energy of a photon is $E = h\nu$, where ν is the frequency of the light and $h = 6.62 \times 10^{-27}$ erg · s is Planck's constant.[2]

Bohr made the following postulates:

1. Atoms cannot possess arbitrary amounts of energy but must exist only in certain *stationary states*. While in a stationary state, an atom does not radiate.

2. An atom can pass from one stationary state a to a lower state by emitting a photon with energy $E_a - E_b$. The frequency of the emitted photon is

$$\nu = \frac{E_a - E_b}{h}. \qquad 6.22$$

3. While in a stationary state, the motion of the atom is given accurately by classical physics.

4. The angular momentum of the atom is $nh/2\pi$, where n is an integer.

Assumption 1, the most drastic, was absolutely necessary to account for the fact that atoms are stable. According to classical theory, an orbiting electron would continuously lose energy by radiation and spiral into the nucleus.

In view of the fact that assumption 1 breaks completely with classical physics, assumption 3 hardly seems justified. Bohr recognized this difficulty and justified the assumption on the ground that the electrodynamical forces connected with the emission of radiation would be very small in comparison with the

[1] Rutherford had earlier been a student of J. J. Thomson and in 1919 succeeded Thomson as director of the Cavendish laboratory. Bohr in turn studied with Rutherford while working out the Bohr theory.

[2] Max Planck had introduced h in 1901 in his theory of radiation from hot bodies.

electrostatic attraction of the charged particles. Possibly the real reason that Bohr continued to apply classical physics to this nonclassical situation was that he felt that at least some of the fundamental concepts of classical physics should carry over into the new physics, and that they should not be discarded until proven to be unworkable.

Bohr did not utilize postulate 4, known as the quantization of angular momentum, in his original work, although he pointed out the possibility of doing so. It has become traditional to treat this postulate as a fundamental assumption.

Let us apply these four postulates to hydrogen. The hydrogen atom consists of a single electron of charge $-e$ and mass m_0, and a nucleus of charge $+e$ and mass M. We assume that the massive nucleus is essentially at rest and that the electron is in a circular orbit of radius r with velocity v. The radial equation of motion is

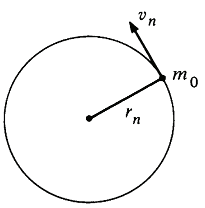
$$-\frac{m_0v^2}{r} = -\frac{e^2}{r^2}, \qquad 6.23$$

where $-e^2/r^2$ is the attractive Coulomb force between the charges The energy is

$$E = K + U = \tfrac{1}{2}m_0v^2 - \frac{e^2}{r}. \qquad 6.24$$

Equations (6.23) and (6.24) yield

$$E = -\frac{1}{2}\frac{e^2}{r}. \qquad 6.25$$



By postulate 4, the angular momentum is $nh/2\pi$, where n is an integer. Labeling the orbit parameters by n, we have

$$\frac{nh}{2\pi} = m_0r_nv_n. \qquad 6.26$$

Equations (6.26) and (6.23) yield

$$r_n = \frac{n^2h^2}{m_0e^2}\frac{1}{(2\pi)^2}, \qquad 6.27$$

and Eq. (6.25) gives

$$E_n = -\frac{1}{2}\frac{(2\pi)^2m_0e^4}{n^2h^2}. \qquad 6.28$$

If the electron makes a transition from state n to state m, the emitted photon has frequency

$$\nu = \frac{E_n - E_m}{h} = \frac{(2\pi)^2}{2}\frac{m_0 e^4}{h^3}\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \qquad 6.29$$

The wavelength of the radiation is given by

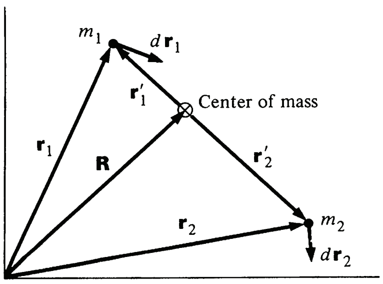
$$\frac{1}{\lambda} = \frac{\nu}{c} = \frac{2\pi^2}{c}\frac{m_0 e^4}{h^3}\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \qquad 6.30$$

This is identical in form to the Balmer formula, Eq. 6.21. What is even more impressive is that the numerical coefficients agree extemely well; Bohr was able to calculate the Rydberg constant from the fundamental atomic constants.

The Bohr theory, with its strong flavor of elementary classical mechanics, formed an important bridge between classical physics and present-day atomic theory. Although the Bohr theory was unsuccessful in explaining more complicated atoms, the impetus provided by Bohr's work led to the development of modern quantum mechanics in the 1920s.

Note 6.1 Chasles' Theorem

Chasles' theorem asserts that is always possible to represent an arbitrary displacement of a rigid body by a translation of its center of mass plus a rotation about its center of mass. This appendix is rather detailed and an understanding of it is not necessary for following the development of the text. However, the result is interesting and its proof provides a nice exercise in vector methods for those interested.



To avoid algebraic complexities, we consider here a simple rigid body consisting of two masses m_1 and m_2 joined by a rigid rod of length l. The position vectors of m_1 and m_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, as shown in the sketch. The position vector of the center of mass of the body is $\mathbf{R}$, and $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are the position vectors of m_1 and m_2 with respect to the center of mass. The vectors $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are back to back along the same line.

In an arbitrary displacement of the body, m_1 is displaced by $d\mathbf{r}_1$ and m_2 is displaced by $d\mathbf{r}_2$. Because the body is rigid, $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are not independent, and we begin our analysis by finding their relation. The distance between m_1 and m_2 is fixed and of length l. Therefore,

$$|\mathbf{r}_1 - \mathbf{r}_2| = l$$

or

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) = l^2. \quad 1$$

Taking differentials of Eq. (1),[1]

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) = 0. \quad 2$$

Equation (2) is the "rigid body condition" we seek. There are evidently two ways of satisfying Eq. (2): either $d\mathbf{r}_1 = d\mathbf{r}_2$, or $(d\mathbf{r}_1 - d\mathbf{r}_2)$ is perpendicular to $(\mathbf{r}_1 - \mathbf{r}_2)$.

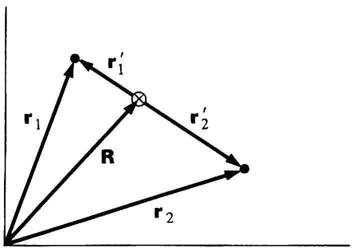
We now turn to the translational motion of the center of mass. By definition,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$

Therefore, the displacement $d\mathbf{R}$ of the center of mass is

$$d\mathbf{R} = \frac{m_1 d\mathbf{r}_1 + m_2 d\mathbf{r}_2}{m_1 + m_2}. \quad 3$$

If we subtract this translational displacement from $d\mathbf{r}_1$ and $d\mathbf{r}_2$, the residual displacements $d\mathbf{r}_1 - d\mathbf{R}$ and $d\mathbf{r}_2 - d\mathbf{R}$ should give a pure rotation about the center of mass. Before investigating this point, we notice that since



$$\mathbf{r}_1 - \mathbf{R} = \mathbf{r}_1'$$
$$\mathbf{r}_2 - \mathbf{R} = \mathbf{r}_2',$$

the residual displacements are

$$d\mathbf{r}_1 - d\mathbf{R} = d\mathbf{r}_1'$$
$$d\mathbf{r}_2 - d\mathbf{R} = d\mathbf{r}_2'. \quad 4$$

Using Eq. (3) in Eq. (4) we have

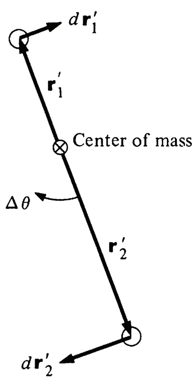
$$d\mathbf{r}_1' = d\mathbf{r}_1 - d\mathbf{R}$$
$$= \left(\frac{m_2}{m_1 + m_2}\right)(d\mathbf{r}_1 - d\mathbf{r}_2) \quad 5$$

and

$$d\mathbf{r}_2' = d\mathbf{r}_2 - d\mathbf{R}$$
$$= -\left(\frac{m_1}{m_1 + m_2}\right)(d\mathbf{r}_1 - d\mathbf{r}_2). \quad 6$$

Note that if $d\mathbf{r}_1 = d\mathbf{r}_2$, the residual displacements $d\mathbf{r}_1'$ and $d\mathbf{r}_2'$ are zero and the rigid body translates without rotating.

[1] Remember that $d(\mathbf{A} \cdot \mathbf{A}) = 2\mathbf{A} \cdot d\mathbf{A}$.



We must show that the residual displacements represent a pure rotation about the center of mass to complete the theorem. The sketch shows what a pure rotation would look like. First we show that $d\mathbf{r}_1'$ and $d\mathbf{r}_2'$ are perpendicular to the line $\mathbf{r}_1' - \mathbf{r}_2'$.

$$\begin{aligned} d\mathbf{r}_1' \cdot (\mathbf{r}_1' - \mathbf{r}_2') &= d\mathbf{r}_1' \cdot (\mathbf{r}_1 - \mathbf{r}_2) \\ &= \left(\frac{m_2}{m_1 + m_2}\right)(d\mathbf{r}_1 - d\mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \\ &= 0, \end{aligned}$$

where we have used Eq. (5) and the rigid body condition, Eq. (2). Similarly,

$$d\mathbf{r}_2' \cdot (\mathbf{r}_1' - \mathbf{r}_2') = 0.$$

Finally, we require that the residual displacements correspond to rotation through the same angle, $\Delta\theta$. With reference to our sketch, this condition in vector form is

$$\frac{d\mathbf{r}_1'}{r_1'} = -\frac{d\mathbf{r}_2'}{r_2'}.$$

Keeping in mind that

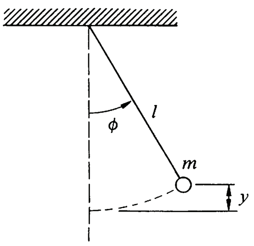
$$\frac{r_1'}{r_2'} = \frac{m_2}{m_1}$$

by definition of center of mass, and using Eq. (5) and (6), we have

$$\begin{aligned} \frac{d\mathbf{r}_1'}{r_1'} &= \left(\frac{m_2}{m_1 + m_2}\right)\frac{(d\mathbf{r}_1 - d\mathbf{r}_2)}{r_1'} \\ &= \left(\frac{m_1}{m_1 + m_2}\right)\frac{(d\mathbf{r}_1 - d\mathbf{r}_2)}{r_2'} \\ &= -\frac{d\mathbf{r}_2'}{r_2'}, \end{aligned}$$

completing the proof.

Note 6.2 Pendulum Motion



The motion of a body moving under conservative forces can always be solved formally by energy methods, and it is natural to use this approach to find the motion of a pendulum.

The total energy of the pendulum is

$$\begin{aligned} E &= K + U \\ &= \tfrac{1}{2}l^2\dot{\phi}^2 + mgy, \end{aligned}$$

where l is the length of the pendulum and y is the vertical distance from the lowest point. From the sketch we have $y = l(1 - \cos\phi)$.

At the end of the swing, $\phi = \phi_0$ and $\dot{\phi} = 0$. The total energy is

$E = mgl(1 - \cos\phi_0)$.

The energy equation is

$$\tfrac{1}{2}ml^2\dot{\phi}^2 + mgl(1 - \cos\phi) = mgl(1 - \cos\phi_0),$$

$$\frac{d\phi}{dt} = \sqrt{\frac{2g}{l}(\cos\phi - \cos\phi_0)},$$

and

$$\int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} = \sqrt{\frac{2g}{l}} \int dt. \qquad 1$$

Before looking at the general solution, let us find the solution for the case of small amplitudes. With the approximation $\cos\phi \approx 1 - \frac{1}{2}\phi^2$, we have

$$\int \frac{d\phi}{\sqrt{\frac{1}{2}}\sqrt{\phi_0{}^2 - \phi^2}} = \sqrt{\frac{2g}{l}} \int dt.$$

Let us integrate over one-fourth of the swing, from $\phi = 0$ to $\phi = \phi_0$. The time varies between $t = 0$ and $t = T/4$, where T is the period. We have

$$\int_0^{\phi_0} \frac{d\phi}{\sqrt{\frac{1}{2}}\,\phi_0\sqrt{1 - (\phi/\phi_0)^2}} = \sqrt{\frac{2g}{l}} \int_0^{T/4} dt$$

or

$$\arcsin\frac{\phi}{\phi_0}\Big|_0^{\phi_0} = \sqrt{\frac{g}{l}}\,\frac{T}{4}$$

$$\frac{\pi}{2} - 0 = \sqrt{\frac{g}{l}}\,\frac{T}{4}$$

$$T = 2\pi\sqrt{\frac{l}{g}},$$

as we found in the text.

To obtain a more accurate solution to Eq. (1), it is helpful to use the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$. Then

$$\cos\phi - \cos\phi_0 = 2[\sin^2(\phi_0/2) - \sin^2(\phi/2)]. \qquad 2$$

Introducing Eq. (2) in Eq. (1) gives

$$\int \frac{d\phi}{\sqrt{2}\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}} = \sqrt{\frac{2g}{l}} \int dt. \qquad 3$$

Now let us change variables as follows:

$$\sin u = \frac{\sin(\phi/2)}{\sin(\phi_0/2)}. \qquad 4$$

As the pendulum swings through a cycle, ϕ varies between $-\phi_0$ and $+\phi_0$. At the same time, u varies between $-\pi$ and $+\pi$. If we let

$$\mathrm{K} = \sin\frac{\phi_0}{2},$$

then

$$\sin\frac{\phi}{2} = \mathrm{K}\sin u$$

$$\frac{1}{2}\cos\frac{\phi}{2}\,d\phi = \mathrm{K}\cos u\,du$$

and

$$d\phi = \left(\frac{1-\sin^2 u}{1-\mathrm{K}^2\sin^2 u}\right)^{\frac{1}{2}} 2\mathrm{K}\,du. \qquad 5$$

Substituting Eqs. (4) and (5) in Eq. (3) gives

$$\int \frac{du}{\sqrt{1-\mathrm{K}^2\sin^2 u}} = \sqrt{\frac{g}{l}}\int dt.$$

Let us take the integral over one period. The limits on u are 0 and 2π, while t ranges from 0 to T. We have

$$\int_0^{2\pi} \frac{du}{\sqrt{1-\mathrm{K}^2\sin^2 u}} = \sqrt{\frac{g}{l}}\,T. \qquad 6$$

The integral on the left is an *elliptic integral*: specifically, it is a complete elliptic integral of the first kind. Values for this function are available from computed tables. However, for our purposes it is more convenient to expand the integrand:

$$(1-\mathrm{K}^2\sin^2 u)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\mathrm{K}^2\sin^2 u + \cdots$$

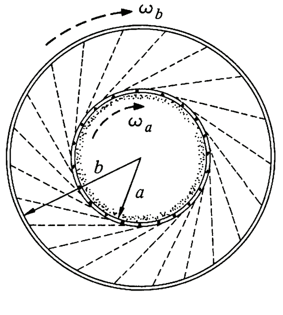
and

$$\begin{aligned} T &= \sqrt{\frac{l}{g}}\int_0^{2\pi} du(1+\tfrac{1}{2}\mathrm{K}^2\sin^2 u + \cdots) \\ &= \sqrt{\frac{l}{g}}\left(2\pi + \frac{2\pi}{4}\mathrm{K}^2 + \cdots\right) \\ &= 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\phi_0}{2} + \cdots\right). \end{aligned}$$

If $\phi_0 \ll 1$, then $\sin^2(\phi_0/2) \approx \phi_0^2/4$, and we have

$$T = 2\pi\sqrt{\frac{l}{g}}(1+\tfrac{1}{16}\phi_0^2 + \cdots). \qquad 7$$

Problems



6.1 *a*. Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins.

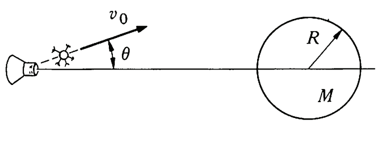
b. Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins.

6.2 A drum of mass M_A and radius a rotates freely with initial angular velocity $\omega_A(0)$. A second drum with mass M_B and radius $b > a$ is mounted on the same axis and is at rest, although it is free to rotate. A thin layer of sand with mass M_S is distributed on the inner surface of the smaller drum. At $t = 0$, small perforations in the inner drum are opened. The sand starts to fly out at a constant rate λ and sticks to the outer drum. Find the subsequent angular velocities of the two drums ω_A and ω_B. Ignore the transit time of the sand.

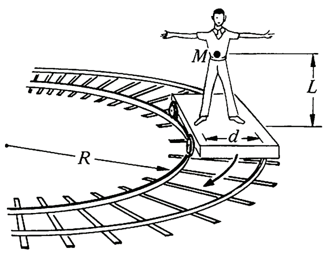
Ans. clue. If $\lambda t = M_b$ and $b = 2a$, then $\omega_B = \omega_A(0)/8$

6.3 A ring of mass M and radius R lies on its side on a frictionless table. It is pivoted to the table at its rim. A bug of mass m walks around the ring with speed v, starting at the pivot. What is the rotational velocity of the ring when the bug is (*a*) halfway around and (*b*) back at the pivot.

Ans. clue. (*a*) If $m = M$, $\omega = v/3R$



6.4 A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance $5R$ from the center of the planet, the ship fires an instrument package with speed v_0, as shown in the sketch. The package has mass m, which is much smaller than the mass of the spaceship. For what angle θ will the package just graze the surface of the planet?



6.5 A 3,000-lb car is parked on a 30° slope, facing uphill. The center of mass of the car is halfway between the front and rear wheels and is 2 ft above the ground. The wheels are 8 ft apart. Find the normal force exerted by the road on the front wheels and on the rear wheels.

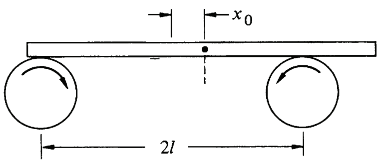
6.6 A man of mass M stands on a railroad car which is rounding an unbanked turn of radius R at speed v. His center of mass is height L above the car, and his feet are distance d apart. The man is facing the direction of motion. How much weight is on each of his feet?

6.7 Find the moment of inertia of a thin sheet of mass M in the shape of an equilateral triangle about an axis through a vertex, perpendicular to the sheet. The length of each side is L.

6.8 Find the moment of inertia of a uniform sphere of mass M and radius R about an axis through the center.

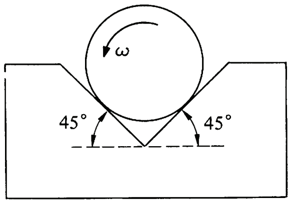
Ans. $I_0 = \frac{2}{5}MR^2$

6.9 A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown.



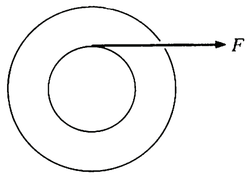
The centers of the rollers are a distance $2l$ apart. The coefficient of friction between the bar and the roller surfaces is μ, a constant independent of the relative speed of the two surfaces.

Initially the bar is held at rest with its center at distance x_0 from the midpoint of the rollers. At time $t = 0$ it is released. Find the subsequent motion of the bar.



6.10 A cylinder of mass M and radius R is rotated in a uniform V groove with constant angular velocity ω. The coefficient of friction between the cylinder and each surface is μ. What torque must be applied to the cylinder to keep it rotating?

Ans. clue. If $\mu = 0.5$, $R = 0.1$ m, $W = 100$ N, then $\tau \approx 5.7$ N·m



6.11 A wheel is attached to a fixed shaft, and the system is free to rotate without friction. To measure the moment of inertia of the wheel-shaft system, a tape of negligible mass wrapped around the shaft is pulled with a known constant force F. When a length L of tape has unwound, the system is rotating with angular speed ω_0. Find the moment of inertia of the system, I_0.

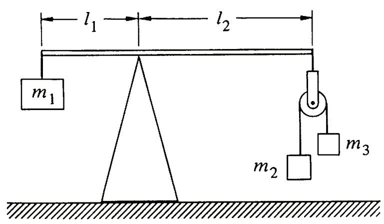
Ans. clue. If $F = 10$ N, $L = 5$ m, $\omega_0 = 0.5$ rad/s, then $I_0 = 400$ kg·m^2

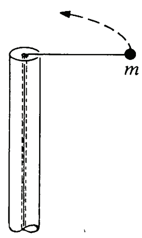
6.12 A pivoted beam has a mass M_1 suspended from one end and an Atwood's machine suspended from the other (see sketch at left below). The frictionless pulley has negligible mass and dimension. Gravity is directed downward, and $M_2 > M_3$.

Find a relation between M_1, M_2, M_3, l_1, and l_2 which will ensure that the beam has no tendency to rotate just after the masses are released.

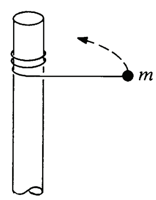
6.13 Mass m is attached to a post of radius R by a string (see right hand sketch below). Initially it is distance r from the center of the post and is moving tangentially with speed v_0. In case (*a*) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (*b*) the string wraps around the outside of the post.

What quantities are conserved in each case? Find the final speed of the mass when it hits the post for each case.

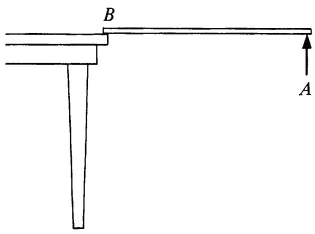




(*a*)



(*b*)



6.14 A uniform stick of mass M and length l is suspended horizontally with end B on the edge of a table, and the other end, A is held by hand. Point A is suddenly released. At the instant after release:

a. What is the torque about B?

b. What is the angular acceleration about B?

c. What is the vertical acceleration of the center of mass?

Ans. $3g/4$

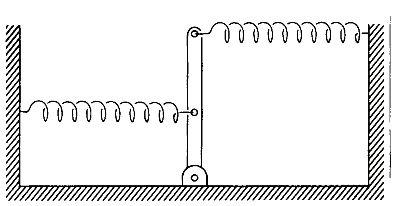
d. From part c, find by inspection the vertical force at B.

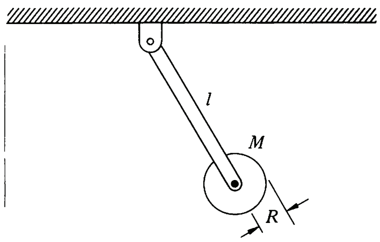
Ans. $mg/4$

6.15 A pendulum is made of two disks each of mass M and radius R separated by a massless rod. One of the disks is pivoted through its center by a small pin. The disks hang in the same plane and their centers are a distance l apart. Find the period for small oscillations.

6.16 A physical pendulum is made of a uniform disk of mass M and radius R suspended from a rod of negligible mass. The distance from the pivot to the center of the disk is l. What value of l makes the period a minimum?

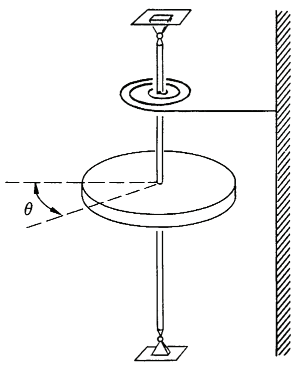
6.17 A rod of length l and mass m, pivoted at one end, is held by a spring at its midpoint and a spring at its far end, both pulling in opposite directions. The springs have spring constant k, and at equilibrium their pull is perpendicular to the rod. Find the frequency of small oscillations about the equilibrium position. See figure below left





6.18 Find the period of a pendulum consisting of a disk of mass M and radius R fixed to the end of a rod of length l and mass m. How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin? See figure above right

6.19 A solid disk of mass M and radius R is on a vertical shaft. The shaft is attached to a coil spring which exerts a linear restoring torque of magnitude $C\theta$, where θ is the angle measured from the static equilibrium position and C is a constant. Neglect the mass of the shaft and the spring, and assume the bearings to be frictionless.

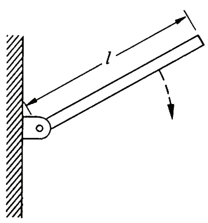


a. Show that the disk can undergo simple harmonic motion, and find the frequency of the motion.

b. Suppose that the disk is moving according to $\theta = \theta_0 \sin(\omega t)$, where ω is the frequency found in part a. At time $t_1 = \pi/\omega$, a ring of sticky putty of mass M and radius R is dropped concentrically on the disk. Find:

(1) The new frequency of the motion
(2) The new amplitude of the motion

6.20 A thin plank of mass M and length l is pivoted at one end (see figure below). The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?

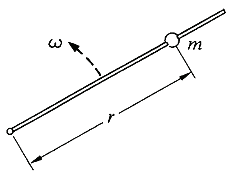


6.21 A cylinder of radius R and mass M rolls without slipping down a plane inclined at angle θ. The coefficient of friction is μ.

What is the maximum value of θ for the cylinder to roll without slipping?

Ans. $\theta = \arctan 3\mu$

6.22 A bead of mass m slides without friction on a rod that is made to rotate at a constant angular velocity ω. Neglect gravity.



a. Show that $r = r_0 e^{\omega t}$ is a possible motion of the bead, where r_0 is the initial distance of the bead from the pivot.

b. For the motion described in part a, find the force exerted on the bead by the rod.

c. For the motion described above, find the power exerted by the agency which is turning the rod and show by direct calculation that this power equals the rate of change of kinetic energy of the bead.

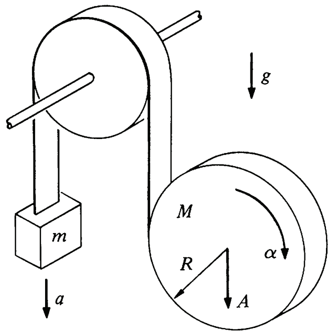
6.23 A disk of mass M and radius R unwinds from a tape wrapped around it (see figure below at left). The tape passes over a frictionless pulley, and a mass m is suspended from the other end. Assume that the disk drops vertically.

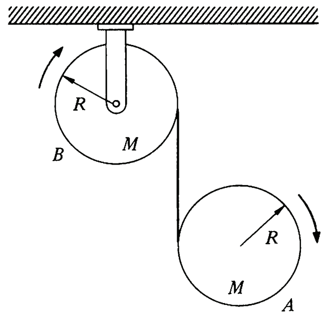
a. Relate the accelerations of m and the disk, a and A, respectively, to the angular acceleration of the disk.

Ans. clue. If $A = 2a$, then $\alpha = 3A/R$

b. Find a, A and α.

6.24 Drum A of mass M and radius R is suspended from a drum B also of mass M and radius R, which is free to rotate about its axis (see sketch below right). The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind, as shown. Gravity is directed downward. Both drums are initially at rest. Find the initial acceleration of drum A, assuming that it moves straight down.

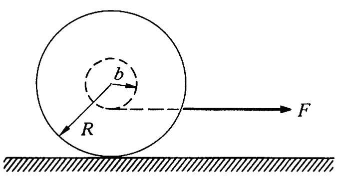




6.25 A marble of mass M and radius R is rolled up a plane of angle θ. If the initial velocity of the marble is v_0, what is the distance l it travels up the plane before it begins to roll back down?

Ans. clue. If $v_0 = 3$ m/s, $\theta = 30°$, then $l \approx 1.3$ m

6.26 A uniform sphere of mass M and radius R and a uniform cylinder of mass M and radius R are released simultaneously from rest at the top of an inclined plane. Which body reaches the bottom first if they both roll without slipping?



6.27 A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force F as shown. The coefficient of friction between the Yo-Yo and the table is μ.

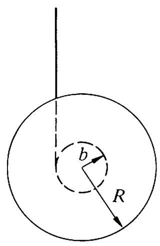
What is the maximum value of F for which the Yo-Yo will roll without slipping?

6.28 The Yo-Yo of the previous problem is pulled so that the string makes an angle θ with the horizontal. For what value of θ does the Yo-Yo have no tendency to rotate?

6.29 A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.

a. What is the tension in the cord as the Yo-Yo descends and as it ascends?

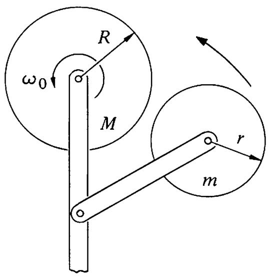
b. The center of the Yo-Yo descends distance h before the string is fully unwound. Assuming that it reverses direction with uniform spin velocity, find the average force on the string while the Yo-Yo turns around.



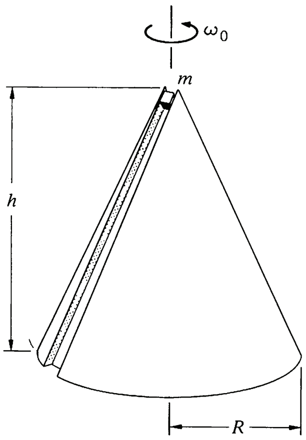
6.30 A bowling ball is thrown down the alley with speed v_0. Initially it slides without rolling, but due to friction it begins to roll. Show that its speed when it rolls without sliding is $\frac{5}{7}v_0$.

6.31 A cylinder of radius R spins with angular velocity ω_0. When the cylinder is gently laid on a plane, it skids for a short time and eventually rolls without slipping. What is the final angular velocity, ω_f?

Ans. clue. If $\omega_0 = 3$ rad/s, $\omega_f = 1$ rad/s



6.32 A solid rubber wheel of radius R and mass M rotates with angular velocity ω_0 about a frictionless pivot (see sketch at left). A second rubber wheel of radius r and mass m, also mounted on a frictionless pivot, is brought into contact with it. What is the final angular velocity of the first wheel?



6.33 A cone of height h and base radius R is free to rotate about a fixed vertical axis. It has a thin groove cut in the surface. The cone is set rotating freely with angular speed ω_0, and a small block of mass m is released in the top of the frictionless groove and allowed to slide under gravity. Assume that the block stays in the groove. Take the moment of inertia of the cone about the vertical axis to be I_0.

a. What is the angular velocity of the cone when the block reaches the bottom?

b. Find the speed of the block in inertial space when it reaches the bottom.

6.34 A marble of radius b rolls back and forth in a shallow dish of radius R. Find the frequency of small oscillations. $R \gg b$.

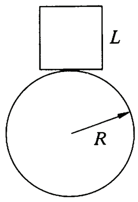
Ans. $\omega = \sqrt{5g/7R}$

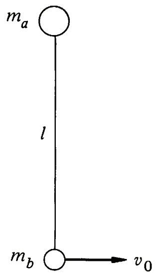
6.35 A cubical block of side L rests on a fixed cylindrical drum of radius R. Find the largest value of L for which the block is stable. See figure below left.

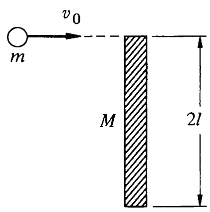
6.36 Two masses m_A and m_B are connected by a string of length l and lie on a frictionless table. The system is twirled and released with m_A instantaneously at rest and m_B moving with instantaneous velocity v_0 at right angles to the line of centers, as shown below right.

Find the subsequent motion of the system and the tension in the string.

Ans. clue. If $m_A = m_B = 2$ kg, $v_0 = 3$ m/s, $l = 0.5$ m, then $T = 18$ N



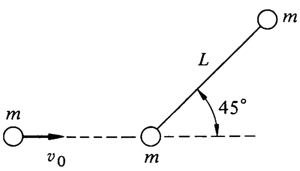




6.37 *a.* A plank of length $2l$ and mass M lies on a frictionless plane. A ball of mass m and speed v_0 strikes its end as shown. Find the final velocity of the ball, v_f, assuming that mechanical energy is conserved and that v_f is along the original line of motion.

b. Find v_f assuming that the stick is pivoted at the lower end.

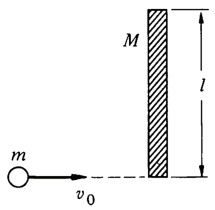
Ans. clue. For $m = M$, (*a*) $v_f = 3v_0/5$; (*b*) $v_f = v_0/2$



6.38 A rigid massless rod of length L joins two particles each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_0, moving as shown. After the collision, the projectile moves straight back.

Find the angular velocity of the rod about its center of mass after the collision, assuming that mechanical energy is conserved.

Ans. $\omega = (4\sqrt{2}/7)(v_0/L)$



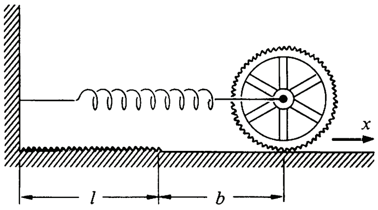
6.39 A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.

a. Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

b. One point on the plank is at rest immediately after the collision. Where is it?

Ans. $2l/3$ from the boy

6.40 A wheel with fine teeth is attached to the end of a spring with constant k and unstretched length l. For $x > l$, the wheel slips freely on

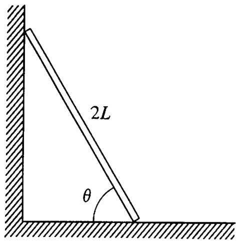


the surface, but for $x < l$ the teeth mesh with the teeth on the ground so that it cannot slip. Assume that all the mass of the wheel is in its rim.

a. The wheel is pulled to $x = l + b$ and released. How close will it come to the wall on its first trip?

b. How far out will it go as it leaves the wall?

c. What happens when the wheel next hits the gear track?



6.41 This problem utilizes most of the important laws introduced so far and it is worth a substantial effort. However, the problem is tricky (although not really complicated), so don't be alarmed if the solution eludes you.

A plank of length $2L$ leans against a wall. It starts to slip downward without friction. Show that the top of the plank loses contact with the wall when it is at two-thirds of its initial height.

Hint: Only a single variable is needed to describe the system. Note the motion of the center of mass.