b and α are constants and v is the velocity. At $t = 0$ it is moving with velocity v_0. Find the velocity at later times.

Ans. $v(t) = (1/\alpha) \ln [1/(\alpha bt/m + e^{-\alpha v_0})]$

2.37 The Eureka Hovercraft Corporation wanted to hold hovercraft races as an advertising stunt. The hovercraft supports itself by blowing air downward, and has a big fixed propeller on the top deck for forward propulsion. Unfortunately, it has no steering equipment, so that the pilots found that making high speed turns was very difficult. The company decided to overcome this problem by designing a bowl shaped track in which the hovercraft, once up to speed, would coast along in a circular path with no need to steer. They hired an engineer to design and build the track, and when he finished, he hastily left the country. When the company held their first race, they found to their dismay that the craft took exactly the same time T to circle the track, no matter what its speed. Find the equation for the cross section of the bowl in terms of T.

3 MOMENTUM

3.1 Introduction

In the last chapter we made a gross simplification by treating nature as if it were composed of point particles rather than real, *extended* bodies. Sometimes this simplification is justified—as in the study of planetary motion, where the size of the planets is of little consequence compared with the vast distances which characterize our solar system, or in the case of elementary particles moving through an accelerator, where the size of the particles, about 10^{-15} m, is minute compared with the size of the machine. However, these cases are unusual. Much of the time we deal with large bodies which may have elaborate structure. For instance, consider the landing of a spacecraft on the moon. Even if we could calculate the gravitational field of such an irregular and inhomogeneous body as the moon, the spacecraft itself is certainly not a point particle—it has spiderlike legs, gawky antennas, and a lumpy body.

Furthermore, the methods of the last chapter fail us when we try to analyze systems such as rockets in which there is a flow of mass. Rockets accelerate forward by ejecting mass backward; it is hard to see how to apply $\mathbf{F} = M\mathbf{a}$ to such a system.

In this chapter we shall generalize the laws of motion to overcome these difficulties. We begin by restating Newton's second law in a slightly modified form. In Chap. 2 we wrote the law in the familiar form

$$\mathbf{F} = M\mathbf{a}. \qquad 3.1$$

This is not quite the way Newton wrote it. He chose to write

$$\mathbf{F} = \frac{d}{dt} M\mathbf{v}. \qquad 3.2$$

For a particle in newtonian mechanics, M is a constant and $(d/dt)(M\mathbf{v}) = M(d\mathbf{v}/dt) = M\mathbf{a}$, as before. The quantity $M\mathbf{v}$, which plays a prominent role in mechanics, is called *momentum.* Momentum is the product of a vector $\mathbf{v}$ and a scalar M. Denoting momentum by $\mathbf{p}$, Newton's second law becomes

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \qquad 3.3$$

This form is preferable to $\mathbf{F} = M\mathbf{a}$ because it is readily generalized to complex systems, as we shall soon see, and because momentum turns out to be more fundamental than mass or velocity separately.

3.2 Dynamics of a System of Particles

Consider a system of interacting particles. One example of such a system is the sun and planets, which are so far apart compared with their diameters that they can be treated as simple particles to good approximation. All particles in the solar system interact via gravitational attraction; the chief interaction is with the sun, although the interaction of the planets with each other also influences their motion. In addition, the entire solar system is attracted by far off matter.

At the other extreme, the system could be a billiard ball resting on a table. Here the particles are atoms (disregarding for now the fact that atoms are not point particles but are themselves composed of smaller particles) and the interactions are primarily interatomic electric forces. The external forces on the billiard ball include the gravitational force of the earth and the contact force of the tabletop.

We shall now prove some simple properties of physical systems. We are free to choose the boundaries of the system as we please, but once the choice is made, we must be consistent about which particles are included in the system and which are not. We suppose that the particles in the system interact with particles outside the system as well as with each other. To make the argument general, consider a system of N interacting particles with masses $m_1, m_2, m_3, \ldots, m_N$. The position of the jth particle is $\mathbf{r}_j$, the force on it is $\mathbf{f}_j$, and its momentum is $\mathbf{p}_j = m_j\dot{\mathbf{r}}_j$. The equation of motion for the jth particle is

$$\mathbf{f}_j = \frac{d\mathbf{p}_j}{dt}. \tag{3.4}$$

The force on particle j can be split into two terms:

$$\mathbf{f}_j = \mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}}. \tag{3.5}$$

Here $\mathbf{f}_j^{\text{int}}$, the *internal* force on particle j, is the force due to all other particles in the system, and $\mathbf{f}_j^{\text{ext}}$, the *external* force on particle j, is the force due to sources outside the system. The equation of motion becomes

$$\mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}} = \frac{d\mathbf{p}_j}{dt}. \tag{3.6}$$

Now let us focus on the system as a whole by the following stratagem: add all the equations of motion of all the particles in the system.

$$\mathbf{f}_1^{\text{int}} + \mathbf{f}_1^{\text{ext}} = \frac{d\mathbf{p}_1}{dt}$$

$$\cdots\cdots\cdots$$

$$\mathbf{f}_j^{\text{int}} + \mathbf{f}_j^{\text{ext}} = \frac{d\mathbf{p}_j}{dt} \qquad 3.7$$

$$\cdots\cdots\cdots$$

$$\mathbf{f}_N^{\text{int}} + \mathbf{f}_N^{\text{ext}} = \frac{d\mathbf{p}_N}{dt}.$$

The result of adding these equations can be written

$$\Sigma\mathbf{f}_j^{\text{int}} + \Sigma\mathbf{f}_j^{\text{ext}} = \sum \frac{d\mathbf{p}_j}{dt}. \qquad 3.8$$

The summations extend over all particles, $j = 1, \ldots, N$.

The second term, $\Sigma\mathbf{f}_j^{\text{ext}}$, is the sum of all external forces acting on all the particles. It is the *total external force* acting on the system, $\mathbf{F}_{\text{ext}}$.

$$\Sigma\mathbf{f}_j^{\text{ext}} \equiv \mathbf{F}_{\text{ext}}.$$

The first term in Eq. (3.8), $\Sigma\mathbf{f}_j^{\text{int}}$, is the sum of all internal forces acting on all the particles. According to Newton's third law, the forces between any two particles are equal and opposite so that their sum is zero. It follows that the sum of all the forces between all the particles is also zero; the internal forces cancel in pairs. Hence

$$\Sigma\mathbf{f}_j^{\text{int}} = 0.$$

Equation (3.8) then simplifies to

$$\mathbf{F}_{\text{ext}} = \sum \frac{d\mathbf{p}_j}{dt}. \qquad 3.9$$

The right hand side can be written $\Sigma(d\mathbf{p}_j/dt) = (d/dt)\Sigma\mathbf{p}_j$, since the derivative of a sum is the sum of the derivatives. $\Sigma\mathbf{p}_j$ is the *total momentum* of the system, which we designate by $\mathbf{P}$.

$$\mathbf{P} \equiv \Sigma\mathbf{p}_j. \qquad 3.10$$

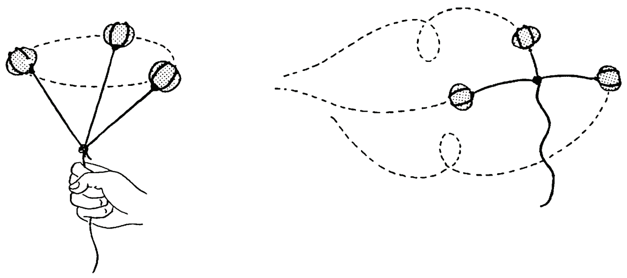
With this substitution, Eq. (3.9) becomes

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt}. \qquad 3.11$$

In words, the total external force applied to a system equals the rate of change of the system's momentum. This is true irrespective of the details of the interaction; $\mathbf{F}_{\text{ext}}$ could be a single force acting on a single particle, or it could be the resultant of many tiny interactions involving each particle of the system.

Example 3.1 The Bola

The bola is a weapon used by gauchos for entangling animals. It consists of three balls of stone or iron connected by thongs. The gaucho whirls the bola in the air and hurls it at the animal. What can we say about its motion?



Consider a bola with masses m_1, m_2, and m_3. The balls are pulled by the binding thong and by gravity. (We neglect air resistance.) Since the constraining forces depend on the instantaneous positions of all three balls, it is a real problem even to write the equation of motion of one ball. However, the total momentum obeys the simple equation

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} = \mathbf{f}_1^{\text{ext}} + \mathbf{f}_2^{\text{ext}} + \mathbf{f}_3^{\text{ext}}$$

$$= m_1\mathbf{g} + m_2\mathbf{g} + m_3\mathbf{g}$$

or

$$\frac{d\mathbf{P}}{dt} = M\mathbf{g},$$

where M is the total mass. This equation represents an important first step in finding the detailed motion. The equation is identical to that of a single particle of mass M with momentum $\mathbf{P}$. This is a familiar fact to the gaucho who forgets that he has a complicated system when he hurls the bola; he instinctively aims it like a single mass.

Center of Mass

According to Eq. (3.11),

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}, \qquad 3.12$$

where we have dropped the subscript *ext* with the understanding that **F** stands for the external force. This result is identical to the equation of motion of a single particle, although in fact it refers to a system of particles. It is tempting to push the analogy between Eq. (3.12) and single particle motion even further by writing

$$\mathbf{F} = M\ddot{\mathbf{R}}, \qquad 3.13$$

where M is the total mass of the system and **R** is a vector yet to be defined. Since $\mathbf{P} = \Sigma m_j\dot{\mathbf{r}}_j$, Eq. (3.12) and (3.13) give

$$M\ddot{\mathbf{R}} = \frac{d\mathbf{P}}{dt} = \Sigma m_j\ddot{\mathbf{r}}_j,$$

which is true if

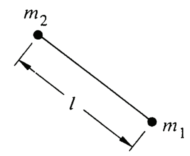
$$\mathbf{R} = \frac{1}{M}\Sigma m_j\mathbf{r}_j. \qquad 3.14$$

R is a vector from the origin to the point called the *center of mass*. The system behaves as if all the mass is concentrated at the center of mass and all the external forces act at that point.

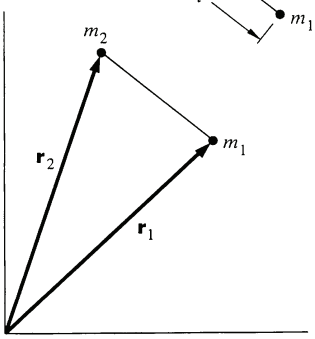
We are often interested in the motion of comparatively rigid bodies like baseballs or automobiles. Such a body is merely a system of particles which are fixed relative to each other by strong internal forces; Eq. (3.13) shows that with respect to external forces, the body behaves as if it were a point particle. In Chap. 2, we casually treated every body as if it were a particle; we see now that this is justified provided that we focus attention on the center of mass.

You may wonder whether this description of center of mass motion isn't a gross oversimplification—experience tells us that an extended body like a plank behaves differently from a compact body like a rock, even if the masses are the same and we apply the same force. We are indeed oversimplifying. The relation $\mathbf{F} = M\ddot{\mathbf{R}}$ describes only the translation of the body (the motion of its center of mass); it does not describe the body's orientation in space. In Chaps. 6 and 7 we shall investigate the rotation of extended bodies, and it will turn out that the rotational motion of a body depends both on its shape and the point where the forces are applied. Nevertheless, as far as translation of the center of mass is concerned, $\mathbf{F} = M\ddot{\mathbf{R}}$ tells the whole story. This result is true for any system of particles, not just for those fixed in rigid objects, as long as the forces between the particles obey Newton's third law. It is immaterial whether or not the particles move relative to each other and whether or not there happens to be any matter at the center of mass.

Example 3.2 Drum Major's Baton



A drum major's baton consists of two masses m_1 and m_2 separated by a thin rod of length l. The baton is thrown into the air. The problem is to find the baton's center of mass and the equation of motion for the center of mass.

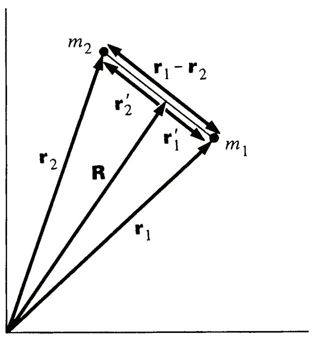


Let the position vectors of m_1 and m_2 be $\mathbf{r}_1$ and $\mathbf{r}_2$. The position vector of the center of mass, measured from the same origin, is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad 1$$

where we have neglected the mass of the thin rod. The center of mass lies on the line joining m_1 and m_2. To show this, suppose first that the tip of $\mathbf{R}$ does not lie on the line, and consider the vectors $\mathbf{r}_1'$, $\mathbf{r}_2'$ from the tip of $\mathbf{R}$ to m_1 and m_2. From the sketch we see that

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R}$$
$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R}.$$



Using Eq. (1) gives

$$\mathbf{r}_1' = \mathbf{r}_1 - \frac{m_1\mathbf{r}_1}{m_1 + m_2} - \frac{m_2\mathbf{r}_2}{m_1 + m_2}$$
$$= \frac{m_2}{m_1 + m_2}(\mathbf{r}_1 - \mathbf{r}_2)$$
$$\mathbf{r}_2' = \mathbf{r}_2 - \frac{m_1\mathbf{r}_1}{m_1 + m_2} - \frac{m_2\mathbf{r}_2}{m_1 + m_2}$$
$$= -\left(\frac{m_1}{m_1 + m_2}\right)(\mathbf{r}_1 - \mathbf{r}_2).$$

$\mathbf{r}_1'$ and $\mathbf{r}_2'$ are proportional to $\mathbf{r}_1 - \mathbf{r}_2$, the vector from m_1 to m_2. Hence $\mathbf{r}_1'$ and $\mathbf{r}_2'$ lie along the line joining m_1 and m_2, as shown. Furthermore,

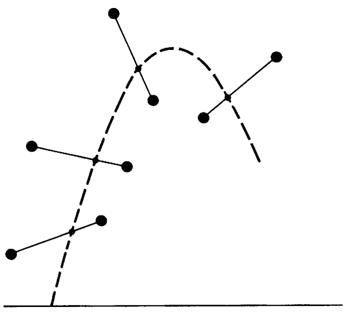
$$r_1' = \frac{m_2}{m_1 + m_2} |\mathbf{r}_1 - \mathbf{r}_2|$$

$$= \frac{m_2}{m_1 + m_2} l$$

and

$$r_2' = \frac{m_1}{m_1 + m_2} |\mathbf{r}_1 - \mathbf{r}_2|$$

$$= \frac{m_1}{m_1 + m_2} l.$$



Assuming that friction is negligible, the external force on the baton is

$\mathbf{F} = m_1\mathbf{g} + m_2\mathbf{g}.$

The equation of motion of the center of mass is

$(m_1 + m_2)\ddot{\mathbf{R}} = (m_1 + m_2)\mathbf{g}$

or

$\ddot{\mathbf{R}} = \mathbf{g}.$

The center of mass follows the parabolic trajectory of a single mass in a uniform gravitational field. With the methods developed in Chap. 6, we shall be able to find the motion of m_1 and m_2 about the center of mass, completing the solution to the problem.

Although it is a simple matter to find the center of mass of a system of particles, the procedure for locating the center of mass of an extended body is not so apparent. However, it is a straightforward task with the help of calculus. We proceed by dividing the body into N mass elements. If $\mathbf{r}_j$ is the position of the jth element, and m_j is its mass, then

$$\mathbf{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \mathbf{r}_j.$$

The result is not rigorous, since the mass elements are not true particles. However, in the limit where N approaches infinity, the size of each element approaches zero and the approximation becomes exact.

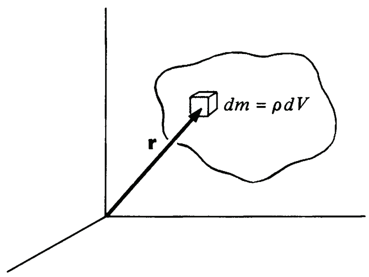
$$\mathbf{R} = \lim_{N\to\infty} \frac{1}{M} \sum_{j=1}^{N} m_j \mathbf{r}_j.$$

This limiting process defines an integral. Formally

$$\lim_{N\to\infty} \sum_{j=1}^{\infty} m_j \mathbf{r}_j = \int \mathbf{r}\, dm,$$

where dm is a differential mass element. Then

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,dm. \qquad 3.15$$



To visualize this integral, think of dm as the mass in an element of volume dV located at position $\mathbf{r}$. If the mass density at the element is ρ, then $dm = \rho\, dV$ and

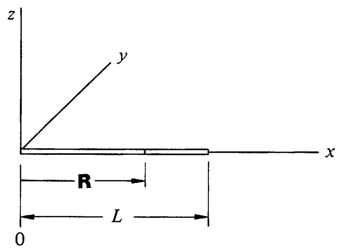
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\rho\, dV.$$

This integral is called a volume integral. Although it is important to know how to find the center of mass of rigid bodies, we shall only be concerned with a few simple cases here, as illustrated by the following two examples. Further examples are given in Note 3.1 at the end of the chapter.

Example 3.3 Center of Mass of a Nonuniform Rod

A rod of length L has a nonuniform density. λ, the mass per unit length of the rod, varies as $\lambda = \lambda_0(s/L)$, where λ_0 is a constant and s is the distance from the end marked 0. Find the center of mass.

It is apparent that $\mathbf{R}$ lies on the rod. Let the origin of the coordinate system coincide with the end of the rod, 0, and let the x axis lie along the rod so that $s = x$. The mass in an element of length dx is $dm = \lambda\, dx = \lambda_0 x\, dx/L$. The rod extends from $x = 0$ to $x = L$ and the total mass is

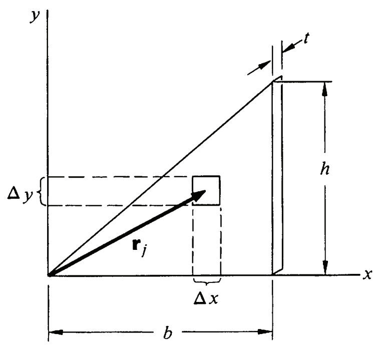


$$\begin{aligned} M &= \int dm \\ &= \int_0^L \lambda\, dx \\ &= \int_0^L \frac{\lambda_0 x\, dx}{L} \\ &= \tfrac{1}{2}\lambda_0 L. \end{aligned}$$

The center of mass is at

$$\begin{aligned} \mathbf{R} &= \frac{1}{M}\int \mathbf{r}\lambda\, dM \\ &= \frac{2}{\lambda_0 L}\int_0^L (x\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}})\,\frac{\lambda_0 x\, dx}{L} \\ &= \frac{2}{L^2}\frac{\hat{\mathbf{i}}}{3}x^3\Big|_0^L \\ &= \tfrac{2}{3}L\hat{\mathbf{i}}. \end{aligned}$$

Example 3.4 Center of Mass of a Triangular Sheet



Consider the two dimensional case of a uniform right triangular sheet of mass M, base b, height h, and small thickness t. If we divide the sheet into small rectangular areas of side Δx and Δy, as shown, then the volume of each element is $\Delta V = t\,\Delta x\,\Delta y$, and

$$\mathbf{R} \approx \frac{\Sigma m_j \mathbf{r}_j}{M}$$
$$= \frac{\Sigma \rho_j t\,\Delta x\,\Delta y \mathbf{r}_j}{M},$$

where j is the label of one of the volume elements and ρ_j is the density. Because the sheet is uniform,

$$\rho_j = \text{constant} = \frac{M}{V} = \frac{M}{At},$$

where A is the area of the sheet.

We can carry out the sum by summing first over the Δx's and then over the Δy's, instead of over the single index j. This gives a double sum which can be converted to a double integral by taking the limit, as follows:

$$\mathbf{R} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{M}{At}\right)\left(\frac{t}{M}\right) \Sigma\Sigma \mathbf{r}_j\,\Delta x\,\Delta y$$
$$= \frac{1}{A}\iint \mathbf{r}\,dx\,dy.$$

Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ be the position vector of an element $dx\,dy$. Then, writing $\mathbf{R} = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}$, we have

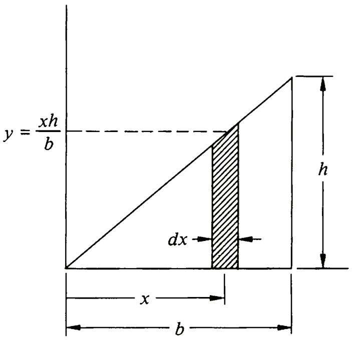
$$\mathbf{R} = X\hat{\mathbf{i}} + Y\hat{\mathbf{j}}$$
$$= \frac{1}{A}\iint (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\,dx\,dy$$
$$= \frac{1}{A}\left(\iint x\,dx\,dy\right)\hat{\mathbf{i}} + \frac{1}{A}\left(\iint y\,dx\,dy\right)\hat{\mathbf{j}}.$$

Hence the coordinates of the center of mass are given by

$$X = \frac{1}{A}\iint x\,dx\,dy$$
$$Y = \frac{1}{A}\iint y\,dx\,dy.$$

The double integrals may look strange, but they are easily evaluated. Consider first the double integral

$$X = \frac{1}{A}\iint x\,dx\,dy.$$

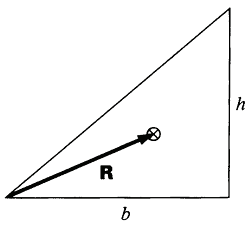


This integral instructs us to take each element, multiply its area by its x coordinate, and sum the results. We can do this in stages by first considering the elements in a strip parallel to the y axis. The strip runs from $y = 0$ to $y = xh/b$. Each element in the strip has the same x coordinate, and the contribution of the strip to the double integral is

$$\frac{1}{A}\,x\,dx\int_0^{xh/b} dy = \frac{h}{bA}\,x^2\,dx.$$

Finally, we sum the contributions of all such strips $x = 0$ to $x = b$ to find

$$X = \frac{h}{bA}\int_0^b x^2\,dx = \frac{h}{bA}\frac{b^3}{3}$$
$$= \frac{hb^2}{3A}.$$



Since $A = \frac{1}{2}bh$,

$$X = \tfrac{2}{3}b.$$

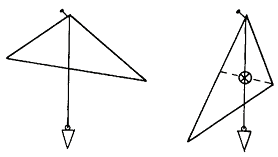
Similarly,

$$Y = \frac{1}{A}\int_0^b\left(\int_0^{xh/b} y\,dy\right)dx$$
$$= \frac{h^2}{2Ab^2}\int_0^b x^2\,dx = \frac{h^2b}{6A}$$
$$= \tfrac{1}{3}h.$$

Hence

$$\mathbf{R} = \tfrac{2}{3}b\hat{\mathbf{i}} + \tfrac{1}{3}h\hat{\mathbf{j}}.$$

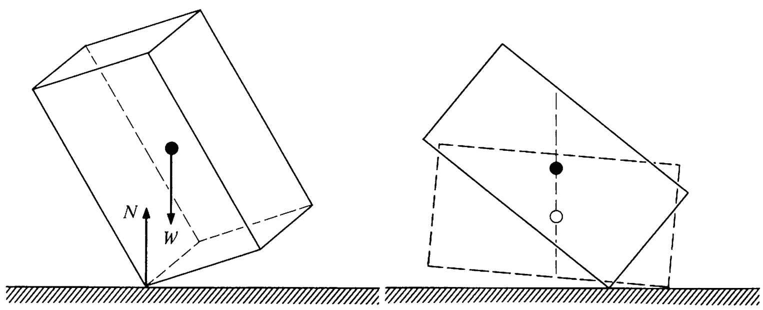
Although the coordinates of **R** depend on the particular coordinate system we choose, the position of the center of mass with respect to the triangular plate is, of course, independent of the coordinate system.



Often physical arguments are more useful than mathematical analysis. For instance, to find the center of mass of an irregular plane object, let it hang from a pivot and draw a plumb line from the pivot. The center of mass will hang directly below the pivot (this may be intuitively be obvious, and it can easily be proved with the methods of Chap. 6), and it is somewhere on the plumb line. Repeat the procedure with a different pivot point. The two lines intersect at the center of mass.

Example 3.5 Center of Mass Motion

A rectangular box is held with one corner resting on a frictionless table and is gently released. It falls in a complex tumbling motion, which we are not yet prepared to solve because it involves rotation. However, there is no difficulty in finding the trajectory of the center of mass.



The external forces acting on the box are gravity and the normal force of the table. Neither of these has a horizontal component, and so the center of mass must accelerate vertically. For a uniform box, the center of mass is at the geometrical center. If the box is released from rest, then its center falls straight down.

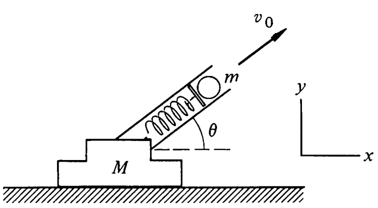
3.3 Conservation of Momentum

In the last section we found that the total external force **F** acting on a system is related to the total momentum **P** of the system by

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}.$$

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case $\mathbf{F} = 0$, and $d\mathbf{P}/dt = 0$. The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system is constant. This is the law of conservation of momentum. As we shall show, this apparently simple law can provide powerful insights into complicated systems.

Example 3.6 **Spring Gun Recoil**



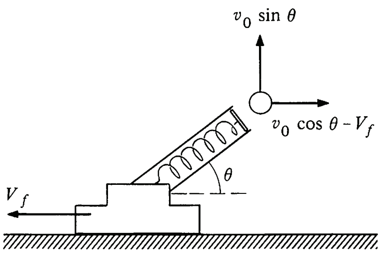
A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is v_0. What is the final motion of the gun?

Take the physical system to be the gun and marble. Gravity and the normal force of the table act on the system. Both these forces are vertical. Since there are no horizontal external forces, the x component of the vector equation $\mathbf{F} = d\mathbf{P}/dt$ is

$$0 = \frac{dP_x}{dt}. \qquad 1$$

According to Eq. (1), P_x is conserved:

$$P_{x,\text{initial}} = P_{x,\text{final}}. \qquad 2$$



Let the initial time be prior to firing the gun. Then $P_{x,\text{initial}} = 0$, since the system is initially at rest. After the marble has left the muzzle, the gun recoils with some speed V_f, and its final horizontal momentum is MV_f, to the left. Finding the final velocity of the marble involves a subtle point, however. Physically, the marble's acceleration is due to the force of the gun, and the gun's recoil is due to the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so that at the instant the marble and the gun part company, the gun has its final speed V_f. At that same instant the speed of the marble *relative to the gun* is v_0. Hence, the final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$. By conservation of horizontal momentum, we therefore have

$$0 = m(v_0 \cos\theta - V_f) - MV_f$$

or

$$V_f = \frac{mv_0 \cos\theta}{M + m}.$$

By using conservation of momentum we found the final motion of the system in a few steps. To show the advantage of this method, let us repeat the problem using Newton's laws directly.

Let $\mathbf{v}(t)$ be the velocity of marble at time t and let $\mathbf{V}(t)$ be the velocity of the gun. While the marble is being fired, it is acted on by the spring, by gravity, and by friction forces with the muzzle wall. Let the net force on the marble be $\mathbf{f}(t)$. The x equation of motion for the marble is

$$m\frac{dv_x}{dt} = f_x(t). \qquad 3$$

Formal integration of Eq. (3) gives

$$mv_x(t) = mv_x(0) + \int_0^t f_x\,dt. \qquad 4$$

The external forces are all vertical, and therefore the horizontal force f_x on the marble is due entirely to the gun. By Newton's third law, there is a reaction force $-f_x$ on the gun due to the marble. No other horizontal forces act on the gun, and the horizontal equation of motion for the gun is therefore

$$M\frac{dV_x}{dt} = -f_x(t),$$

which can be integrated to give

$$MV_x(t) = MV_x(0) - \int_0^t f_x\,dt. \qquad 5$$

We can eliminate the integral by combining Eqs. (4) and (5):

$$MV_x(t) + mv_x(t) = MV_x(0) + mv_x(0). \qquad 6$$

We have rediscovered that the horizontal component of momentum is conserved.

What about the motion of the center of mass? Its horizontal velocity is

$$\dot{R}_x(t) = \frac{MV_x(t) + mv_x(t)}{M + m}.$$

Using Eq. (6), the numerator can be rewritten to give

$$\dot{R}_x(t) = \frac{MV_x(0) + mv_x(0)}{M + m} = 0,$$

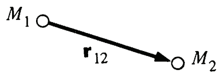
since the system is initially at rest. R_x is constant, as we expect.

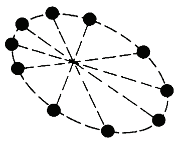
We did not include the small force of air friction. Would the center of mass remain at rest if we had included it?

The essential step in our derivation of the law of conservation of momentum was to use Newton's third law. Thus, conservation of momentum appears to be a natural consequence of newtonian mechanics. It has been found, however, that conservation of momentum holds true even in areas where newtonian mechanics proves inadequate, including the realms of quantum mechanics and relativity. In addition, conservation of momentum can be generalized to apply to systems like the electromagnetic field, which possess momentum but not mass. For these reasons, conservation of momentum is generally regarded as being more fundamental than newtonian mechanics. From this point of view, Newton's third law is a simple consequence of conservation of momentum for interacting particles. For our present purposes it is purely a matter of taste whether we wish to regard Newton's third law or conservation of momentum as more fundamental.

Example 3.7 Earth, Moon, and Sun—a Three Body System

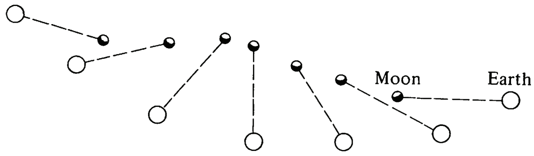
Newton was the first to calculate the motion of two gravitating bodies. As we shall discuss in Chap. 9, two bodies of mass M_1 and M_2 bound by gravity move so that $\mathbf{r}_{12}$ traces out an ellipse. The sketch shows the motion in a frame in which the center of mass is at rest. (Note that the center of mass of two particles lies on the line joining them.)

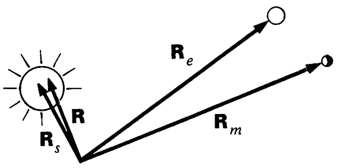




There is no general analytical solution for the motion of three gravitating bodies, however. In spite of this, we can explain many of the important features of the motion with the help of the concept of center of mass.

At first glance, the motion of the earth-moon-sun system appears to be quite complex. In the absence of the sun, the earth and moon would execute elliptical motion about their center of mass. As we shall now show, that center of mass orbits the sun like a single planet, to good approximation. The total motion is the simple result of two simultaneous elliptical orbits.

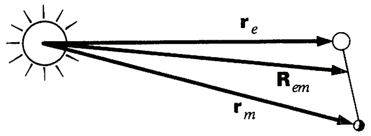




The center of mass of the earth-moon-sun system lies at

$$\mathbf{R} = \frac{M_e\mathbf{R}_e + M_m\mathbf{R}_m + M_s\mathbf{R}_s}{M_e + M_m + M_s},$$

where M_e, M_m, and M_s are the masses of the earth, moon, and sun, respectively. The sun's mass is so large compared with the mass of the earth or the moon that $\mathbf{R}_0 \approx \mathbf{R}_s$, and to good approximation the center of mass of the three body system lies at the center of the sun. Since external forces are negligible, the sun is effectively at rest in an inertial frame and it is natural to use a coordinate system with its origin at the center of the sun so that $\mathbf{R} = 0$.



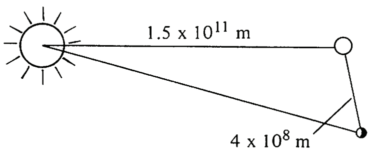
Let r_e and r_m be the positions of the earth and moon with respect to the sun, and let us focus for the moment on the system composed of the earth and moon. Their center of mass lies at

$$\mathbf{R}_{em} = \frac{M_e\mathbf{r}_e + M_m\mathbf{r}_m}{M_e + M_m}.$$

The external force on the earth-moon system is the gravitational pull of the sun:

$$\mathbf{F} = -GM_s\left(\frac{M_e}{r_e^2}\hat{\mathbf{r}}_e + \frac{M_m}{r_m^2}\hat{\mathbf{r}}_m\right).$$

The equation of motion of the center of mass is



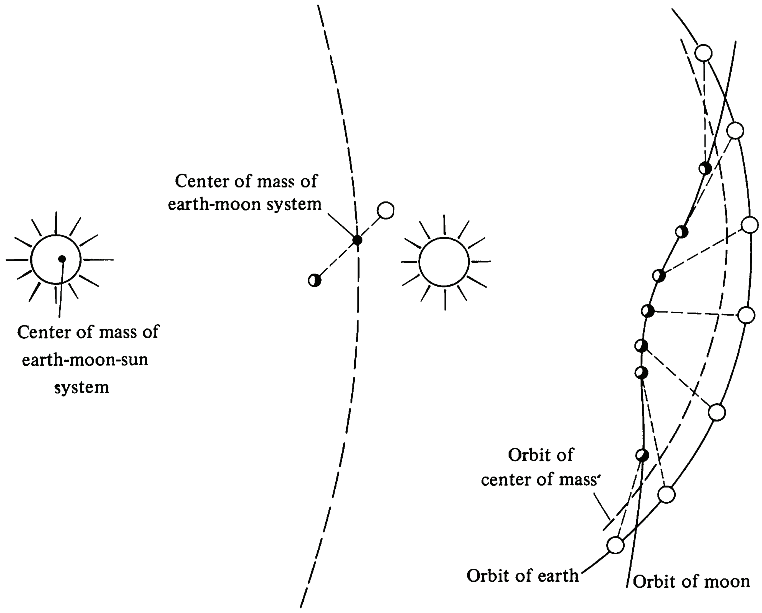
$$(M_e + M_m)\ddot{\mathbf{R}}_{em} = \mathbf{F}.$$

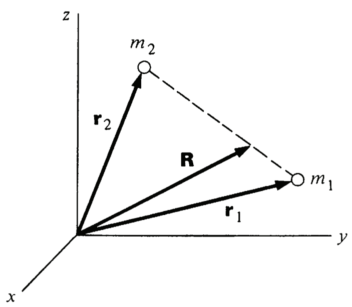
The earth and moon are so close compared with their distance from the sun that we shall not make a large error if we assume $r_e \approx r_m \approx R_{em}$. With this approximation,

$$(M_e + M_m)\ddot{\mathbf{R}}_{em} \approx \frac{-GM_s}{R^2}(M_e\hat{\mathbf{r}}_e + M_m\hat{\mathbf{r}}_m)$$

$$= \frac{-GM_s(M_e + M_m)\hat{\mathbf{R}}_{em}}{R^2}.$$

The center of mass of the earth and moon moves like a planet of mass $M_e + M_m$ about the sun. The total motion is the combination of this elliptical motion and the elliptical motion of the earth and moon about their center of mass, as illustrated on the opposite page. (The drawing is not to scale: the center of mass of the earth-moon system lies within the earth, and the moon's orbit is always concave toward the sun. Also, the plane of the moon's orbit is inclined by 5° with respect to the earth's orbit around the sun.)

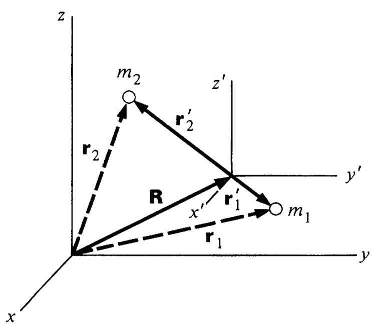




Center of Mass Coordinates

Often a problem can be simplified by the right choice of coordinates. The center of mass coordinate system, in which the origin lies at the center of mass, is particularly useful. The drawing illustrates the case of a two particle system with masses m_1 and m_2. In the initial coordinate system, x, y, z, the particles are located at $\mathbf{r}_1$ and $\mathbf{r}_2$ and their center of mass is at

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}.$$



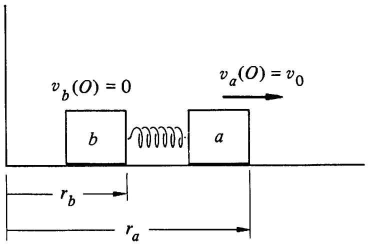
We now set up the center of mass coordinate system, x', y', z', with its origin at the center of mass. The origins of the old and new system are displaced by **R**. The center of mass coordinates of the two particles are

$$\mathbf{r}'_1 = \mathbf{r}_1 - \mathbf{R}$$
$$\mathbf{r}'_2 = \mathbf{r}_2 - \mathbf{R}.$$

Center of mass coordinates are the natural coordinates for an isolated two body system. For such a system the motion of the center of mass is trivial—it moves uniformly. Furthermore,

$m_1\mathbf{r}_1' + m_2\mathbf{r}_2' = 0$ by the definition of center of mass, so that if the motion of one particle is known, the motion of the other particle follows directly. Here is an example.

Example 3.8 **The Push Me-Pull You**



Two identical blocks a and b both of mass m slide without friction on a straight track. They are attached by a spring of length l and spring constant k. Initially they are at rest. At $t = 0$, block a is hit sharply, giving it an instantaneous velocity v_0 to the right. Find the velocities for subsequent times. (Try this yourself if there is a linear air track available—the motion is quite unexpected.)

Since the system slides freely after the collision, the center of mass moves uniformly and therefore defines an inertial frame.

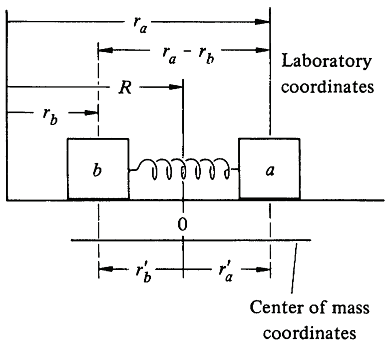
Let us transform to center of mass coordinates. The center of mass lies at

$$R = \frac{mr_a + mr_b}{m + m}$$
$$= \frac{1}{2}(r_a + r_b).$$

As expected, R is always halfway between a and b. The center of mass coordinates of a and b are

$$r_a' = r_a - R$$
$$= \tfrac{1}{2}(r_a - r_b)$$
$$r_b' = r_b - R$$
$$= -\tfrac{1}{2}(r_a - r_b)$$
$$= -r_a'.$$

The sketch below shows these coordinates.



The instantaneous length of the spring is $r_a - r_b - l = r_a' - r_b' - l$, where l is the unstretched length of the spring. The magnitude of the spring force is $k(r_a' - r_b' - l)$. The equations of motion in the center of mass system are

$$m\ddot{r}_a' = -k(r_a' - r_b' - l)$$
$$m\ddot{r}_b' = +k(r_a' - r_b' - l),$$

where l is the unstretched length of the spring. The form of these equations suggests that we subtract them, obtaining

$$m(\ddot{r}_a' - \ddot{r}_b') = -2k(r_a' - r_b' - l).$$

It is natural to introduce the departure of the spring from its equilibrium length as a variable. Letting $u = r_a' - r_b' - l$, we have

$$m\ddot{u} + 2ku = 0.$$

This is the equation for simple harmonic motion which we discussed in Example 2.14. The solution is

$$u = A \sin \omega t + B \cos \omega t,$$

where $\omega = \sqrt{2k/m}$. Since the spring is unstretched at $t = 0$, $u(0) = 0$ which requires $B = 0$. Furthermore, since $u = r_a' - r_b' - l = r_a - r_b - l$, we have at $t = 0$

$$\begin{aligned} \dot{u}(0) &= v_a(0) - v_b(0) \\ &= A\omega \cos (0) \\ &= v_0, \end{aligned}$$

so that

$$A = v_0/\omega$$

and

$$u = (v_0/\omega) \sin \omega t.$$

Since $v_a' - v_b' = \dot{u}$, and $v_a' = -v_b'$, we have

$$v_a' = -v_b' = \tfrac{1}{2}v_0 \cos \omega t.$$

The laboratory velocities are

$$v_a = \dot{R} + v_a'$$
$$v_b = \dot{R} + v_b'.$$

Since $\dot{R}$ is constant, it is always equal to its initial value

$$\dot{R} = \tfrac{1}{2}[v_a(0) + v_b(0)]$$
$$= \tfrac{1}{2}v_0.$$

Putting these together gives

$$v_a = \frac{v_0}{2}(1 + \cos \omega t)$$

$$v_b = \frac{v_0}{2}(1 - \cos \omega t).$$

The masses move to the right on the average, but they alternately come to rest in a push me–pull you fashion.

3.4 Impulse and a Restatement of the Momentum Relation

The relation between force and momentum is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}. \qquad 3.16$$

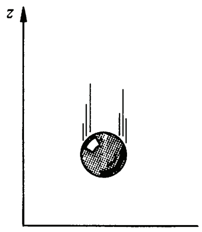
As a general rule, any law of physics which can be expressed in terms of derivatives can also be written in an integral form. The integral form of the force-momentum relationship is

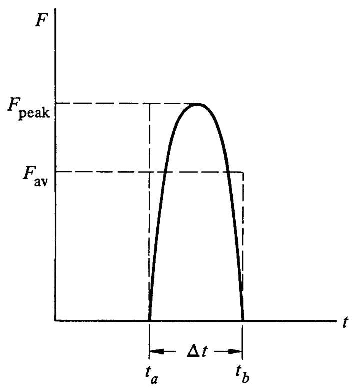
$$\int_0^t \mathbf{F}\, dt = \mathbf{P}(t) - \mathbf{P}(0). \qquad 3.17$$

The change in momentum of a system is given by the integral of force with respect to time. This form contains essentially the same physical information as Eq. (3.16), but it gives a new way of looking at the effect of a force: the change in momentum is the time integral of the force. To produce a given change in the momentum in time interval t requires only that $\int_0^t \mathbf{F}\, dt$ have the appropriate value; we can use a small force acting for much of the time or a large force acting for only part of the interval. The integral $\int_0^t \mathbf{F}\, dt$ is called the *impulse*. The word impulse calls to mind a short, sharp shock, as in Example 3.8, where we talked of giving a blow to a mass at rest so that its final velocity was v_0. However, the physical definition of impulse can just as well be applied to a weak force acting for a long time. Change of momentum depends only on $\int \mathbf{F}\, dt$, independent of the detailed time dependence of the force.

Here are two examples involving impulse.

Example 3.9 Rubber Ball Rebound





A rubber ball of mass 0.2 kg falls to the floor. The ball hits with a speed of 8 m/s and rebounds with approximately the same speed. High speed photographs show that the ball is in contact with the floor for 10^{-3} s. What can we say about the force exerted on the ball by the floor?

The momentum of the ball just before it hits the floor is $\mathbf{P}_a = -1.6\hat{\mathbf{k}}$ kg·m/s and its momentum 10^{-3} s later is $\mathbf{P}_b = +1.6\hat{\mathbf{k}}$ kg·m/s. Since $\int_{t_a}^{t_b} \mathbf{F}\,dt = \mathbf{P}_b - \mathbf{P}_a$, $\int_{t_a}^{t_b} \mathbf{F}\,dt = 1.6\hat{\mathbf{k}} - (-1.6\hat{\mathbf{k}}) = 3.2\hat{\mathbf{k}}$ kg·m/s. Although the exact variation of **F** with time is not known, it is easy to find the average force exerted by the floor on the ball. If the collision time is $\Delta t = t_b - t_a$, the average force $\mathbf{F}_{av}$ acting during the collision is

$$\mathbf{F}_{av}\,\Delta t = \int_{t_a}^{t_a+\Delta t} \mathbf{F}\,dt.$$

Since $\Delta t = 10^{-3}$ s,

$$\mathbf{F}_{av} = \frac{3.2\hat{\mathbf{k}}\ \text{kg·m/s}}{10^{-3}\ \text{s}} = 3{,}200\hat{\mathbf{k}}\ \text{N}.$$

The average force is directed upward, as we expect. In more familiar units, 3,200 N ≈ 720 lb—a sizable force. The instantaneous force on the ball is even larger at the peak, as the sketch shows. If the ball hits a resilient surface, the collision time is longer and the peak force is less.

Actually, there is a weakness in our treatment of the rubber ball rebound. In calculating the impulse $\int \mathbf{F}\,dt$, **F** is the total force. This includes the gravitational force, which we have neglected. Proceeding more carefully, we write

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_{floor} + \mathbf{F}_{grav}\\ &= \mathbf{F}_{floor} - Mg\hat{\mathbf{k}}.\end{aligned}$$

The impulse equation then becomes

$$\int_0^{10^{-3}} \mathbf{F}_{floor}\,dt - \int_0^{10^{-3}} Mg\hat{\mathbf{k}}\,dt = 3.2\hat{\mathbf{k}}\ \text{kg·m/s}.$$

The impulse due to the gravitational force is

$$\begin{aligned}-\int_0^{10^{-3}} Mg\hat{\mathbf{k}}\,dt = -Mg\hat{\mathbf{k}}\int_0^{10^{-3}} dt &= -(0.2)(9.8)(10^{-3})\hat{\mathbf{k}}\\ &= -1.96 \times 10^{-3}\hat{\mathbf{k}}\ \text{kg·m/s}.\end{aligned}$$

This is less than one-thousandth of the total impulse, and we can neglect it with little error. Over a long period of time, gravity can produce a large change in the ball's momentum (the ball gains speed as it falls, for example). In the short time of contact, however, gravity contributes little momentum change compared with the tremendous force exerted by the floor. Contact forces during a short collision are generally so huge that we can neglect the impulse due to other forces of moderate strength, such as gravity or friction.

The last example reveals why a quick collision is more violent than a slow collision, even when the initial and final velocities are identical. This is the reason that a hammer can produce a force far greater than the carpenter could produce on his own; the hard hammerhead rebounds in a very short time compared with the time of the hammer swing, and the force driving the hammer is correspondingly amplified. Many devices to prevent bodily injury in accidents are based on the same considerations, but applied in reverse—they essentially prolong the time of the collision. This is the rationale for the hockey player's helmet, as well as the automobile seat belt. The following example shows what can happen in even a relatively mild collision, as when you jump to the ground.

Example 3.10 How to Avoid Broken Ankles

Animals, including humans, instinctively reduce the force of impact with the ground by flexing while running or jumping. Consider what happens to someone who hits the ground with his legs rigid.

Suppose a man of mass M jumps to the ground from height h, and that his center of mass moves downward a distance s during the time of collision with the ground. The average force during the collision is

$$F = \frac{Mv_0}{t}, \qquad 1$$

where t is the time of the collision and v_0 is the velocity with which he hits the ground. As a reasonable approximation, we can take his acceleration due to the force of impact to be constant, so that the man comes uniformly to rest. In this case the collision time is given by $v_0 = 2s/t$, or

$$t = \frac{2s}{v_0}.$$

Inserting this in Eq. (1) gives

$$F = \frac{Mv_0^2}{2s}. \qquad 2$$

For a body in free fall for distance h,

$$v_0^2 = 2gh.$$

Inserting this in Eq. (2) gives

$$F = Mg\frac{h}{s}.$$

If the man hits the ground rigidly in a vertical position, his center of mass will not move far during the collision. Suppose that his center of mass moves 1 cm, which roughly means that his height momentarily decreases by approximately 2 cm. If he jumps from a height of 2 m, the force is 200 times his weight!

Consider the force on a 90-kg ($\approx$200-lb) man jumping from a height of 2 m. The force is

$$\begin{aligned} F &= 90 \text{ kg} \times 9.8 \text{ m/s}^2 \times 200 \\ &= 1.8 \times 10^5 \text{ N.} \end{aligned}$$

Where is a bone fracture most likely to occur? The force is a maximum at the feet, since the mass above a horizontal plane through the man decreases with height. Thus his ankles will break, not his neck. If the area of contact of bone at each ankle is 5 cm^2, then the force per unit area is

$$\begin{aligned} \frac{F}{A} &= \frac{1.8 \times 10^5 \text{ N}}{10 \text{ cm}^2} \\ &= 1.8 \times 10^4 \text{ N/cm}^2. \end{aligned}$$

This is approximately the compressive strength of human bone, and so there is a good probability that his ankles will snap.

Of course, no one would be so rash as to jump rigidly. We instinctively cushion the impact when jumping by flexing as we hit the ground, in the extreme case collapsing to the ground. If the man's center of mass drops 50 cm, instead of 1 cm, during the collision, the force is only one-fiftieth as much as we calculated, and there is no danger of compressive fracture.

3.5 Momentum and the Flow of Mass

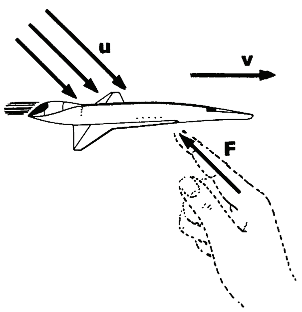
Analyzing the forces on a system in which there is a flow of mass becomes terribly confusing if we try to apply Newton's laws blindly. A rocket provides the most dramatic example of such a system, although there are many other everyday problems where the same considerations apply—for instance, the problem of calculating the reaction force on a fire hose, or of calculating the acceleration of a snowball which grows larger as it rolls downhill.

There is no fundamental difficulty in handling any of these problems provided that we keep clearly in mind exactly what is included in the system. Recall that $\mathbf{F} = d\mathbf{P}/dt$ [Eq. (3.12)] was established for a system composed of a certain set of particles. When we apply this equation in the integral form,

$$\int_{t_a}^{t_b} \mathbf{F}\, dt = \mathbf{P}(t_b) - \mathbf{P}(t_a),$$

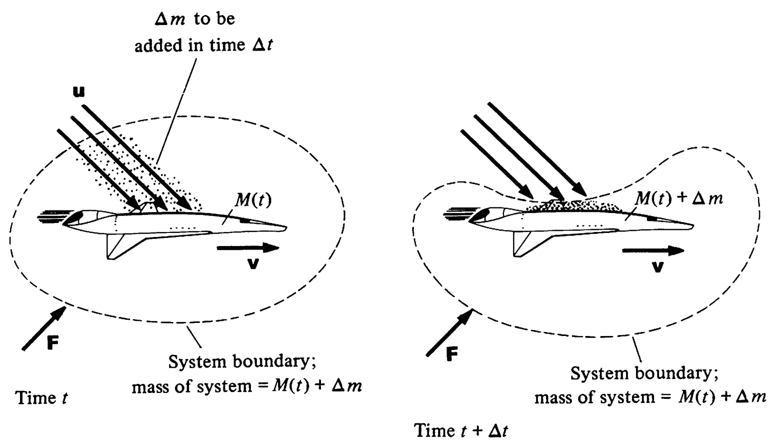
it is essential to deal with the same set of particles throughout the time interval t_a to t_b; we must keep track of all the particles that were originally in the system. Consequently, the mass of the system cannot change during the time of interest.

Example 3.11 **Mass Flow and Momentum**



A spacecraft moves through space with constant velocity **v**. The spacecraft encounters a stream of dust particles which embed themselves in it at rate dm/dt. The dust has velocity **u** just before it hits. At time t the total mass of the spacecraft is $M(t)$. The problem is to find the external force **F** necessary to keep the spacecraft moving uniformly. (In practice, **F** would most likely come from the spacecraft's own rocket engines. For simplicity, we can visualize the source **F** to be completely external—an invisible hand, so to speak.)

Let us focus on the short time interval between t and $t + \Delta t$. The drawings below show the system at the beginning and end of the interval.



Let Δm denote the mass added to the satellite during Δt. The system consists of $M(t)$ and Δm. The initial momentum is

$$\mathbf{P}(t) = M(t)\mathbf{v} + (\Delta m)\mathbf{u}.$$

The final momentum is

$$\mathbf{P}(t + \Delta t) = M(t)\mathbf{v} + (\Delta m)\mathbf{v}.$$

The change in momentum is

$$\begin{aligned}\Delta\mathbf{P} &= \mathbf{P}(t + \Delta t) - \mathbf{P}(t)\\ &= (\mathbf{v} - \mathbf{u})\,\Delta m.\end{aligned}$$

The rate of change of momentum is approximately

$$\frac{\Delta \mathbf{P}}{\Delta t} = (\mathbf{v} - \mathbf{u})\frac{\Delta m}{\Delta t}.$$

In the limit $\Delta t \to 0$, we have the exact result

$$\frac{d\mathbf{P}}{dt} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt}.$$

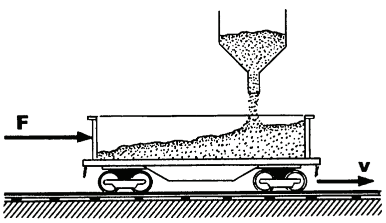
Since $\mathbf{F} = d\mathbf{P}/dt$, the required external force is

$$\mathbf{F} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt}.$$

Note that $\mathbf{F}$ can be either positive or negative, depending on the direction of the stream of mass. If $\mathbf{u} = \mathbf{v}$, the momentum of the system is constant, and $\mathbf{F} = 0$.

The procedure of isolating the system, focusing on differentials, and taking the limit may appear a trifle formal. However, the procedure is helpful in avoiding errors in a subject where it is easy to become confused. For instance, a frequent error is to argue that $\mathbf{F} = (d/dt)(m\mathbf{v}) = m(d\mathbf{v}/dt) + \mathbf{v}(dm/dt)$. In the last example $\mathbf{v}$ is constant, and the result would be $\mathbf{F} = \mathbf{v}(dm/dt)$ rather than $(\mathbf{v} - \mathbf{u})(dm/dt)$. The difficulty arises from the fact that there are several contributions to the momentum, so that the expression for the momentum of a single particle, $\mathbf{p} = m\mathbf{v}$, is not appropriate. The limiting procedure illustrated in the last example avoids such ambiguities.

Example 3.12 Freight Car and Hopper



Sand falls from a stationary hopper onto a freight car which is moving with uniform velocity v. The sand falls at the rate dm/dt. How much force is needed to keep the freight car moving at the speed v?

In this case, the initial speed of the sand is 0, and

$$\frac{dP}{dt} = (v - u)\left(\frac{dm}{dt}\right) = v\frac{dm}{dt}.$$

The required force is $F = v\,dm/dt$. We can understand why this force is needed by considering in detail just what happens to a sand grain as it lands on the surface of the freight car. What would happen if the surface of the freight car were slippery?

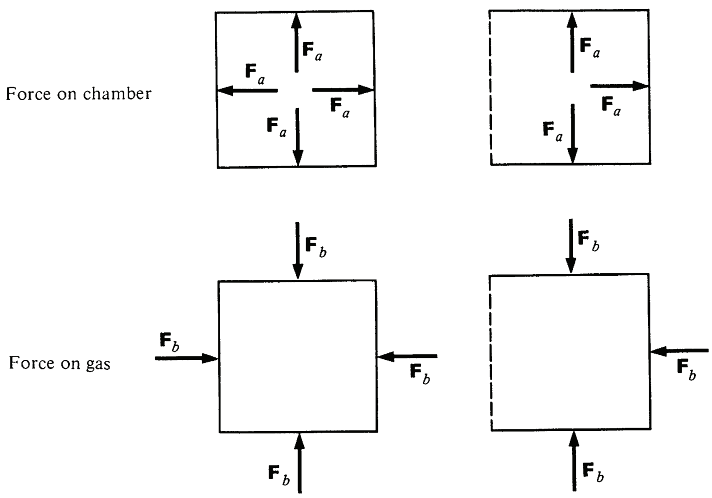
Example 3.13 Leaky Freight Car

Now consider a related case. The same freight car is leaking sand at the rate dm/dt; what force is needed to keep the freight car moving uniformly with speed **v**?

Here the mass is decreasing. However, the velocity of the sand after leaving the freight car is identical to its initial velocity, and its momentum does not change. Since $dP/dt = 0$, no force is required. (The sand does change its momentum when it hits the ground, and there is a resulting force on the ground, but that does not affect the motion of the freight car.)

The concept of momentum is invaluable in understanding the motion of a rocket. A rocket accelerates by expelling gas at a high velocity; the reaction force of the gas on the rocket accelerates the rocket in the opposite direction. The mechanism is illustrated by the drawings of the cubical chamber containing gas at high pressure.

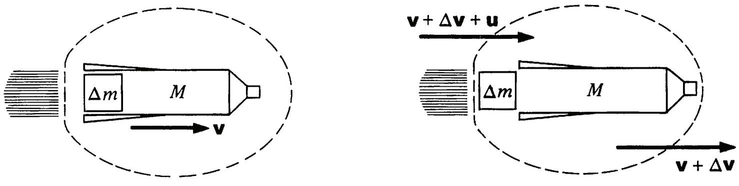
The gas presses outward on each wall with the force $\mathbf{F}_a$. (We show only four walls for clarity.) The vector sum of the $\mathbf{F}_a$'s is zero, giving zero net force on the chamber. Similarly each wall of the chamber exerts a force on the gas $\mathbf{F}_b = -\mathbf{F}_a$; the net force on the gas is also zero. In the right hand drawings below, one wall



has been removed. The net force on the chamber is F_a, to the right. The net force on the gas is F_b, to the left. Hence the gas accelerates to the left, and the chamber accelerates to the right.

To analyze the motion of the rocket in detail, we must equate the external force on the system, **F**, with the rate of change of momentum, $d\mathbf{P}/dt$. Consider the rocket at time t. Between t and $t + \Delta t$ a mass of fuel Δm is burned and expelled as gas with velocity **u** *relative to the rocket.*. The exhaust velocity **u** is determined by the nature of the propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket.

The sketches below show the system at time t and at time



$t + \Delta t$. The system consists of Δm plus the remaining mass of the rocket M. Hence the total mass is $M + \Delta m$.

The velocity of the rocket at time t is $\mathbf{v}(t)$, and at $t + \Delta t$, it is $\mathbf{v} + \Delta\mathbf{v}$. The initial momentum is

$$\mathbf{P}(t) = (M + \Delta m)\mathbf{v}$$

and the final momentum is

$$\mathbf{P}(t + \Delta t) = M(\mathbf{v} + \Delta\mathbf{v}) + \Delta m(\mathbf{v} + \Delta\mathbf{v} + \mathbf{u}).$$

The change in momentum is

$$\begin{aligned}\Delta\mathbf{P} &= \mathbf{P}(t + \Delta t) - \mathbf{P}(t)\\ &= M\,\Delta\mathbf{v} + (\Delta m)\mathbf{u}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d\mathbf{P}}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta\mathbf{P}}{\Delta t}\\ &= M\frac{d\mathbf{v}}{dt} + \mathbf{u}\frac{dm}{dt}\cdot\end{aligned} \qquad 3.18$$

Note that we have defined **u** to be positive in the direction of **v**. In most rocket applications, **u** is negative, opposite to **v**. It is inconvenient to have both m and M in the equation. dm/dt is the rate of increase of the exhaust mass. Since this mass comes from the rocket,

$$\frac{dm}{dt} = -\frac{dM}{dt}.$$

Using this in Eq. (3.18), and equating the external force to $d\mathbf{P}/dt$, we obtain the fundamental rocket equation

$$\mathbf{F} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}. \qquad 3.19$$

It may be useful to point out two minor subtleties in our development. The first is that the velocities have been expressed with respect to an inertial frame, not a frame attached to the rocket. The second is that we took the final velocity of the element of exhaust gas to be $\mathbf{v} + \Delta\mathbf{v} + \mathbf{u}$ rather than $\mathbf{v} + \mathbf{u}$. This is correct (consult Example 3.6 on spring gun recoil if you need help in seeing the reason), but actually it makes no difference here, since either expression yields the same final result when the limit is taken. Here are two examples on rockets.

Example 3.14 Rocket in Free Space

If there is no external force on a rocket, $\mathbf{F} = 0$ and its motion is given by

$$M\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dM}{dt}$$

or

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt}.$$

Generally the exhaust velocity $\mathbf{u}$ is constant, in which case it is easy to integrate the equation of motion.

$$\int_{t_0}^{t_f}\frac{d\mathbf{v}}{dt}\,dt = \mathbf{u}\int_{t_0}^{t_f}\frac{1}{M}\frac{dM}{dt}\,dt = \mathbf{u}\int_{M_0}^{M_f}\frac{dM}{M}$$

or

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\frac{M_f}{M_0} = -\mathbf{u}\ln\frac{M_0}{M_f}.$$

If $\mathbf{v}_0 = 0$, then

$$\mathbf{v}_f = -\mathbf{u} \ln \frac{M_0}{M_f}.$$

The final velocity is independent of how the mass is released—the fuel can be expended rapidly or slowly without affecting $\mathbf{v}_f$. The only important quantities are the exhaust velocity and the ratio of initial to final mass.

The situation is quite different if a gravitational field is present, as shown by the next example.

Example 3.15 Rocket in a Gravitational Field

If a rocket takes off in a constant gravitational field, Eq. (3.19) becomes

$$M\mathbf{g} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt},$$

where **u** and **g** are directed down and are assumed to be constant.

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt} + \mathbf{g}.$$

Integrating with respect to time, we obtain

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u} \ln\left(\frac{M_f}{M_0}\right) + \mathbf{g}(t_f - t_0).$$

Let $\mathbf{v}_0 = 0$, $t_0 = 0$, and take velocity positive upward.

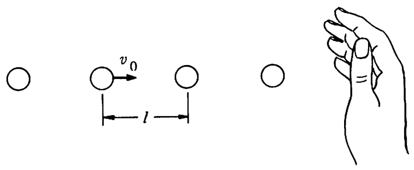
$$v_f = u \ln\left(\frac{M_0}{M_f}\right) - gt_f.$$

Now there is a premium attached to burning the fuel rapidly. The shorter the burn time, the greater the velocity. This is why the takeoff of a large rocket is so spectacular—it is essential to burn the fuel as quickly as possible.

3.6 Momentum Transport

Nearly everyone has at one time or another been on the receiving end of a stream of water from a hose. You feel a push. If the stream is intense, as in the case of a fire hose, the push can be dramatic—a jet of high pressure water can be used to break through the wall of a burning building.

The push of a water stream arises from the momentum it transfers to you. Unless another external force gives you equal momentum in the opposite direction, off you go. How can a column of water flying through the air exert a force which is every bit as real as a force transmitted by a rigid steel rod? The reason is easy to see if we picture the stream of water as a series of small uniform droplets of mass m, traveling with velocity v_0. Let the droplets be distance l apart and suppose that the stream is directed against your hand. Assume that the drops collide without rebound and simply run down your arm. Consider the force exerted by your hand on the stream. As each drop hits there is a large force for a short time. Although we do not know the instantaneous force, we can find the impulse I_{droplet} on each drop due to your hand.

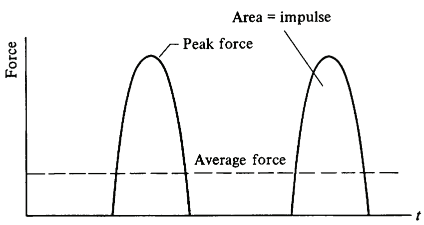


$$\begin{aligned} I_{\text{droplet}} &= \int_{1 \text{ collision}} F\, dt \\ &= \Delta p \\ &= m(v_f - v_0) \\ &= -mv_0. \end{aligned}$$

The impulse on your hand is equal and opposite.

$$I_{\text{hand}} = mv_0.$$

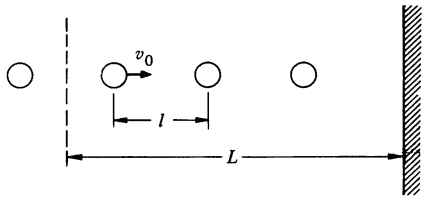
The positive sign means that the impulse on the hand is in the same direction as the velocity of the drop. The impulse equals the area under one of the peaks shown in the drawing. If there are many collisions per second, you do not feel the shock of each drop. Rather, you feel the average force F_{av} indicated by the dashed line in the drawing. The area under F_{av} during one collision period T (the time between collisions) is identical to the impulse due to one drop.



$$F_{\text{av}}T = \int_{1 \text{ collision}} F\, dt$$

Since $T = l/v_0$ and $\int F\, dt = mv_0$, the average force is

$$\begin{aligned} F_{\text{av}} &= \frac{mv_0}{T} \\ &= \frac{m}{l} v_0^2. \end{aligned}$$



Here is another way to find the average force. Consider length L of the stream just about to hit the surface. The number of drops in L is L/l, and since each drop has momentum mv_0, the total momentum is

$$\Delta p = \frac{L}{l} mv_0.$$

All these drops will strike the wall in time

$$\Delta t = \frac{L}{v_0}.$$

The average force is

$$F_{av} = \frac{\Delta p}{\Delta t}$$

$$= \frac{m}{l} v_0^2.$$

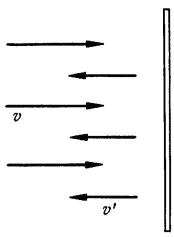
To apply this model to a fluid, consider a stream moving with speed v. If the mass per unit length is $m/l \equiv \lambda$, the momentum per unit length is λv and the rate at which the stream transports momentum to the surface is

$$\frac{dp}{dt} = \lambda v^2. \qquad 3.20$$

If the stream comes to rest at the surface, the force on the surface is

$$F = \lambda v^2. \qquad 3.21$$

Example 3.16 **Momentum Transport to a Surface**



A stream of particles of mass m and separation l hits a perpendicular surface with velocity v. The stream rebounds along the original line of motion with velocity v'. The mass per unit length of the incident stream is $\lambda = m/l$. What is the force on the surface?

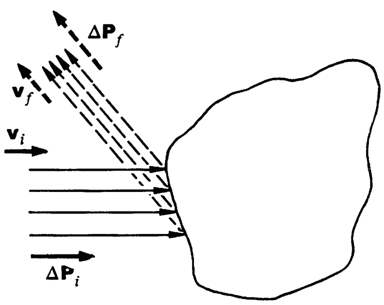
The incident stream transfers momentum to the surface at the rate λv^2. However, the reflected stream does not carry it away at the rate $\lambda v'^2$, since the density of the stream must change at the surface. The number of particles incident on the surface in time Δt is $v\,\Delta t/l$ and their total mass is $\Delta m = mv\,\Delta t/l$. Hence, the rate at which mass arrives at the surface is

$$\frac{dm}{dt} = \frac{m}{l} v = \lambda v.$$

The rate at which mass is carried away from the surface is $\lambda' v'$. Since mass does not accumulate on the surface, these rates must be equal. Hence $\lambda' v' = \lambda v$, and the force on the surface is

$$F = \frac{dp'}{dt} + \frac{dp}{dt} = \lambda' v'^2 + \lambda v^2$$
$$= \lambda v(v' + v).$$

If the stream collides without rebound, then $v' = 0$ and $F = \lambda v^2$, in agreement with our previous result. If the particles undergo perfect reflection, then $v' = v$, and $F = 2\lambda v^2$. The actual force lies somewhere between these extremes.



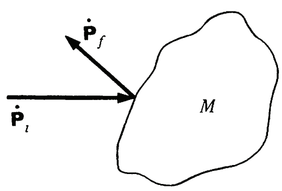
We can generalize the idea of momentum transport to three dimensions. Consider a stream of fluid which strikes an object and rebounds in some arbitrary direction. For simplicity we assume that the incident stream is uniform and that in time Δt it transports momentum $\Delta \mathbf{P}_i$. The direction of $\Delta \mathbf{P}_i$ is parallel to the initial velocity $\mathbf{v}_i$ and $\Delta P_i = \lambda_i v_i^2 \, \Delta t$. During the same interval Δt the rebounding stream carries away momentum $\Delta \mathbf{P}_f$, where $\Delta P_f = \lambda_f v_f^2 \, \Delta t$; the direction of $\Delta \mathbf{P}_f$ is parallel to the final velocity $\mathbf{v}_f$. The vectors are shown in the sketch.

The net momentum change of the fluid in Δt is

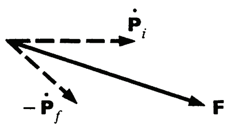
$$\Delta \mathbf{P}_{\text{fluid}} = \Delta \mathbf{P}_f - \Delta \mathbf{P}_i.$$

The rate of change of the fluid's momentum is

$$\left(\frac{d\mathbf{P}}{dt}\right)_{\text{fluid}} = \left(\frac{d\mathbf{P}}{dt}\right)_f - \left(\frac{d\mathbf{P}}{dt}\right)_i.$$



By Newton's second law, $(d\mathbf{P}/dt)_{\text{fluid}}$ equals the force on the fluid due to the object. By Newton's third law, the force on the object due to the fluid is



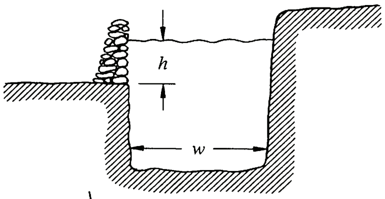
$$\mathbf{F} = -\left(\frac{d\mathbf{P}}{dt}\right)_{\text{fluid}}$$
$$= \left(\frac{d\mathbf{P}}{dt}\right)_i - \left(\frac{d\mathbf{P}}{dt}\right)_f$$
$$= \dot{\mathbf{P}}_i - \dot{\mathbf{P}}_f. \qquad 3.22$$

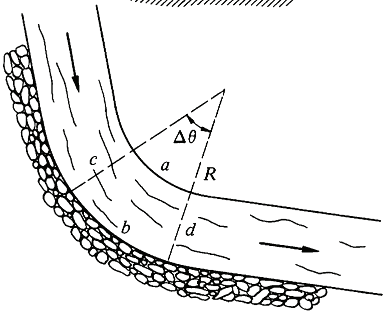
The sketches illustrate this result.

Unless there is some opposing force, the object will begin to accelerate. If $\dot{\mathbf{P}}_f = \dot{\mathbf{P}}_i$, the stream transfers no momentum and $\mathbf{F} = 0$.

The force on a moving airplane or boat can be found by considering the effect of a multitude of streams hitting the surface, each with its own velocity. Although the mathematical formalism for analyzing this would lead us too far afield, the physical principle is the same: momentum transport.

Example 3.17 A Dike at the Bend of a River

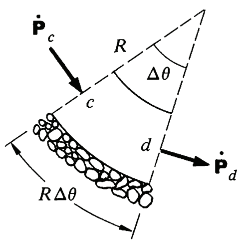




The problem is to build a dike at the bend of a river to prevent flooding when the river rises. Obviously the dike has to be strong enough to withstand the static pressure of the river $\rho g h$, where ρ is the density of the water and h is the height from the base of the dike to the surface of the water. However, because of the bend there is an additional pressure, the dynamic pressure due to the rush of water. How does this compare with the static pressure?

We approximate the bend by a circular curve with radius R, and focus our attention on a short length of the curve subtending angle $\Delta\theta$. We need only concern ourselves with that section of the river above the base of the dike, and we consider the volume of the river bounded by the bank a, the dike b, and two imaginary surfaces c and d. Momentum is transferred into the volume through surface c and out through surface d at rate $\dot{P} = \lambda v^2 = \rho A v^2$. Here A is the cross sectional area of the river lying above the base of the dike, $A = hw$. (Note that $\rho A = \lambda =$ mass per unit length of the river.)

However, surfaces c and d are not parallel. The rate of change of the stream's momentum is

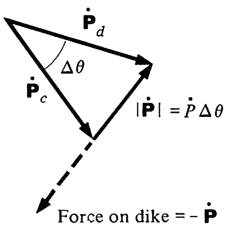


$$\dot{\mathbf{P}} = \dot{\mathbf{P}}_d - \dot{\mathbf{P}}_c.$$

As we can see from the vector drawing below, $\dot{\mathbf{P}}$ is radially inward and has magnitude

$$|\dot{\mathbf{P}}| = \dot{P}\,\Delta\theta.$$

The dynamic force on the dike is radially outward, and has the same magnitude, $\dot{P}\,\Delta\theta$. The force is exerted over the area $(R\,\Delta\theta)h$, and the dynamic pressure is therefore



$$\text{pressure} = \frac{\dot{P}\,\Delta\theta}{R\,\Delta\theta h} = \frac{\rho A v^2}{Rh} = \frac{\rho w v^2}{R}.$$

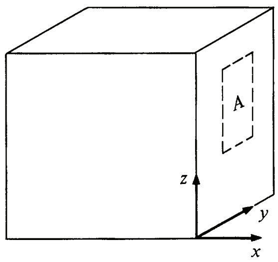
The ratio of dynamic to static pressure is

$$\frac{\text{dynamic pressure}}{\text{static pressure}} = \frac{\rho w v^2}{R}\frac{1}{\rho g h} = \frac{w}{h}\frac{v^2}{Rg}$$
$$= \frac{\text{width}}{\text{depth}} \times \frac{\text{centripetal acceleration}}{g}.$$

For a river in flood with a speed of 10 mi/h (approximately 14 ft/s), a radius of 2,000 ft, a flood height of 3 ft, and a width of 200 ft, the ratio is 0.22, so that the dynamic pressure is by no means negligible. The ratio is even larger near the surface of the river where the static pressure is small.

Example 3.18 **Pressure of a Gas**

As a further application of the idea of momentum transport, let us find the pressure exerted by a gas. Although our argument will be somewhat simpleminded, it exhibits the essential ideas and gives the same result as more refined arguments.



Assume that there are n atoms per unit volume of the gas, each having mass m, and that they move randomly. Let us find the force exerted on an area A in the yz plane due to motion of the atoms in the x direction. We make the plausible assumption that it is permissible to neglect motion in the y and z direction, and treat only motion parallel to the x axis. Suppose that all atoms have the same speed, v_x. The rate at which they hit the surface is $\frac{1}{2}nAv_x$, where the factor of $\frac{1}{2}$ is introduced because the atoms can move in either direction with equal probability. The momentum carried by each atom is mv_x. It is unlikely that the atoms come to rest after the collision; this would correspond to the freezing of the gas on the walls. On the average, they must leave at the same rate as they arrive, which means that the average change in momentum is $2mv_x$. Hence, the rate at which momentum changes due to collisions with area A is

$$\frac{dp}{dt} = \left(\frac{1}{2}nAv_x\right)(2mv_x)$$
$$= mnAv_x^2.$$

The force is

$$F = \frac{dp}{dt}$$
$$= mnAv_x^2$$

and the pressure P_x on the x surface is

$$P_x = \frac{F}{A}$$
$$= mnv_x^2.$$

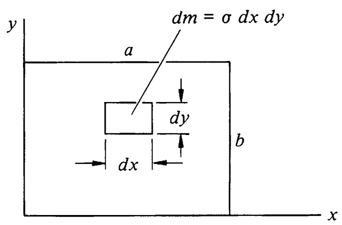
The assumption that v_x has a fixed value is actually unnecessary. If the atoms have many different instantaneous speeds, then it can be shown that v_x^2 should be replaced by its average $\overline{v_x^2}$, and $P_x = nm\overline{v_x^2}$. By an identical argument we have $P_y = mn\overline{v_y^2}$ and $P_z = nm\overline{v_z^2}$. However, since the pressure of a gas should not depend on direction, we have $P_x = P_y = P_z$, which implies that $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$. The mean squared velocity is $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$, so that $\overline{v_x^2} = \frac{1}{3}\overline{v^2}$ and the pressure is

$$P = \tfrac{1}{3}nm\overline{v^2}.$$

This is a famous result of the kinetic theory of gas, and it is a crucial point in the argument connecting heat and kinetic energy.

Note 3.1 Center of Mass

In this Note we shall find the center of mass of some nonsymmetrical objects. These examples are trivial if you have had experience evaluating two or three dimensional integrals. Otherwise, read on.



1. Find the center of mass of a thin rectangular plate with sides of length a and b, whose mass per unit area σ varies in the following fashion: $\sigma = \sigma_0(xy/ab)$, where σ_0 is a constant.

$$\mathbf{R} = \frac{1}{M}\iint (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\sigma \, dx \, dy$$

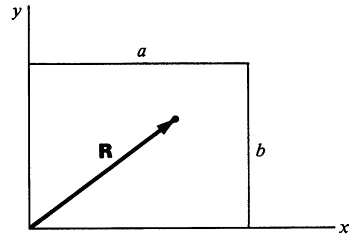
We find M, the mass of the plate, as follows:

$$\begin{aligned} M &= \int_0^b \int_0^a \sigma \, dx \, dy \\ &= \int_0^b \int_0^a \sigma_0 \frac{x}{a}\frac{y}{b} \, dx \, dy. \end{aligned}$$

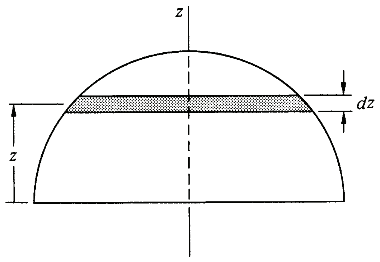
We first integrate over x, treating y as a constant.

$$\begin{aligned} M &= \int_0^b \left(\int_0^a \sigma_0 \frac{x}{a}\frac{y}{b} \, dx \right) dy \\ &= \int_0^b \left(\sigma_0 \frac{y}{b}\frac{x^2}{2a} \Big|_{x=0}^{x=a} \right) dy \\ &= \int_0^b \sigma_0 \frac{y}{b}\frac{a}{2} \, dy \\ &= \frac{\sigma_0 a}{2}\frac{y^2}{2b}\Big|_{y=0}^{y=b} = \frac{1}{4}\sigma_0 ab. \end{aligned}$$

The x component of **R** is



$$
\begin{aligned}
X &= \frac{1}{M}\iint x\sigma\,dx\,dy \\
&= \frac{1}{M}\int_0^b \left(\int_0^a x\sigma_0\frac{xy}{ab}\,dx\right)dy \\
&= \frac{1}{M}\int_0^b \left(\frac{\sigma_0 y}{ab}\frac{x^3}{3}\Big|_0^a\right)dy \\
&= \frac{1}{M}\frac{\sigma_0}{ab}\int_0^b \frac{ya^3}{3}\,dy \\
&= \frac{1}{M}\frac{\sigma_0}{ab}\frac{a^3}{3}\frac{b^2}{2} \\
&= \frac{4}{\sigma_0 ab}\frac{\sigma_0 a^2 b}{6} \\
&= \frac{2}{3}a.
\end{aligned}
$$



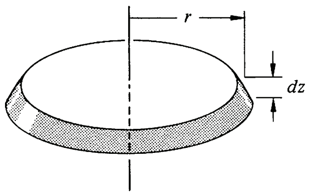
Similarly, $Y = \frac{2}{3}b$.

2. Find the center of mass of a uniform solid hemisphere of radius R and mass M.

From symmetry it is apparent that the center of mass lies on the z axis, as illustrated. Its height above the equatorial plane is

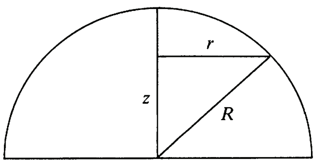
$$Z = \frac{1}{M}\int z\,dM.$$

The integral is over three dimensions, but the symmetry of the situation lets us treat it as a one dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider the circular disk of radius r and thickness dz. Its volume is $dV = \pi r^2\,dz$, and its mass is $dM = \rho\,dV = (M/V)(dV)$, where $V = \frac{2}{3}\pi R^3$. Hence,

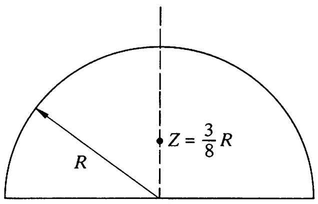


$$
\begin{aligned}
Z &= \frac{1}{M}\int \frac{M}{V}z\,dV \\
&= \frac{1}{V}\int_{z=0}^{R} \pi r^2 z\,dz.
\end{aligned}
$$

To evaluate the integral we need to find r in terms of z. Since $r^2 = R^2 - z^2$, we have



$$
\begin{aligned}
Z &= \frac{\pi}{V}\int_0^R z(R^2 - z^2)\,dz \\
&= \frac{\pi}{V}\left(\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\right)\Big|_0^R
\end{aligned}
$$



$$= \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right)$$
$$= \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3}$$
$$= \frac{3}{8}R.$$

Problems

3.1 The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 x^2/l^2$. Find the position of the center of mass.

Ans. $X = 3l/4$

3.2 Find the center of mass of a thin uniform plate in the shape of an equilateral triangle with sides a.

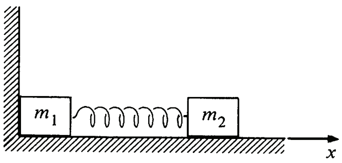
3.3 Suppose that a system consists of several bodies, and that the position of the center of mass of each body is known. Prove that the center of mass of the system can be found by treating each body as a particle concentrated at its center of mass.

3.4 An instrument-carrying projectile accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the earth's curvature.

3.5 A circus acrobat of mass M leaps straight up with initial velocity v_0 from a trampoline. As he rises up, he takes a trained monkey of mass m off a perch at a height h above the trampoline.

What is the maximum height attained by the pair?

3.6 A light plane weighing 2,500 lb makes an emergency landing on a short runway. With its engine off, it lands on the runway at 120 ft/s. A hook on the plane snags a cable attached to a 250-lb sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is 0.4, and if the plane's brakes give an additional retarding force of 300 lb, how far does the plane go before it comes to a stop?

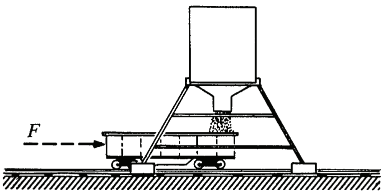


3.7 A system is composed of two blocks of mass m_1 and m_2 connected by a massless spring with spring constant k. The blocks slide on a frictionless plane. The unstretched length of the spring is l. Initially m_2 is held so that the spring is compressed to $l/2$ and m_1 is forced against a stop, as shown. m_2 is released at $t = 0$.

Find the motion of the center of mass of the system as a function of time.

3.8 A 50-kg woman jumps straight into the air, rising 0.8 m from the ground. What impulse does she receive from the ground to attain this height?

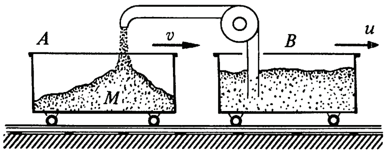
3.9 A freight car of mass M contains a mass of sand m. At $t = 0$ a constant horizontal force F is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at constant rate dm/dt. Find the speed of the freight car when all the sand is gone. Assume the freight car is at rest at $t = 0$.



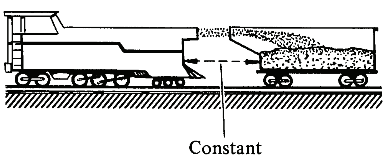
3.10 An empty freight car of mass M starts from rest under an applied force F. At the same time, sand begins to run into the car at steady rate b from a hopper at rest along the track.

Find the speed when a mass of sand, m, has been transferred. (*Hint:* There is a way to do this problem in one or two lines.)

Ans. clue. If $M = 500$ kg, $b = 20$ kg/s, $F = 100$ N, then $v = 1.4$ m/s at $t = 10$ s



3.11 Material is blown into cart A from cart B at a rate b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity as cart B, u. At the moment of interest, cart A has mass M and velocity v, as shown. Find dv/dt, the instantaneous acceleration of A.



3.12 A sand-spraying locomotive sprays sand horizontally into a freight car as shown in the sketch. The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate $dm/dt = 10$ kg/s with a velocity of 5 m/s relative to the locomotive. The car starts from rest with an initial mass of 2,000 kg. Find its speed after 100 s.

3.13 A ski tow consists of a long belt of rope around two pulleys, one at the bottom of a slope and the other at the top. The pulleys are driven by a husky electric motor so that the rope moves at a steady speed of 1.5 m/s. The pulleys are separated by a distance of 100 m, and the angle of the slope is 20°.

Skiers take hold of the rope and are pulled up to the top, where they release the rope and glide off. If a skier of mass 70 kg takes the tow every 5 s on the average, what is the average force required to pull the rope? Neglect friction between the skis and the snow.

3.14 N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

a. What is the final velocity of the flatcar if all the men jump at the same time?

b. What is the final velocity of the flatcar if they jump off one at a time? (The answer can be left in the form of a sum of terms.)

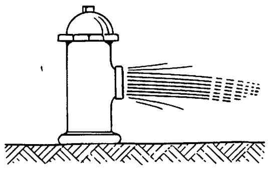
c. Does case a or case b yield the largest final velocity of the flat car? Can you give a simple physical explanation for your answer?

3.15 A rope of mass M and length l lies on a frictionless table, with a short portion, l_0, hanging through a hole. Initially the rope is at rest.

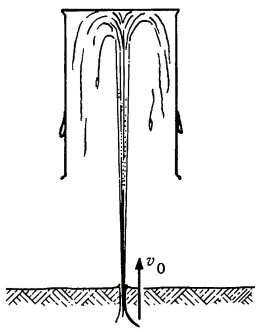
a. Find a general equation for $x(t)$, the length of rope through the hole.

Ans. $x = Ae^{\gamma t} + Be^{-\gamma t}$, $\gamma^2 = g/l$

b. Evaluate the constants A and B so that the initial conditions are satisfied.



3.16 Water shoots out of a fire hydrant having nozzle diameter D with nozzle speed V_0. What is the reaction force on the hydrant?



3.17 An inverted garbage can of weight W is suspended in air by water from a geyser. The water shoots up from the ground with a speed v_0, at a constant rate dm/dt. The problem is to find the maximum height at which the garbage can rides. What assumption must be fulfilled for the maximum height to be reached?

Ans. clue. If $v_0 = 20$ m/s, $W = 10$ kg, $dm/dt = 0.5$ kg/s, then $h_{\max} \approx 17$ m

3.18 A raindrop of initial mass M_0 starts falling from rest under the influence of gravity. Assume that the drop gains mass from the cloud at a rate proportional to the product of its instantaneous mass and its instantaneous velocity:

$$\frac{dM}{dt} = kMV,$$

where k is a constant.

Show that the speed of the drop eventually becomes effectively constant, and give an expression for the terminal speed. Neglect air resistance.

3.19 A bowl full of water is sitting out in a pouring rainstorm. Its surface area is 500 cm^2. The rain is coming straight down at 5 m/s at a rate of 10^{-3} g/cm^2·s. If the excess water drips out of the bowl with negligible velocity, find the force on the bowl due to the falling rain.

What is the force if the bowl is moving uniformly upward at 2 m/s?

3.20 A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u. Assume that the rate at which mass is expelled is given by $dm/dt = \gamma m$, where m is the instantaneous mass of the rocket and γ is a constant, and that the rocket is retarded by air resistance with a force mbv, where b is a constant. Find the velocity of the rocket as a function of time.

Ans. clue. The terminal velocity is $(\gamma u - g)/b$.

4 WORK AND ENERGY

4.1 Introduction

In this chapter we make another attack on the fundamental problem of classical mechanics—predicting the motion of a system under known interactions. We shall encounter two important new concepts, work and energy, which first appear to be mere computational aids, mathematical crutches so to speak, but which turn out to have very real physical significance.

As first glance there seems to be no problem in finding the motion of a particle if we know the force; starting with Newton's second law, we obtain the acceleration, and by integrating we can find first the velocity and then the position. It sounds simple, but there is a problem; in order to carry out these calculations we must know the force as a function of time, whereas force is usually known as a function of position as, for example, the spring force or the gravitational force. The problem is serious because physicists are generally interested in interactions between systems, which means knowing how the force varies with position, not how it varies with time.

The task, then, is to find $\mathbf{v}(t)$ from the equation

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{r}), \qquad 4.1$$

where the notation emphasizes that $\mathbf{F}$ is a known function of position. A physicist with a penchant for mathematical formalism might stop at this point and say that what we are dealing with is a problem in differential equations and that what we ought to do now is study the schemes available, including numerical methods, for solving such equations. From the strict calculational point of view, he is right. However, such an approach is too narrow and affords too little physical understanding.

Fortunately, the solution to Eq. (4.1) is simple for the important case of one dimensional motion in a single variable. The general case is more complex, but we shall see that it is not too difficult to integrate Eq. (4.1) for three dimensional motion provided that we are content with less than a complete solution. By way of compensation we shall obtain a very helpful physical relation, the work-energy theorem; its generalization, the law of conservation of energy, is among the most useful conservation laws in physics.

Let's consider the one dimensional problem before tackling the general case.

4.2 Integrating the Equation of Motion in One Dimension

A large class of important problems involves only a single variable to describe the motion. The one dimensional harmonic oscillator provides a good example. For such problems the equation of motion reduces to

$$m\frac{d^2x}{dt^2} = F(x)$$

or

$$m\frac{dv}{dt} = F(x). \tag{4.2}$$

We can solve this equation for v by a mathematical trick. First, formally integrate $m\,dv/dt = F(x)$ with respect to x:

$$m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx = \int_{x_a}^{x_b}F(x)\,dx.$$

The integral on the right can be evaluated by standard methods since $F(x)$ is known. The integral on the left is intractable as it stands, but it can be integrated by changing the variable from x to t. The trick is to use[1]

$$\begin{aligned} dx &= \left(\frac{dx}{dt}\right)dt \\ &= v\,dt. \end{aligned}$$

Then

$$\begin{aligned} m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx &= m\int_{t_a}^{t_b}\frac{dv}{dt}\,v\,dt \\ &= m\int_{t_a}^{t_b}\frac{d}{dt}\left(\frac{1}{2}v^2\right)dt \\ &= \frac{1}{2}mv^2\Big|_{t_a}^{t_b} \\ &= \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2, \end{aligned}$$

where $x_a \equiv x(t_a)$, $v_a \equiv v(t_a)$, etc.

Putting these results together yields

$$\tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2 = \int_{x_a}^{x_b}F(x)\,dx. \tag{4.3}$$

[1] Change of variables using differentials is discussed in Note 1.1.

Alternatively, we can use indefinite upper limits in Eq. (4.3):

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_a{}^2 = \int_{x_a}^{x} F(x)\,dx, \qquad 4.4$$

where v is the speed of the particle when it is at position x. Equation (4.4) gives us v as a function of x. Since $v = dx/dt$, we could solve Eq. (4.4) for dx/dt and integrate again to find $x(t)$. Rather than write out the general formula, it is easier to see the method by studying a few examples.

Example 4.1 Mass Thrown Upward in a Uniform Gravitational Field

A mass m is thrown vertically upward with initial speed v_0. How high does it rise, assuming the gravitational force to be constant, and neglecting air friction?

Taking the z axis to be directed vertically upward,

$$F = -mg.$$

Equation (4.3) gives

$$\begin{aligned}\tfrac{1}{2}mv_1{}^2 - \tfrac{1}{2}mv_0{}^2 &= \int_{z_0}^{z_1} F\,dz \\ &= -mg\int_{z_0}^{z_1} dz \\ &= -mg(z_1 - z_0).\end{aligned}$$

At the peak, $v_1 = 0$ and we obtain the answer

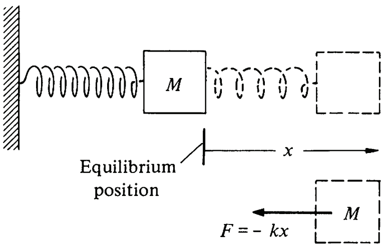
$$z_1 = z_0 + \frac{v_0{}^2}{2g}.$$

It is interesting to note that the solution makes no reference to time at all. We could have solved the problem by applying Newton's second law, but we would have had to eliminate t to obtain the result.

Here is an example that is not easy to solve by direct application of Newton's second law.

Example 4.2 Solving the Equation of Simple Harmonic Motion

In Example 2.17 we discussed the equation of simple harmonic motion and pulled the solution out of a hat without proof. Now we shall derive the solution using Eq. (4.4).



Consider a mass M attached to a spring. Using the coordinate x measured from the equilibrium point, the spring force is $F = -kx$. Then Eq. (4.4) becomes

$$\begin{aligned}\tfrac{1}{2}Mv^2 - \tfrac{1}{2}Mv_0^2 &= -k\int_{x_0}^{x} x\,dx \\ &= -\tfrac{1}{2}kx^2 + \tfrac{1}{2}kx_0^2.\end{aligned}$$

The initial coordinates are labeled by the subscript 0.

In order to find x and v, we must know their values at some time t_0. Physically, this arises because the equation of motion by itself cannot completely specify the motion; we also need to know a set of initial conditions, in this case the initial position and velocity.[1] We are free to choose any initial conditions we wish. Let us consider the case where at $t = 0$ the mass is released from rest, $v_0 = 0$, at a distance x_0 from the origin. Then

$$v^2 = -\frac{k}{M}x^2 + \frac{k}{M}x_0^2$$

and

$$\begin{aligned}\frac{dx}{dt} &= v \\ &= \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}.\end{aligned}$$

Separating the variables gives

$$\begin{aligned}\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2 - x^2}} &= \sqrt{\frac{k}{M}}\int_0^t dt \\ &= \sqrt{\frac{k}{M}}\,t.\end{aligned}$$

The integral on the left hand side is arcsin (x/x_0). (The integral is listed in standard tables. Consulting a table of integrals is just as respectable for a physicist as consulting a dictionary is for a writer. Of course, in both cases one hopes that experience gradually reduces dependence.) Denoting $\sqrt{k/M}$ by ω, we obtain

$$\arcsin\left(\frac{x}{x_0}\right)\Bigg|_{x_0}^{x} = \omega t$$

or

$$\arcsin\left(\frac{x}{x_0}\right) - \arcsin 1 = \omega t.$$

[1] In the language of differential equations, Newton's second law is a "second order" equation in the position; the highest order derivative it involves is the acceleration, which is the second derivative of the position with respect to time. The theory of differential equations shows that the complete solution of a differential equation of nth order must involve n initial conditions.

Since arcsin 1 = $\pi/2$, we obtain

$$x = x_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$
$$= x_0 \cos \omega t.$$

Note that the solution indeed satisfies the given initial conditions: at $t = 0$, $x = x_0 \cos 0 = x_0$, and $\dot{x} = x_0\omega \sin 0 = 0$. For these conditions our result agrees with the general solution given in Example 2.14.

4.3 The Work-energy Theorem in One Dimension

In Sec. 4.2 we demonstrated the formal procedure for integrating Newton's second law with respect to position. The result was

$$\tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)\, dx,$$

which we now wish to interpret in physical terms.

The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy* K, and the left hand side can be written $K_b - K_a$. The integral $\int_{x_a}^{x_b} F(x)\, dx$ is called the *work* W_{ba} done *by* the force F *on* the particle as the particle moves from a to b. Our relation now takes the form

$$W_{ba} = K_b - K_a. \qquad 4.5$$

This result is known as the work-energy theorem or, more precisely, the work-energy theorem in one dimension. (We shall shortly see a more general statement.) The unit of work and energy in the SI system is the *joule* (J):

1 J = 1 kg·m²/s².

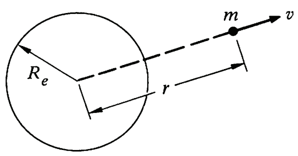
The unit of work and energy in the cgs system is the *erg*:

1 erg = 1 gm·cm²/s²
= 10^{-7} J.

The unit work in the English system is the *foot-pound*:

1 ft·lb ≈ 1.336 J.

Example 4.3 **Vertical Motion in an Inverse Square Field**



A mass m is shot vertically upward from the surface of the earth with initial speed v_0. Assuming that the only force is gravity, find its maximum altitude and the minimum value of v_0 for the mass to escape the earth completely.

The force on m is

$$F = -\frac{GM_e m}{r^2}.$$

The problem is one dimensional in the variable r, and it is simple to find the kinetic energy at distance r by the work-energy theorem.

Let the particle start at $r = R_e$ with initial velocity v_0.

$$\begin{aligned} K(r) - K(r_e) &= \int_{R_e}^{r} F(r)\,dr \\ &= -GM_e m \int_{R_e}^{r} \frac{dr}{r^2} \end{aligned}$$

or

$$\tfrac{1}{2}mv(r)^2 - \tfrac{1}{2}mv_0^2 = GM_e m\left(\frac{1}{r} - \frac{1}{R_e}\right).$$

We can immediately find the maximum height of m. At the highest point, $v(r) = 0$ and we have

$$v_0^2 = 2GM_e\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right).$$

It is a good idea to introduce known familiar constants whenever possible. For example, since $g = GM_e/R_e^2$, we can write

$$\begin{aligned} v_0^2 &= 2gR_e^2\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right) \\ &= 2gR_e\left(1 - \frac{R_e}{r_{\max}}\right) \end{aligned}$$

or

$$r_{\max} = \frac{R_e}{1 - \dfrac{v_0^2}{2gR_e}}.$$

The escape velocity from the earth is the initial velocity needed to move $r_{\max}$ to infinity. The escape velocity is therefore

$$\begin{aligned} v_{\text{escape}} &= \sqrt{2gR_e} \\ &= \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \\ &= 1.1 \times 10^4 \text{ m/s}. \end{aligned}$$

The energy needed to eject a 50-kg spacecraft from the surface of the earth is

$$\begin{aligned} W &= \tfrac{1}{2}Mv_{\text{escape}}^2 \\ &= \tfrac{1}{2}(50)(1.1 \times 10^4)^2 = 3.0 \times 10^9 \text{ J}. \end{aligned}$$

4.4 Integrating the Equation of Motion in Several Dimensions

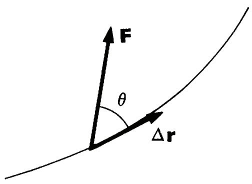
Returning to the central problem of this chapter, let us try to integrate the equation of motion of a particle acted on by a force which depends on position.

$$\mathbf{F}(\mathbf{r}) = m\frac{d\mathbf{v}}{dt}. \qquad 4.6$$

In the case of one dimensional motion we integrated with respect to position. To generalize this, consider what happens when the particle moves a short distance $\Delta\mathbf{r}$.

We assume that $\Delta\mathbf{r}$ is so small that $\mathbf{F}$ is effectively constant over this displacement. If we take the scalar product of Eq. (4.6) with $\Delta\mathbf{r}$, we obtain

$$\mathbf{F}\cdot\Delta\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot\Delta\mathbf{r}. \qquad 4.7$$



The sketch shows the trajectory and the force at some point along the trajectory. At this point,

$$\mathbf{F}\cdot\Delta\mathbf{r} = F\,\Delta r\cos\theta.$$

Perhaps you are wondering how we know $\Delta\mathbf{r}$, since this requires knowing the trajectory, which is what we are trying to find. Let us overlook this problem for a few moments and pretend we know the trajectory.

Now consider the right hand side of Eq. (4.7), $m(d\mathbf{v}/dt)\cdot\Delta\mathbf{r}$. We can transform this by noting that $\mathbf{v}$ and $\Delta\mathbf{r}$ are not independent; for a sufficiently short length of path, $\mathbf{v}$ is approximately constant. Hence $\Delta\mathbf{r} = \mathbf{v}\,\Delta t$, where Δt is the time the particle requires to travel $\Delta\mathbf{r}$, and therefore

$$m\frac{d\mathbf{v}}{dt}\cdot\Delta\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,\Delta t. \qquad 4.8$$

We can transform Eq. (4.7) with the vector identity[1]

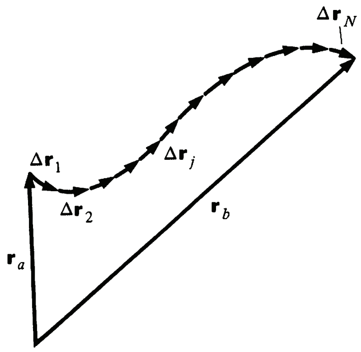
$$\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{1}{2}\frac{d}{dt}(v^2).$$

[1] The identity $\mathbf{A}\cdot(d\mathbf{A}/dt) = \frac{1}{2}(d/dt)(A^2)$ is easily proved:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(A^2) &= \frac{1}{2}\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A})\\ &= \frac{1}{2}\left(\mathbf{A}\cdot\frac{d\mathbf{A}}{dt}+\frac{d\mathbf{A}}{dt}\cdot\mathbf{A}\right)\\ &= \mathbf{A}\cdot\frac{d\mathbf{A}}{dt}.\end{aligned}$$

Equation (4.7) becomes

$$\mathbf{F} \cdot \Delta\mathbf{r} = \frac{m}{2}\frac{d}{dt}(v^2)\,\Delta t. \qquad 4.9$$



The next step is to divide the entire trajectory from the initial position $\mathbf{r}_a$ to the final position $\mathbf{r}_b$ into N short segments of length $\Delta\mathbf{r}_j$, where j is an index numbering the segments. (It makes no difference whether all the pieces have the same length.) For each segment we can write a relation similar to Eq. (4.9):

$$\mathbf{F}(\mathbf{r}_j) \cdot \Delta\mathbf{r}_j = \frac{m}{2}\frac{d}{dt}(v_j^2)\,\Delta t_j, \qquad 4.10$$

where $\mathbf{r}_j$ is the location of segment j, $\mathbf{v}_j$ is the velocity the particle has there, and Δt_j is the time it spends in traversing it. If we add together the equations of all the segments, we have

$$\sum_{j=1}^{N} \mathbf{F}(\mathbf{r}_j) \cdot \Delta\mathbf{r}_j = \sum_{j=1}^{N} \frac{m}{2}\frac{d}{dt}(v_j^2)\,\Delta t_j. \qquad 4.11$$

Next we take the limiting process where the length of each segment approaches zero, and the number of segments approaches infinity. We have

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_a}^{t_b} \frac{m}{2}\frac{d}{dt}(v^2)\,dt, \qquad 4.12$$

where t_a and t_b are the times corresponding to $\mathbf{r}_a$ and $\mathbf{r}_b$. In converting the sum to an integral, we have dropped the numerical index j and have indicated the location of the first segment $\Delta\mathbf{r}_1$ by $\mathbf{r}_a$, and the location of the last section $\Delta\mathbf{r}_N$ by $\mathbf{r}_b$.

The integral on the right in Eq. (4.12) is

$$\frac{m}{2}\int_{t_a}^{t_b} \frac{d}{dt}(v^2)\,dt = \tfrac{1}{2}mv^2\Big|_{t_a}^{t_b}$$
$$= \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2.$$

This represents a simple generalization of the result we found for one dimension. Here, however, $v^2 = v_x^2 + v_y^2 + v_z^2$, whereas for the one dimensional case we had $v^2 = v_x^2$.

Equation (4.12) becomes

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} = \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2. \qquad 4.13$$

The integral on the left is called a *line integral*. We shall see how to evaluate line integrals in the next two sections, and we shall also see how to interpret Eq. (4.13) physically. However, before proceeding, let's pause for a moment to summarize.

Our starting point was $\mathbf{F}(\mathbf{r}) = m\, d\mathbf{v}/dt$. All we have done is to integrate this equation with respect to distance, but because we described each step carefully, it looks like many operations are involved. This is not really the case; the whole argument can be stated in a few lines as follows:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot d\mathbf{r} &= \int_a^b m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}\\ &= \int_a^b m\frac{d\mathbf{v}}{dt}\cdot \mathbf{v}\,dt\\ &= \int_a^b \frac{m}{2}\frac{d}{dt}(v^2)\,dt\\ &= \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2.\end{aligned}$$

4.5 The Work-energy Theorem

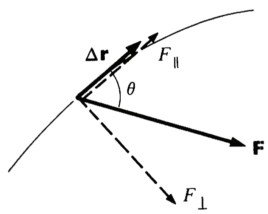
We now want to interpret Eq. (4.13) in physical terms. The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy* K, and the right hand side of Eq. (4.13) can be written as $K_b - K_a$. The integral $\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r}$ is called the *work* W_{ba} done by the force $\mathbf{F}$ on the particle as the particle moves from a to b. Equation (4.13) now takes the form

$$W_{ba} = K_b - K_a. \qquad 4.14$$

This result is the general statement of the *work-energy theorem* which we met in restricted form in our discussion of one dimensional motion.

The work ΔW done by a force $\mathbf{F}$ in a small displacement $\Delta\mathbf{r}$ is

$$\Delta W = \mathbf{F}\cdot\Delta\mathbf{r} = F\cos\theta\,\Delta r = F_\parallel\,\Delta r,$$



where $F_\parallel = F\cos\theta$ is the component of $\mathbf{F}$ along the direction of $\Delta\mathbf{r}$. The component of $\mathbf{F}$ perpendicular to $\Delta\mathbf{r}$ does no work. For a finite displacement from $\mathbf{r}_a$ to $\mathbf{r}_b$, the work on the particle, $\int_a^b \mathbf{F}\cdot d\mathbf{r}$, is the sum of the contributions $\Delta W = F_\parallel\,\Delta r$ from each segment of the path, in the limit where the size of each segment approaches zero.

In the work-energy theorem, $W_{ba} = K_b - K_a$, W_{ba} is the work done on the particle by the total force **F**. If **F** is the sum of several forces $\mathbf{F} = \Sigma \mathbf{F}_i$, we can write

$$\begin{aligned} W_{ba} &= \sum_i (W_i)_{ba} \\ &= K_b - K_a, \end{aligned}$$

where

$$(W_i)_{ba} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}_i \cdot d\mathbf{r}$$

is the work done by the ith force $\mathbf{F}_i$.

Our discussion so far has been restricted to the case of a single particle. However, we showed in Chap. 3 that the center of mass of an extended system moves according to the equation of motion

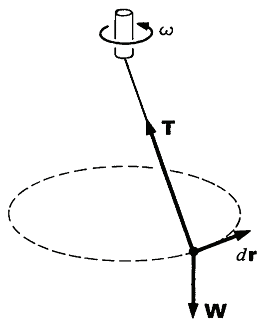
$$\begin{aligned} \mathbf{F} &= M\ddot{\mathbf{R}} \\ &= M\frac{d\mathbf{V}}{dt}, \end{aligned} \qquad 4.15$$

where $\mathbf{V} = \mathbf{R}$ is the velocity of the center of mass. Integrating Eq. (4.15) with respect to position gives

$$\int_{\mathbf{R}_a}^{\mathbf{R}_b} \mathbf{F} \cdot d\mathbf{R} = \tfrac{1}{2}MV_b^2 - \tfrac{1}{2}MV_a^2, \qquad 4.16$$

where $d\mathbf{R} = \mathbf{V}\,dt$ is the displacement of the center of mass in time dt. Equation (4.16) is the work-energy theorem for the translational motion of an extended system; in Chaps. 6 and 7 we shall extend the ideas of work and kinetic energy to include rotational motion. Note, however, that Eq. (4.16) holds regardless of the rotational motion of the system.

Example 4.4 The Conical Pendulum

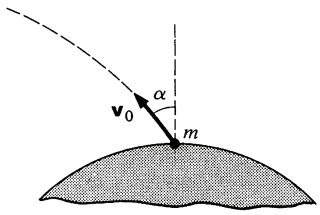


We discussed the motion of the conical pendulum in Example 2.8. Since the mass moves with constant angular velocity ω in a circle of constant radius R, the kinetic energy of the mass, $\frac{1}{2}mR\omega^2$, is constant. The work-energy theorem then tells us that no net work is being done on the mass.

Furthermore, in the conical pendulum the string force and the weight force separately do no work, since each of these forces is perpendicular to the path of the particle, making the integrand of the work integral zero.

It is important to realize that in the work integral $\int \mathbf{F} \cdot d\mathbf{r}$, the vector $d\mathbf{r}$ is along the path of the particle. Since $\mathbf{v} = d\mathbf{r}/dt$, $d\mathbf{r} = \mathbf{v}\,dt$ and $d\mathbf{r}$ is always parallel to $\mathbf{v}$.

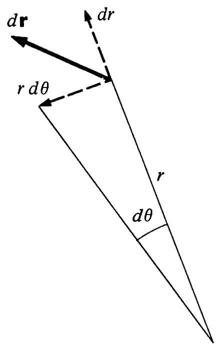
Example 4.5 **Escape Velocity—the General Case**



In Example 4.3 we discussed the one dimensional motion of a mass m projected vertically upward from the earth. We found that if the initial speed is greater than $v_0 = \sqrt{2gR_e}$, the mass will escape from the earth. Suppose that we look at the problem once again, but now allow the mass to be projected at angle α from the vertical.

The force on m, neglecting air resistance, is

$$\mathbf{F} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}}$$
$$= -mg\frac{R_e^2}{r^2}\hat{\mathbf{r}},$$



where $g = GM_e/R_e^2$ is the acceleration due to gravity at the earth's surface. We do not know the trajectory of the particle without solving the problem in detail. However, any element of the path $d\mathbf{r}$ can be written

$$d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}.$$

Hence

$$\mathbf{F}\cdot d\mathbf{r} = -mg\frac{R_e^2}{r^2}\hat{\mathbf{r}}\cdot(dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}})$$
$$= -mg\frac{R_e^2}{r^2}dr.$$

The work-energy theorem becomes

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = -mgR_e^2\int_{R_e}^{r}\frac{dr}{r^2}$$
$$= -mgR_e^2\left(\frac{1}{r} - \frac{1}{R_e}\right).$$

The escape velocity is the value of v_0 for which $r = \infty$, $v = 0$. We find

$$v_0 = \sqrt{2gR_e}$$
$$= 1.1 \times 10^4 \text{ m/s},$$

as before. The escape velocity is independent of the launch direction.

We have neglected the earth's rotation in our analysis. In the absence of air resistance the projectile should be fired horizontally to the east, since the rotational speed of the earth's surface is then added to the launch velocity.

4.6 Applying the Work-energy Theorem

In the last section we derived the work-energy theorem

$$W_{ba} = K_b - K_a \qquad 4.17$$

and applied it to a few simple cases. In this section we shall use it to tackle more complicated problems. However, a few comments on the properties of the theorem are in order first.

To begin, we should emphasize that the work-energy theorem is a mathematical consequence of Newton's second law; we have introduced no new physical ideas. The work-energy theorem is merely the statement that the change in kinetic energy is equal to the net work done. This should not be confused with the general law of conservation of energy, an independent physical law which we shall discuss in Sec. 4.12.

Possibly you are troubled by the following problem: to apply the work-energy theorem, we have to evaluate the line integral for work[1]

$$W_{ba} = \oint_a^b \mathbf{F} \cdot d\mathbf{r}$$

and the evaluation of this integral depends on knowing what path the particle actually follows. We seem to need to know everything about the motion even before we use the work-energy theorem, and it is hard to see what use the theorem would be.

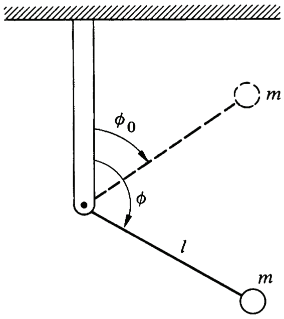
In the most general case, the work integral depends on the path followed, and since we don't know the path without completely solving the problem, the work-energy theorem is useless. There are, fortunately, two special cases of considerable practical importance. For many forces of interest, the work integral does not depend on the particular path but only on the end points. Such forces, which include most of the important forces in physics, are called *conservative* forces. As we shall discuss later in this chapter, the work-energy theorem can be put in a very simple form when the forces are conservative.

The work-energy theorem is also useful in cases where the path is known because the motion is *constrained*. By constrained motion, we mean motion in which external constraints act to keep the particle on a predetermined trajectory. The roller coaster is a perfect example. Except in cases of calamity, the roller coaster follows the track because it is held on by wheels both below and above the track. There are many other examples of constrained motion which come readily to mind—the conical pendulum is one (here the constraint is that the length of the string is fixed)—but all have one feature in common—the constraining force does no work. To see this, note that the effect of the constraint force is

[1] The C through the integral sign reminds us that the integral is to be evaluated along some specific curve.

to assure that the direction of the velocity is always tangential to the predetermined path. Hence, constraint forces change only the direction of **v** and do no work.[1]

Example 4.6 **The Inverted Pendulum**



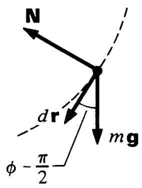
A pendulum consists of a light rigid rod of length l, pivoted at one end and with mass m attached at the other end. The pendulum is released from rest at angle ϕ_0, as shown. What is the velocity of m when the rod is at angle ϕ?

The work-energy theorem gives

$$\tfrac{1}{2}mv(\phi)^2 - \tfrac{1}{2}mv_0^2 = W_{\phi,\phi_0}.$$

Since $v_0 = 0$, we have

$$v(\phi) = \left(\frac{2W_{\phi,\phi_0}}{m}\right)^{\frac{1}{2}}$$



To evaluate W_{ϕ,ϕ_0}, the work done as the bob swings from ϕ_0 to ϕ, we examine the force diagram. $d\mathbf{r}$ lies along the circle of radius l. The forces acting are gravity, directed down, and the force of the rod, **N**. Since **N** lies along the radius, $\mathbf{N} \cdot d\mathbf{r} = 0$, and **N** does no work. The work done by gravity is

$$\begin{aligned} m\mathbf{g} \cdot d\mathbf{r} &= mgl \cos\left(\varphi - \frac{\pi}{2}\right) d\varphi \\ &= mgl \sin \varphi \, d\varphi \end{aligned}$$

where we have used $|d\mathbf{r}| = l\,d\phi$.

$$\begin{aligned} W_{\phi,\phi_0} &= \int_{\phi_0}^{\phi} mgl \sin \phi \, d\phi \\ &= -mgl \cos \phi \Big|_{\phi_0}^{\phi} \\ &= mgl (\cos \phi_0 - \cos \phi). \end{aligned}$$

The speed at ϕ is

$$v(\phi) = [2gl (\cos \phi_0 - \cos \phi)]^{\frac{1}{2}}.$$

The maximum velocity is obtained by letting the pendulum fall from the top, $\phi_0 = 0$, to the bottom, $\phi = \pi$:

$$v_{\max} = 2(gl)^{\frac{1}{2}}.$$

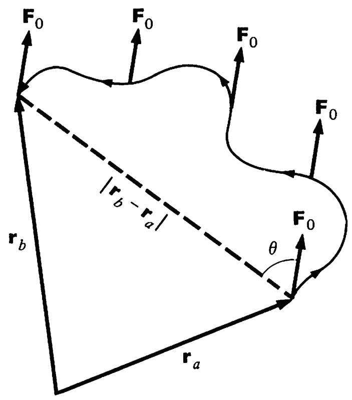
[1] We can prove that constraint forces do no work as follows. Suppose that the constraint force $\mathbf{F}_{\text{constraint}}$ changes the velocity by an amount $\Delta\mathbf{v}_c$ in time Δt. $\Delta\mathbf{v}_c$ is perpendicular to the instantaneous velocity **v**. The work done by $\mathbf{F}_{\text{constraint}}$ is $\mathbf{F}_{\text{constraint}} \cdot \Delta\mathbf{r} = m(\Delta\mathbf{v}_c/\Delta t) \cdot (\mathbf{v}\,\Delta t) = m\Delta\mathbf{v}_c \cdot \mathbf{v} = 0$.

This is the same speed attained by a mass falling through the same vertical distance $2l$. However, the mass on the pendulum is not traveling vertically at the bottom of its path, it is traveling horizontally.

If you doubt the utility of the work-energy theorem, try solving the last example by integrating the equation of motion. However, the example also illustrates one of the shortcomings of the method: we found a simple solution for the speed of the mass at any point on the circle—we have no information on *when* the mass gets there. For instance, if the pendulum is released at $\phi_0 = 0$, in principle it balances there forever, never reaching the bottom. Fortunately, in many problems we are not interested in time, and even when time is important, the work-energy theorem provides a valuable first step toward obtaining a complete solution.

Next we turn to the general problem of evaluating work done by a known force over a given path, the problem of evaluating line integrals. We start by looking at the case of a constant force.

Example 4.7 Work Done by a Uniform Force

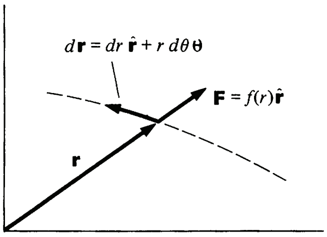


The case of a uniform force is particularly simple. Here is how to find the work done by a force, $\mathbf{F} = F_0\hat{\mathbf{n}}$, where F_0 is a constant and $\hat{\mathbf{n}}$ is a unit vector in some direction, as the particle moves from $\mathbf{r}_a$ to $\mathbf{r}_b$ along some arbitrary path. All the steps are put in to make the procedure clear, but with any practice this problem can be solved by inspection.

$$
\begin{aligned}
W_{ba} &= \oint_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_{\mathbf{r}_a}^{\mathbf{r}_b} F_0\hat{\mathbf{n}} \cdot d\mathbf{r} \\
&= F_0\hat{\mathbf{n}} \cdot \oint_{\mathbf{r}_a}^{\mathbf{r}_b} d\mathbf{r} \\
&= F_0\hat{\mathbf{n}} \cdot \left(\hat{\mathbf{i}} \int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dx + \hat{\mathbf{j}} \int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dy + \hat{\mathbf{k}} \int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dz\right) \\
&= F_0\hat{\mathbf{n}} \cdot [\hat{\mathbf{i}}(x_b - x_a) + \hat{\mathbf{j}}(y_b - y_a) + \hat{\mathbf{k}}(z_b - z_a)] \\
&= F_0\hat{\mathbf{n}} \cdot (\mathbf{r}_b - \mathbf{r}_a) \\
&= F_0 \cos\theta\, |\mathbf{r}_b - \mathbf{r}_a|
\end{aligned}
$$

For a constant force the work depends only on the net displacement, $\mathbf{r}_b - \mathbf{r}_a$, not on the path followed. This is not generally the case, but it holds true for an important group of forces, including central forces, as the next example shows.

Example 4.8 **Work Done by a Central Force**



A *central force* is a radial force which depends only on the distance from the origin. Let us find the work done by the central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$ on a particle which moves from $\mathbf{r}_a$ to $\mathbf{r}_b$. For simplicity we shall consider motion in a plane, for which $d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$. Then

$$\begin{aligned} W_{ba} &= \oint_a^b \mathbf{F}\cdot d\mathbf{r} \\ &= \oint_a^b f(r)\hat{\mathbf{r}}\cdot(dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}) \\ &= \int_a^b f(r)\,dr. \end{aligned}$$

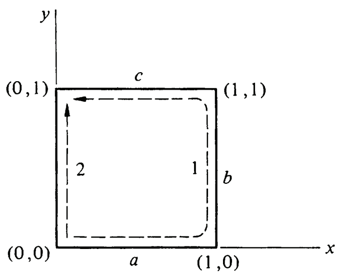
The work is given by a simple one dimensional integral over the variable r. Since θ has disappeared from the problem, it should be obvious that the work depends only on the initial and final radial distances [and, of course, on the particular form of $f(r)$], not on the particular path.

For some forces, the work is different for different paths between the initial and final points. One familiar example is work done by the force of sliding friction. Here the force always opposes the motion, so that the work done by friction in moving through distance dS is $dW = -f\,dS$, where f is the magnitude of the friction force. If we assume that f is constant, then the work done by friction in going from $\mathbf{r}_a$ to $\mathbf{r}_b$ along some path is

$$\begin{aligned} W_{ba} &= -\int_{\mathbf{r}_a}^{\mathbf{r}_b} f\,dS \\ &= -fS, \end{aligned}$$

where S is the total length of the path. The work is negative because the force always retards the particle. W_{ba} is never smaller in magnitude than fS_0, where S_0 is the distance between the two points, but by choosing a sufficiently devious route, S can be made arbitrarily large.

Example 4.9 **A Path-dependent Line Integral**



Here is a second example of a path-dependent line integral. Let $\mathbf{F} = A(xy\hat{\mathbf{i}} + y^2\hat{\mathbf{j}})$, and consider the integral from (0,0) to (0,1), first along path 1 and then along path 2, as shown in the figure. The force $\mathbf{F}$ has no physical significance, but the example illustrates the properties of nonconservative forces. Since the segments of each path lie along a coordinate axis, it is particularly simple to evaluate the integrals. For path 1 we have

$$\oint_1 \mathbf{F}\cdot d\mathbf{r} = \int_a \mathbf{F}\cdot d\mathbf{r} + \int_b \mathbf{F}\cdot d\mathbf{r} + \int_c \mathbf{F}\cdot d\mathbf{r}.$$

Along segment a, $d\mathbf{r} = dx\,\hat{\mathbf{i}}$, $\mathbf{F}\cdot d\mathbf{r} = F_x\,dx = Axy\,dx$. Since $y = 0$ along the line of this integration, $\int_a \mathbf{F}\cdot d\mathbf{r} = 0$. Similarly, for path b,

$$\int_b \mathbf{F}\cdot d\mathbf{r} = A\int_{x=1,y=0}^{x=1,y=1} y^2\,dy = \frac{A}{3},$$

while for path c,

$$\int_c \mathbf{F}\cdot d\mathbf{r} = A\int_{x=1,y=1}^{x=0,y=1} xy\,dx = A\int_1^0 x\,dx = -\frac{A}{2}.$$

Thus

$$\oint_1 \mathbf{F}\cdot d\mathbf{r} = \frac{A}{3} - \frac{A}{2} = -\frac{A}{6}.$$

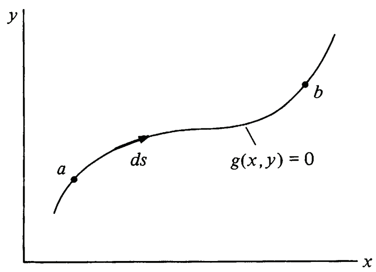
Along path 2 we have

$$\oint_2 \mathbf{F}\cdot d\mathbf{r} = A\int_{0,0}^{0,1} y^2\,dy = \frac{A}{3} \neq \oint_1 \mathbf{F}\cdot d\mathbf{r}.$$

The work done by the applied force is different for the two paths.

Usually the path of a line integral does not lie conveniently along the coordinate axes but along some arbitrary curve. The following method of evaluating a line integral in such a case is quite general; use it if all else fails.

For simplicity we again consider motion in a plane. Generalization to three dimensions is straightforward.



The problem is to evaluate $\oint_a^b \mathbf{F}\cdot d\mathbf{r}$ along a specified path. The path can be characterized by an equation of the form $g(x,y) = 0$. For example, if the path is a unit circle about the origin, then all points on the path obey $x^2 + y^2 - 1 = 0$.

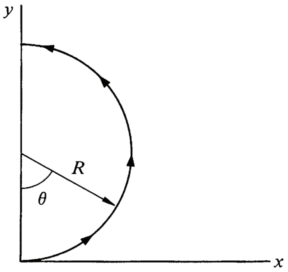
We can characterize every point on the path by a parameter s which in practical problems could be (for example) distance along the path, or angle—anything just as long as each point on the path is associated with a value of s so that we can write

$x = x(s)$, $y = y(s)$. If we move along the path a short way, so that s changes by the amount ds, then the change in x is $dx = (dx/ds)\,ds$, and the change in y is $dy = (dy/ds)\,ds$. Since both x and y are determined by s, so are F_x and F_y. Hence, we can write $\mathbf{F} = F_x(s)\hat{\mathbf{i}} + F_y(s)\hat{\mathbf{j}}$, and we have

$$\oint_a^b \mathbf{F}\cdot d\mathbf{r} = \int_a^b (F_x\,dx + F_y\,dy)$$
$$= \int_{s_a}^{s_b}\left[F_x(s)\frac{dx}{ds} + F_y(s)\frac{dy}{ds}\right] ds.$$

We have reduced the problem to the more familiar problem of evaluating a one dimensional definite integral. The calculation is much simpler in practice than in theory. Here is an example.

Example 4.10 Parametric Evaluation of a Line Integral



Evaluate the line integral of $\mathbf{F} = A(x^3\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}})$ from $(x = 0, y = 0)$ to $(x = 0, y = 2R)$ along the semicircle shown.

The natural parameter to use here is θ, since as θ varies from 0 to π, the radius vector sweeps out the semicircle. We have

$$x = R\sin\theta \qquad dx = R\cos\theta\,d\theta \qquad F_x = AR^3\sin^3\theta$$
$$y = R(1-\cos\theta) \qquad dy = R\sin\theta\,d\theta \qquad F_y = AR^3\sin\theta(1-\cos\theta)^2$$
$$\oint \mathbf{F}\cdot d\mathbf{r} = A\int_0^\pi [(R\sin\theta)^3 R\cos\theta + R^3\sin\theta\,(1-\cos\theta)^2 R\sin\theta]\,d\theta$$
$$= R^4 A\int_0^\pi [\sin^3\theta\cos\theta + \sin^2\theta(1-\cos\theta)^2]\,d\theta.$$

Evaluation of the integral is straightforward. If you are interested in carrying it through, try substituting $u = \cos\theta$.

4.7 Potential Energy

We introduced the idea of a conservative force in the last section. The work done by a conservative force on a particle as it moves from one point to another depends only on the end points, not on the path between them. Hence, for a conservative force,

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = \text{function of } (\mathbf{r}_b) - \text{function of } (\mathbf{r}_a)$$

or

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F}\cdot d\mathbf{r} = -U(\mathbf{r}_b) + U(\mathbf{r}_a), \qquad 4.18$$

where $U(\mathbf{r})$ is a function, defined by the above expression, known as the *potential energy* function. (The reason for the sign convention will be clear in a moment.) Note that we have not proven that $U(\mathbf{r})$ exists. However, we have already seen several cases where the work is indeed path-independent, so that we can assume that U exists for at least a few forces.

The work-energy theorem $W_{ba} = K_b - K_a$ now becomes

$$\begin{aligned} W_{ba} &= -U_b + U_a \\ &= K_b - K_a \end{aligned}$$

or, rearranging,

$$K_a + U_a = K_b + U_b. \qquad 4.19$$

The left hand side of this equation, $K_a + U_a$, depends on the speed of the particle and its potential energy at $\mathbf{r}_a$; it makes no reference to $\mathbf{r}_b$. Similarly, the right hand side depends on the speed and potential energy at $\mathbf{r}_b$; it makes no reference to $\mathbf{r}_a$. This can be true only if each side of the equation equals a constant, since $\mathbf{r}_a$ and $\mathbf{r}_b$ are arbitrary and not specially chosen points. Denoting this constant by E, we have

$$K_a + U_a = K_b + U_b = E. \qquad 4.20$$

E is called the *total mechanical energy* of the particle, or, somewhat less precisely, the total energy. We have shown that if the force is conservative, the total energy is independent of the position of the particle—it remains constant, or, in the language of physics, the energy is *conserved.* Although the conservation of mechanical energy is a derived law, which means that it has basically no new physical content, it presents such a different way of looking at a physical process compared with applying Newton's laws that we have what amounts to a completely new tool. Furthermore, although the conservation of mechanical energy follows directly from Newton's laws, it is an important key to understanding the more general law of conservation of energy, which is independent of Newton's laws and which vastly increases our understanding of nature. When we discuss this in greater detail in Sec. 4.12, we shall see that the conservation law for mechanical energy turns out to be a special case of the more general law.

A peculiar property of energy is that the value of E is to a certain extent arbitrary; only changes in E have physical significance. This comes about because the equation

$$U_b - U_a = -\int_a^b \mathbf{F} \cdot d\mathbf{r}$$

defines only the difference in potential energy between a and b and not the potential energy itself. We could add a constant to U_b and the same constant to U_a and still satisfy the defining equation. However, since $E = K + U$, adding a constant to U increases E by the same amount.

Illustrations of Potential Energy

We have already seen that for a uniform force or a central force the work is path-independent. There are many other conservative forces, but by way of illustrating potential energy, here are two examples involving these forces.

Example 4.11 **Potential Energy of a Uniform Force Field**

From Example 4.7, the work done by a uniform force is $W_{ba} = \mathbf{F}_0 \cdot (\mathbf{r}_b - \mathbf{r}_a)$ For instance, the force on a particle of mass m due to a uniform gravitational field is $-mg\hat{\mathbf{k}}$, so that if the particle moves from $\mathbf{r}_a$ to $\mathbf{r}_b$, the change in potential energy is

$$U_b - U_a = -\int_{z_a}^{z_b} (-mg)\,dz = mg(z_b - z_a).$$

If we adopt the convention $U = 0$ at ground level where $z = 0$, then $U(h) = mgh$, where h is the height above the ground. However, a potential energy of the form $mgh + C$, where C is any constant, is just as suitable.

In Example 4.1 we considered the problem of a mass projected upward with a given initial velocity in a region of constant gravity. Here is how to solve the same problem by using conservation of energy.

Suppose that a mass is projected upward with initial velocity $\mathbf{v}_0 = v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}} + v_{0z}\hat{\mathbf{k}}$. Find the speed at height h.

$$K_0 + U_0 = K(h) + U(h)$$
$$\tfrac{1}{2}mv_0^2 + 0 = \tfrac{1}{2}mv(h)^2 + mgh$$

or

$$v(h) = \sqrt{v_0^2 - 2gh}.$$

Example 4.11 is trivial, since motion in a uniform force field is easily found from $\mathbf{F} = m\mathbf{a}$. However, it does illustrate the ease with which the energy method handles the problem. For instance, motion in all three directions is handled at once, whereas Newton's law involves one equation for each component of motion.

Example 4.12 **Potential Energy of an Inverse Square Force**

Frequently we encounter central forces $\mathbf{F} = f(r)\hat{\mathbf{r}}$, where $f(r)$ is some function of the distance to the origin. For instance, in the case of the Coulomb electrostatic force, $\mathbf{F} \propto (q_1q_2/r^2)\hat{\mathbf{r}}$, where q_1 and q_2 are the charges of two interacting particles. The gravitational force between two particles provides another example.

The potential energy of a particle in a central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$ obeys

$$U_b - U_a = -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{F} \cdot d\mathbf{r}$$
$$= -\int_{r_a}^{r_b} f(r)\,dr.$$

For an inverse square force, $f(r) = A/r^2$, and we have

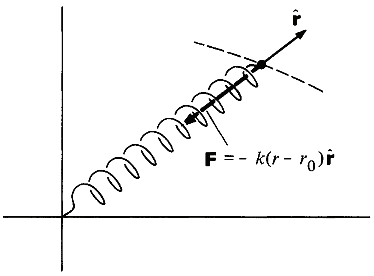
$$U_b - U_a = -\int_{r_a}^{r_b} \frac{A}{r^2}\,dr$$
$$= \frac{A}{r_b} - \frac{A}{r_a}.$$

To obtain the general potential energy function, we replace r_b by the radial variable r. Then

$$U(r) = \frac{A}{r} + \left(U_a - \frac{A}{r_a}\right)$$
$$= \frac{A}{r} + C.$$

The constant C has no physical meaning, since only changes in U are significant. We are free to give C any value we like. A convenient choice in this case is $C = 0$, which corresponds to taking $U(\infty) = 0$. With this convention we have

$$U(r) = \frac{A}{r}.$$



One of the most important forces in physics is the linear restoring force, the spring force. To show that the spring force is conservative, consider a spring of equilibrium length r_0 with one end attached at the origin. If the spring is stretched to length r along direction $\hat{\mathbf{r}}$, it exerts a force

$$\mathbf{F}(r) = -k(r - r_0)\hat{\mathbf{r}}.$$

Since the force is central, it is conservative. The potential energy is given by

$$U(r) - U(a) = -\int_a^r (-k)(r - r_0)\,dr$$
$$= \tfrac{1}{2}k(r - r_0)^2 \Big|_a^r.$$

Hence

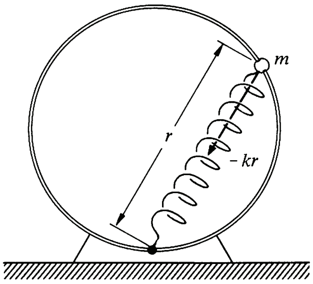
$$U(r) = \tfrac{1}{2}k(r - r_0)^2 + C.$$

Conventionally, we choose the potential energy to be zero at equilibrium: $U(r_0) = 0$. This gives

$$U(r) = \tfrac{1}{2}k(r - r_0)^2. \qquad 4.21$$

When several conservative forces act on a particle, the potential energy is the sum of the potential energies for each force. In the next example, two conservative forces act.

Example 4.13 **Bead, Hoop, and Spring**

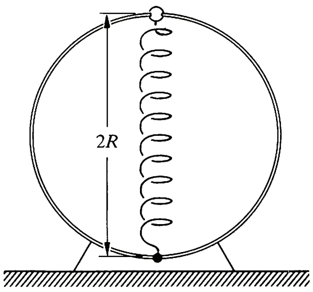


A bead of mass m slides without friction on a vertical hoop of radius R. The bead moves under the combined action of gravity and a spring attached to the bottom of the hoop. For simplicity, we assume that the equilibrium length of the spring is zero, so that the force due to the spring is $-kr$, where r is the instantaneous length of the spring, as shown.

The bead is released at the top of the hoop with negligible speed. How fast is the bead moving at the bottom of the hoop?

At the top of the hoop, the gravitational potential energy of the bead is $mg(2R)$ and the potential energy due to the spring is $\tfrac{1}{2}k(2R)^2 = 2kR^2$. Hence the initial potential energy is

$$U_i = 2mgR + 2kR^2.$$



The potential energy at the bottom of the hoop is

$$U_f = 0.$$

Since all the forces are conservative, the mechanical energy is constant and we have

$$K_i + U_i = K_f + U_f.$$

The initial kinetic energy is zero and we obtain

$$K_f = U_i - U_f$$

or

$$\tfrac{1}{2}mv_f^2 = 2mgR + 2kR^2.$$

Hence

$$v_f = 2\sqrt{gR + \frac{kR^2}{m}}.$$

4.8 What Potential Energy Tells Us about Force

If we are given a conservative force, it is a straightforward matter to find the potential energy from the defining equation

$$U_b - U_a = -\int_a^b \mathbf{F} \cdot d\mathbf{r},$$

where the integral is over any path from $\mathbf{r}_a$ to $\mathbf{r}_b$. However, in many cases it is easier to characterize a force by giving its potential energy function rather than by specifying each of its components. In such cases we would like to use our knowledge of the potential energy to determine what force is acting. The procedure for finding the force turns out to be simple. In this section we shall learn how to find the force from the potential energy in a one dimensional system. The general case of three dimensions can be treated by a straightforward extension of the method developed here, but since it involves some new notation which is more readily introduced in the next chapter, let us defer the three dimensional case until then.

Suppose that we have a one dimensional system, such as a mass on a spring, in which the force is $F(x)$ and the potential energy is

$$U_b - U_a = -\int_{x_a}^{x_b} F(x)\,dx.$$

Consider the change in potential energy ΔU as the particle moves from some point x to $x + \Delta x$.

$$\begin{aligned} U(x + \Delta x) - U(x) &\equiv \Delta U \\ &= -\int_x^{x+\Delta x} F(x)\,dx. \end{aligned}$$

For Δx sufficiently small, $F(x)$ can be considered constant over the range of integration and we have

$$\begin{aligned} \Delta U &\approx -F(x)(x + \Delta x - x) \\ &= -F(x)\,\Delta x \end{aligned}$$

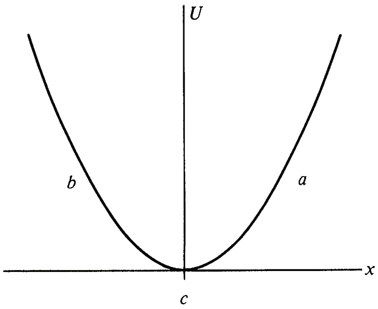
or

$$F(x) \approx -\frac{\Delta U}{\Delta x}.$$

In the limit $\Delta x \rightarrow 0$ we have

$$F(x) = -\frac{dU}{dx}. \qquad 4.22$$

The result is quite reasonable: potential energy is the negative integral of the force, and it follows that force is the negative derivative of the potential energy.

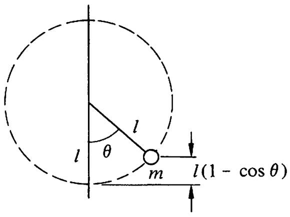


Stability

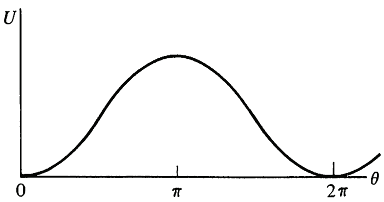
The result $F = -dU/dx$ is useful not only for computing the force but also for visualizing the stability of a system from a diagram of the potential energy. For instance, in the case of a harmonic oscillator the potential energy $U = kx^2/2$ is described by a parabola.

At point a, $dU/dx > 0$ and so the force is negative. At point b, $dU/dx < 0$ and the force is positive. At c, $dU/dx = 0$ and the force is zero. The force is directed toward the origin no matter which way the particle is displaced, and the force vanishes only when the particle is at the origin. The minimum of the potential energy curve coincides with the equilibrium position of the system. Evidently this is a stable equilibrium, since any displacement of the system produces a force which tends to push the particle toward its resting point.

Whenever $dU/dx = 0$, a system is in equilibrium. However, if this occurs at a maximum of U, the equilibrium is not stable, since a positive displacement produces a positive force, which tends to increase the displacement, and a negative displacement produces a negative force, which again causes the displacement to become larger. A pendulum of length l supporting mass m offers a good illustration of this. If we take the potential energy to be zero at the bottom of its swing, we see that



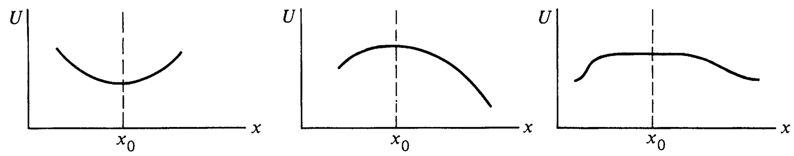
$$
\begin{aligned}
U(\theta) &= mgz \\
&= mgl(1 - \cos\theta).
\end{aligned}
$$



The pendulum is in equilibrium for $\theta = 0$ and $\theta = \pi$. However, although the pendulum will quite happily hang downward for as long as you please, it will not hang vertically up for long. $dU/dx = 0$ at $\theta = \pi$, but U has a maximum there and the equilibrium is not stable.

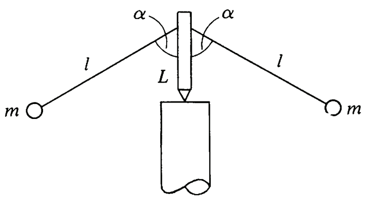
The sketch of a potential energy function makes the idea of stability almost intuitively obvious. A minimum of a potential energy curve is a point of stable equilibrium, and a maximum is a point of unstable equilibrium. In more descriptive terms, the system is stable at the bottom of a potential energy "valley," and unstable at the top of a potential energy "hill."

Alternatively, we can use a simple mathematical test to determine whether or not an equilibrium point is stable. Let $U(x)$ be the potential energy function for a particle. As we have shown, the force on the particle is $F = -dU/dx$, and the system is in equilibrium where $dU/dx = 0$. Suppose that this occurs at some point x_0. To test for stability we must determine whether U has a minimum or a maximum at x_0. To accomplish this we need to examine d^2U/dx^2 at x_0. If the second derivative is positive, the equilibrium is stable; if it is negative, the system is unstable. If $d^2U/dx^2 = 0$, we must look at higher derivatives. If all derivatives vanish so that U is constant in a region about x_0, the system is said to be in a condition of neutral stability—no force results from a displacement; the particle is effectively free.



$\frac{d^2U}{dx^2} > 0$ stable $\frac{d^2U}{dx^2} < 0$ unstable $\frac{d^2U}{dx^2} = 0$ neutral

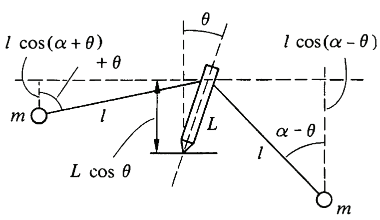
Example 4.14 **Energy and Stability—The Teeter Toy**



The teeter toy consists of two identical weights which hang from a peg on drooping arms, as shown. The arrangement is unexpectedly stable—the toy can be spun or rocked with little danger of toppling over. We can see why this is so by looking at its potential energy. For simplicity, we shall consider only rocking motion in the vertical plane.

Let us evaluate the potential energy when the teeter toy is cocked at angle θ, as shown in the sketch. If we take the zero of gravitational potential at the pivot, we have

$$U(\theta) = mg[L\cos\theta - l\cos(\alpha + \theta)] + mg[L\cos\theta - l\cos(\alpha - \theta)].$$



Using the identity $\cos(\alpha \pm \theta) = \cos\alpha\cos\theta \mp \sin\alpha\sin\theta$, we can rewrite $U(\theta)$ as

$$U(\theta) = 2mg\cos\theta(L - l\cos\alpha).$$

Equilibrium occurs when

$$\frac{dU}{d\theta} = -2mg\sin\theta(L - l\cos\alpha)$$
$$= 0.$$

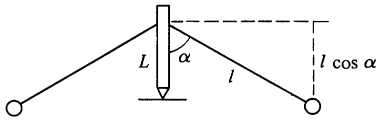
The solution is $\theta = 0$, as we expect from symmetry. (We reject the solution $\theta = \pi$ on the grounds that θ must be limited to values less than

$\pi/2$.) To investigate the stability of the equilibrium position, we must examine the second derivative of the potential energy. We have

$$\frac{d^2U}{d\theta^2} = -2mg\cos\theta(L - l\cos\alpha).$$

At equilibrium,

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -2mg(L - l\cos\alpha).$$



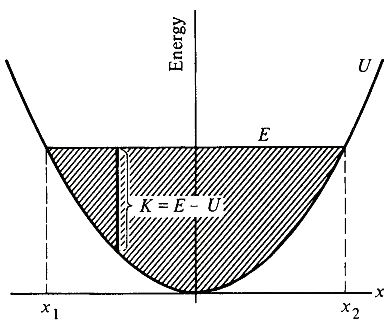
For the second derivative to be positive, we require $L - l\cos\alpha < 0$, or

$L < l\cos\alpha$.

In order for the teeter toy to be stable, the weights must hang below the pivot.

4.9 Energy Diagrams

We can often find the most interesting features of the motion of a one dimensional system by using an *energy diagram*, in which the total energy E and the potential energy U are plotted as functions of position. The kinetic energy $K = E - U$ is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where $U \leq E$.

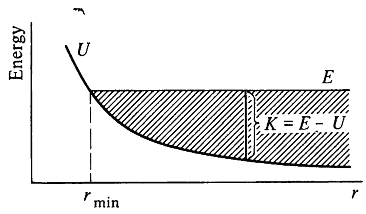


Here is the energy diagram for a harmonic oscillator. The potential energy $U = kx^2/2$ is a parabola centered at the origin. Since the total energy is constant for a conservative system, E is represented by a horizontal straight line. Motion is limited to the shaded region where $E \geq U$; the limits of the motion, x_1 and x_2 in the sketch, are sometimes called the turning points.

Here is what the diagram tells us. The kinetic energy, $K = E - U$, is greatest at the origin. As the particle flies past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points x_1, x_2. The particle then moves toward the origin with increasing kinetic energy, and the cycle is repeated.

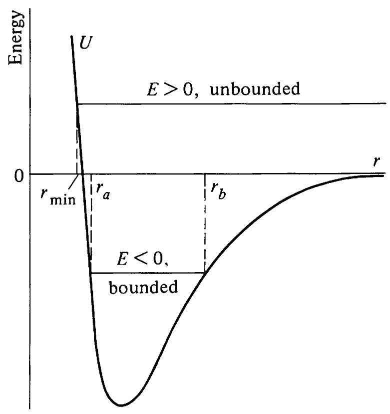
The harmonic oscillator provides a good example of bounded motion. As E increases, the turning points move farther and farther off, but the particle can never move away freely. If E is decreased, the amplitude of motion decreases, until finally for $E = 0$ the particle lies at rest at $x = 0$.

Quite a different behavior occurs if U does not increase indefinitely with distance. For instance, consider the case of a particle constrained to a radial line and acted on by a repulsive inverse



square law force $A\hat{\mathbf{r}}/r^2$. Here $U = A/r$, where A is positive. There is a distance of closest approach, $r_{\min}$, as shown in the diagram, but the motion is not bounded for large r since U decreases with distance. If the particle is shot toward the origin, it gradually loses kinetic energy until it comes momentarily to rest at $r_{\min}$. The motion then reverses and the particle moves out toward infinity. The final and initial speeds at any point are identical; the collision merely reverses the velocity.

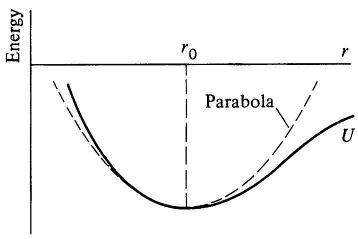
With some potentials, either bounded or unbounded motion can occur depending upon the energy. For instance, consider the interaction between two atoms. At large separations, the atoms attract each other weakly with the van der Waals force, which varies as $1/r^7$. As the atoms approach, the electron clouds begin to overlap, producing strong forces. In this intermediate region the force is either attractive or repulsive depending on the details of the electron configuration. If the force is attractive, the potential energy decreases with decreasing r. At very short distances the atoms always repel each other strongly, so that U increases rapidly as r becomes small.



The energy diagram for a typical attractive two atom system is shown in the sketch. For positive energy, $E > 0$, the motion is unbounded, and the atoms are free to fly apart. As the diagram indicates, the distance of closest approach, $r_{\min}$, does not change appreciably as E is increased. The steep slope of the potential energy curve at small r means that the atoms behave like hard spheres—$r_{\min}$ is not sensitive to the energy of collision.

The situation is quite different if E is negative. Then the motion is bounded for both small and large separations; the atoms never approach closer than r_a or move farther apart than r_b. A bound system of two atoms is, of course, a molecule, and our sketch represents a typical diatomic molecule energy diagram. If two atoms collide with positive energy, they cannot form a molecule unless some means is available for losing enough energy to make E negative. In general, a third body is necessary to carry off the excess energy. Sometimes the third body is a surface, which is the reason surface catalysts are used to speed certain reactions. For instance, atomic hydrogen is quite stable in the gas phase even though the hydrogen molecule is tightly bound. However, if a piece of platinum is inserted in the hydrogen, the atoms immediately join to form molecules. What happens is that hydrogen atoms tightly adhere to the surface of the platinum, and if a collision occurs between two atoms on the surface, the excess energy is released to the surface, and the molecule, which is not strongly attracted to the surface, leaves. The energy delivered to the surface is so large that the platinum glows brightly. A third atom can also carry off the excess energy, but for this to happen the two atoms must collide when a third atom is nearby. This is a rare event at low pressures, but it becomes increasingly important at higher pressures. Another possibility is for the two atoms to lose energy by the emission of light. However, this occurs so rarely that it is usually not important.

4.10 Small Oscillations in a Bound System



The interatomic potential we discussed in the last section illustrates an important feature of all bound systems; at equilibrium the potential energy has a minimum. As a result, nearly every bound system oscillates like a harmonic oscillator if it is slightly perturbed from its equilibrium position. This is suggested by the appearance of the energy diagram near the minimum—U has the parabolic shape of a harmonic oscillator potential. If the total energy is low enough so that the motion is restricted to the region where the curve is nearly parabolic, as illustrated in the sketch, the system must behave like a harmonic oscillator. It is not difficult to prove this.

As we have discussed in Note 1.1, any "well behaved" function $f(x)$ can be expanded in a Taylor's series about a point x_0. Thus

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + \cdots .$$

Suppose that we expand $U(r)$ about r_0, the position of the potential minimum. Then

$$U(r) = U(r_0) + (r - r_0)\left.\frac{dU}{dr}\right|_{r_0} + \frac{1}{2}(r - r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0} + \cdots .$$

However, since U is a minimum at r_0, $(dU/dr)|_{r_0} = 0$. Furthermore, for sufficiently small displacements, we can neglect the terms beyond the third in the power series. In this case,

$$U(r) = U(r_0) + \frac{1}{2}(r - r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

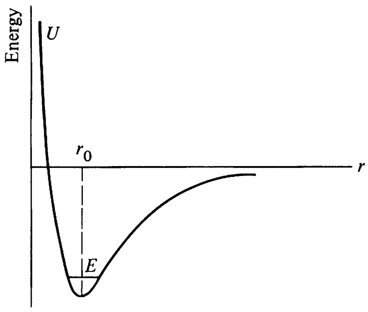
This is the potential energy of a harmonic oscillator,

$$U(x) = \text{constant} + \frac{kx^2}{2}.$$

We can even identify the effective spring constant:

$$k = \left.\frac{d^2U}{dr^2}\right|_{r_0} \qquad 4.23$$

Example 4.15 **Molecular Vibrations**

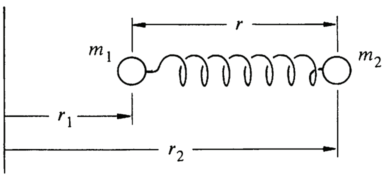


Suppose that two atoms of masses m_1 and m_2 are bound together in a molecule with energy so low that their separation is always close to the equilibrium value r_0. With the parabola approximation, the effective spring constant is $k = (d^2U/dr^2)|_{r_0}$. How can we find the vibration frequency of the molecule?

Consider the two atoms connected by a spring of equilibrium length r_0 and spring constant k, as shown below. The equations of motion are

$$m_1\ddot{r}_1 = k(r - r_0)$$
$$m_2\ddot{r}_2 = -k(r - r_0),$$

where $r = r_2 - r_1$ is the instantaneous separation of the atoms. We can find the equation of motion for r by dividing the first equation by m_1 and the second by m_2, and subtracting. The result is



$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

or

$$\ddot{r} = -\frac{k}{\mu}(r - r_0),$$

where $\mu = m_1m_2/(m_1 + m_2)$. μ has the dimension of mass and is called the *reduced mass.*

By analogy with the harmonic oscillator equation $\ddot{x} = -(k/m)(x - x_0)$ for which the frequency of oscillation is $\omega = \sqrt{k/m}$, the vibrational frequency of the molecule is

$$\omega = \sqrt{\frac{k}{\mu}}$$
$$= \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r_0}\frac{1}{\mu}}.$$

This vibrational motion, characteristic of all molecules, can be identified by the light the molecule radiates. The vibrational frequencies typically lie in the near infrared (3×10^{13} Hz), and by measuring the frequency we can find the value of d^2U/dr^2 at the potential energy minimum. For the HCl molecule, the effective spring constant turns out to be 5×10^5 dynes/cm = 500 N/m (roughly 3 lb/in). For large amplitudes the higher order terms in the Taylor's series start to play a role, and these lead to slight departures of the oscillator from its ideal behavior. The slight "anharmonicities" introduced by this give further details on the shape of the potential energy curve.

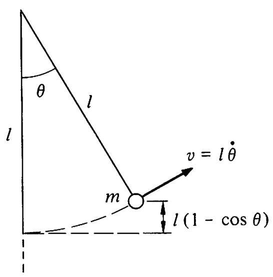
Since all bound systems have a potential energy minimum at equilibrium, we naturally expect that all bound systems behave like harmonic oscillators for small displacements (unless the minimum is so flat that the second derivative vanishes there also). The harmonic oscillator approximation therefore has a wide range of applicability, even down to internal motions in nuclei.

Once we have identified the kinetic and potential energies of a bound system, we can find the frequency of small oscillations by inspection. For the elementary case of a mass on a spring we have

$$U = \tfrac{1}{2}kx^2$$
$$K = \tfrac{1}{2}m\dot{x}^2$$

and

$$\omega = \sqrt{\frac{k}{m}}.$$



In many problems, however, it is more natural to write the energies in terms of a variable other than linear displacement. For instance, the energies of a pendulum are

$$U = mgl(1 - \cos\theta) \approx \tfrac{1}{2}mgl\theta^2$$
$$K = \tfrac{1}{2}ml^2\dot{\theta}^2.$$

More generally, the energies may have the form

$$U = \tfrac{1}{2}Aq^2 + \text{constant}$$
$$K = \tfrac{1}{2}B\dot{q}^2, \qquad 4.24$$

where q represents a variable appropriate to the problem. By analogy with the mass on a spring, we expect that the frequency of motion of the oscillator is

$$\omega = \sqrt{\frac{A}{B}}. \qquad 4.25$$

To show explicitly that any system whose energy has the form of Eq. (4.24) oscillates harmonically with a frequency $\sqrt{A/B}$, note that the total energy of the system is

$$E = K + U$$
$$= \tfrac{1}{2}B\dot{q}^2 + \tfrac{1}{2}Aq^2 + \text{constant}.$$

Since the system is conservative, E is constant. Differentiating the energy equation with respect to time gives

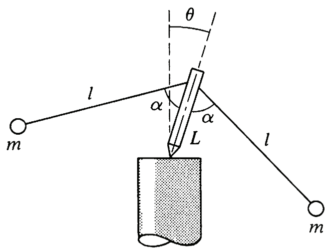
$$\frac{dE}{dt} = B\dot{q}\ddot{q} + Aq\dot{q}$$
$$= 0$$

or

$$\ddot{q} + \frac{A}{B}q = 0.$$

Hence q undergoes harmonic motion with frequency $\sqrt{A/B}$.

Example 4.16 Small Oscillations



In Example 4.14 we determined the stability criterion for a teeter toy. In this example we shall find the period of oscillation of the toy when it is rocking from side to side.

From Example 4.14, the potential energy of the teeter toy is

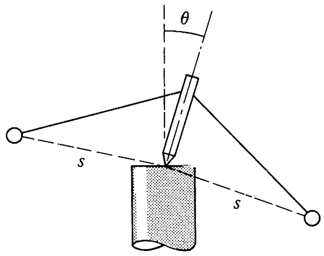
$$U(\theta) = -A\cos\theta,$$

where $A = 2mg(l\cos\alpha - L)$. For stability, $A > 0$. If we expand $U(\theta)$ about $\theta = 0$, we have

$$U(\theta) = -A\left(1 - \frac{\theta^2}{2} + \cdots\right),$$

since $\cos\theta = 1 - \theta^2/2 + \cdots$. Thus,

$$U(\theta) = -A + \tfrac{1}{2}A\theta^2.$$



To find the kinetic energy, let s be the distance of each mass from the pivot, as shown in the sketch. If the toy rocks with angular speed $\dot{\theta}$, the speed of each mass is $s\dot{\theta}$, and the total kinetic energy is

$$K = \tfrac{1}{2}(2m)s^2\dot{\theta}^2$$
$$= \tfrac{1}{2}B\dot{\theta}^2,$$

where $B = 2ms^2$.

Hence the frequency of oscillation is

$$\omega = \sqrt{\frac{A}{B}}$$
$$= \sqrt{\frac{g(l\cos\alpha - L)}{s^2}}. \qquad 1$$

We found in Example 4.14 that for stability $l \cos \alpha - L > 0$. Equation (1) shows that as $l \cos \alpha - L$ approaches zero, ω approaches zero, and the period of oscillation becomes infinite. In the limit $l \cos \alpha - L = 0$, the system is in neutral equilibrium, and if $l \cos \alpha - L < 0$, the system becomes unstable. Thus, a low frequency of oscillation is associated with the system operating near the threshold of stability. This is a general property of stable systems, because a low frequency of oscillation corresponds to a weak restoring force. For instance, a ship rolled by a wave oscillates about equilibrium. For comfort the period of the roll should be long. This can be accomplished by designing the hull so that its center of gravity is as high as possible consistent with stability. Lowering the center of gravity makes the system "stiffer." The roll becomes quicker and less comfortable, but the ship becomes intrinsically more stable.

4.11 Nonconservative Forces

We have stressed conservative forces and potential energy in this chapter because they play an important role in physics. However, in many physical processes nonconservative forces like friction are present. Let's see how to extend the work-energy theorem to include nonconservative forces.

Often both conservative and nonconservative forces act on the same system. For instance, an object falling through the air experiences the conservative gravitational force and the nonconservative force of air friction. We can write the total force $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{F}^c + \mathbf{F}^{nc}$$

where $\mathbf{F}^c$ and $\mathbf{F}^{nc}$ are the conservative and the nonconservative forces respectively. Since the work-energy theorem is true whether or not the forces are conservative, the total work done by $\mathbf{F}$ as the particle moves from a to b is

$$\begin{aligned} W_{ba}{}^{\text{total}} &= \oint_a^b \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_a^b \mathbf{F}^c \cdot d\mathbf{r} + \oint_a^b \mathbf{F}^{nc} \cdot d\mathbf{r} \\ &= -U_b + U_a + W_{ba}{}^{\text{nc}}. \end{aligned}$$

Here U is the potential energy associated with the conservative force and $W_{ba}{}^{\text{nc}}$ is the work done by the nonconservative force. The work-energy theorem, $W_{ba}{}^{\text{total}} = K_b - K_a$, now has the form

$$-U_b + U_a + W_{ba}{}^{\text{nc}} = K_b - K_a$$

or

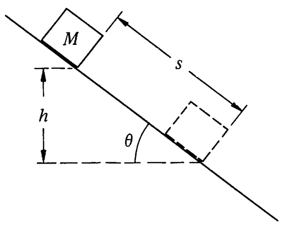
$$K_b + U_b - (K_a + U_a) = W_{ba}^{\mathrm{nc}}. \qquad 4.26$$

If we define the total mechanical energy by $E = K + U$, as before, then E is no longer a constant but instead depends on the state of the system. We have

$$E_b - E_a = W_{ba}^{\mathrm{nc}}. \qquad 4.27$$

This result is a generalization of the statement of conservation of mechanical energy which we discussed in Sec. 4.7. If nonconservative forces do no work, $E_b = E_a$, and mechanical energy is conserved. However, this is a special case, since nonconservative forces are often present. Nevertheless, energy methods continue to be useful; we simply must be careful not to omit the work done by the nonconservative forces, W_{ba}^{nc}. Here is an example.

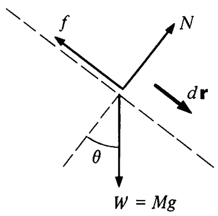
Example 4.17 Block Sliding down Inclined Plane



A block of mass M slides down a plane of angle θ. The problem is to find the speed of the block after it has descended through height h, assuming that it starts from rest and that the coefficient of friction μ is constant.

Initially the block is at rest at height h; finally the block is moving with speed v at height 0. Hence

$$\begin{aligned} U_a &= Mgh & U_b &= 0 \\ K_a &= 0 & K_b &= \tfrac{1}{2}Mv^2 \\ E_a &= Mgh & E_b &= \tfrac{1}{2}Mv^2. \end{aligned}$$



The nonconservative force is $f = \mu N = \mu Mg \cos\theta$. Hence, the nonconservative work is

$$\begin{aligned} W_{ba}^{\mathrm{nc}} &= \int_a^b \mathbf{f}\cdot d\mathbf{r} \\ &= -fs, \end{aligned}$$

where s is the distance the block slides. The negative sign arises because the direction of $\mathbf{f}$ is always opposite to the displacement, so that $\mathbf{f}\cdot d\mathbf{r} = -f\,dr$. Using $s = h/\sin\theta$, we have

$$\begin{aligned} W_{ba}^{\mathrm{nc}} &= -\mu Mg \cos\theta \frac{h}{\sin\theta} \\ &= -\mu \cot\theta\, Mgh. \end{aligned}$$

The energy equation $E_b - E_a = W_{ba}^{\text{nc}}$ becomes

$$\tfrac{1}{2}Mv^2 - Mgh = -\mu \cot \theta \, Mgh,$$

which gives

$$v = [2(1 - \mu \cot \theta)gh]^{\frac{1}{2}}.$$

Since all the forces acting on the block are constant, the expression for v could easily be found by applying our results for motion under uniform acceleration; the energy method does not represent much of a shortcut here. The power of the energy method lies in its generality. For instance, suppose that the coefficient of friction varies along the surface so that the friction force is $f = \mu(x)Mg \cos \theta$. The work done by friction is

$$W_{ba}^{\text{nc}} = -Mg \cos \theta \int_a^b \mu(x) \, dx,$$

and the final speed is easily found. In contrast, there is no simple way to find the speed by integrating the acceleration with respect to time.

4.12 The General Law of Conservation of Energy

As far as we know, the basic forces of nature, such as the force of gravity and the forces of electric and magnetic interactions, are conservative. This leads to a puzzle; if fundamental forces are conservative, how can nonconservative forces arise? The resolution of this problem lies in the point of view we adopt in describing a physical system, and in our willingness to broaden the concept of energy.

Consider friction, the most familiar nonconservative force. Mechanical energy is lost by friction when a block slides across a table, but something else occurs: the block and the table get warmer. However, there was no reference to temperature in our development of the concept of mechanical energy; a block of mass M moving with speed v has kinetic energy $\frac{1}{2}Mv^2$, whether the block is hot or cold. The fact that a block sliding across a table warms up does not affect our conclusion that mechanical energy is lost. Nevertheless, if we look carefully, we find that the heating of the system bears a definite relation to the energy dissipated. The British physicist James Prescott Joule was the first to appreciate that heat itself represents a form of energy.

By a series of meticulous experiments on the heating of water by a paddle wheel driven by a falling weight, he showed that the loss of mechanical energy by friction is accompanied by the appearance of an equivalent amount of heat. Joule concluded that heat must be a form of energy and that the sum of the mechanical energy and the heat energy of a system is conserved.

We now have a more detailed picture of heat energy than was available to Joule. We know that solids are composed of atoms held together by strong interatomic forces. Each atom can oscillate about its equilibrium position and has mechanical energy in the form of kinetic and potential energies. As the solid is heated, the amplitude of oscillation increases and the average energy of each atom grows larger. The heat energy of a solid is the mechanical energy of the random vibrations of the atoms.

There is a fundamental difference between mechanical energy on the atomic level and that on the level of everyday events. The atomic vibrations in a solid are random; at any instant there are atoms moving in all possible directions, and the center of mass of the block has no tendency to move on the average. Kinetic energy of the block represents a collective motion; when the block moves with velocity $\mathbf{v}$, each atom has, on the average, the same velocity $\mathbf{v}$.

Mechanical energy is turned into heat energy by friction, but the reverse process is never observed. No one has ever seen a hot block at rest on a table suddenly cool off and start moving, although this would not violate conservation of energy. The reason is that collective motion can easily become randomized. For instance, when a block hits an obstacle, the collective translational motion ceases and, under the impact, the atoms start to jitter more violently. Kinetic energy has been transformed to heat energy. The reverse process where the random motion of the atoms suddenly turns to collective motion is so improbable that for all practical purposes it never occurs. It is for this reason that we can distinguish between the heat energy and the mechanical energy of a chunk of matter even though on the atomic scale the distinction vanishes.

We now recognize that in addition to mechanical energy and heat there are many other forms of energy. These include the radiant energy of light, the energy of nuclear forces, and, as we shall discuss in Chap. 13, the energy associated with mass. It is apparent that the concept of energy is much wider than the simple idea of kinetic and potential energy of a mechanical system. We believe that the total energy of a system is conserved if all forms of energy are taken into account.

4.13 Power

Power is the time rate of doing work. If a force $\mathbf{F}$ acts on a body which undergoes a displacement $d\mathbf{r}$, the work is $dW = \mathbf{F} \cdot d\mathbf{r}$ and the power delivered by the force is

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$$
$$= \mathbf{F} \cdot \mathbf{v}.$$

The unit of power in the SI system is the watt (W).

$$1 \text{ W} = 1 \text{ J/s}.$$

In the cgs system, the unit of power is the erg/s $= 10^{-7}$ W; it has no special name. The unit of power in the English system is the horsepower (hp). The horsepower is most commonly defined as 550 ft·lb/s, but slightly different definitions are sometimes encountered. The relation between the horsepower and the watt is

$$1 \text{ hp} \approx 746 \text{ W}.$$

This is a discouraging number for builders of electric cars; the average power obtainable from an ordinary automobile storage battery is only about 350 W.

The power rating of an engine is a useful indicator of its performance. For instance, a small motor with a system of reduction gears can raise a large mass M any given height, but the process will take a long time; the average power delivered is low. The power required is Mgv, where v is the weight's upward speed. To raise the mass rapidly the power must be large.

A human being in good condition can develop between $\frac{1}{2}$ to 1 hp for 30 s or so, for example while running upstairs. Over a period of 8 hours (h), however, a husky man can do work only at the rate of about 0.2 hp = 150 W. The total work done in 8 h is then $(150)(8)(3{,}600) = 4.3 \times 10^6 \text{ J} \approx 1{,}000$ kcal. The kilocalorie, approximately equal to 4,200 J, is often used to express the energy available from food. A normally active person requires 2,000 to 3,000 kcal/d. (In dietetic work the kilocalorie is sometimes called the "large" calorie, but more often simply the calorie.)

The power production of modern industrialized nations corresponds to several thousand watts per person (United States: 6,000 W per person; India: 300 W per person). The energy comes primarily from the burning of fossil fuels, which are the chief source of energy at present. In principle, we could use the sun's energy directly. When the sun is overhead, it supplies approximately 1,000 W/m^2 ($\approx$ 1 hp/yd^2) to the earth's surface. Unfortunately, present solar cells are costly and inefficient, and there is no economical way of storing the energy for later use.

4.14 Conservation Laws and Particle Collisions

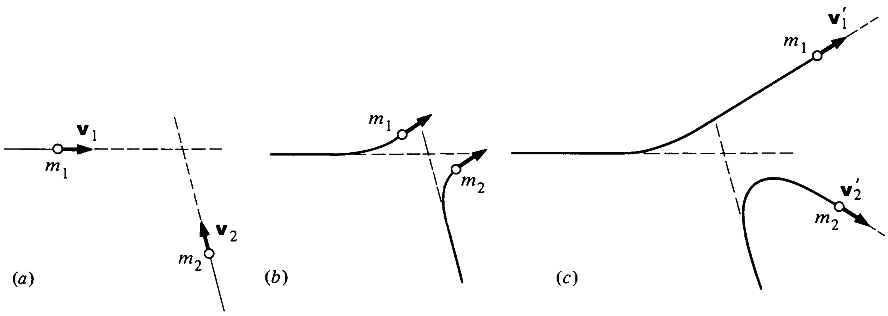
Much of our knowledge of atoms, nuclei, and elementary particles has come from scattering experiments. Perhaps the most dramatic of these was the experiment performed in 1911 by Ernest Rutherford in which alpha particles (doubly ionized helium atoms) were scattered from atoms of gold in a thin foil. By studying how the number of scattered alpha particles varied with the deflection angle, Rutherford was led to the nuclear model of the atom. The techniques of experimental physics have advanced considerably since Rutherford's time. A high energy particle accelerator several miles long may appear to have little in common with Rutherford's tabletop apparatus, but its purpose is the same—to discover the interaction forces between particles by studying how they scatter.

Finding the interaction force from a scattering experiment is a difficult task. Furthermore, the detailed description of collisions on the atomic scale generally requires the use of quantum mechanics. Nevertheless, there are constraints on the motion arising from the conservation laws of momentum and energy which are so strong that they are solely responsible for many of the features of scattering. Since the conservation laws can be applied without knowing the interactions, they play a vital part in the analysis of collision phenomena.

In this section we shall see how to apply the conservation laws of momentum and energy to scattering experiments. No new physical principles are involved; the discussion is intended to illustrate ideas we have already introduced.

Collisions and Conservation Laws

The drawings below show three stages during the collision of two particles. In (a), long before the collision, each particle is effectively free, since the interaction forces are generally important only at very small separations. As the particles approach, (b), the momentum and energy of each particle change due to the interaction forces. Finally, long after the collision, (*c*), the particles are again free and move along straight lines with new directions and velocities. Experimentally, we usually know the initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$; often one particle is initially at rest in a target and is bombarded by particles of known energy. The experiment might consist of measuring the final velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$ with suitable particle detectors.



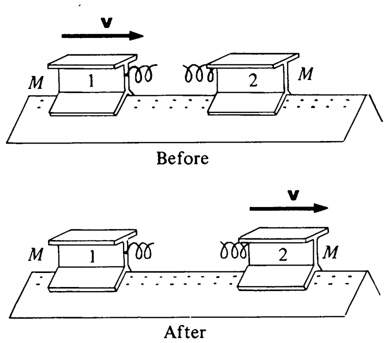
Since external forces are usually negligible, the total momentum is conserved and we have

$$\mathbf{P}_i = \mathbf{P}_f. \tag{4.28}$$

For a two body collision, this becomes

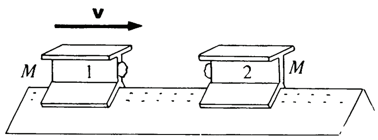
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'. \tag{4.29}$$

Equation (4.29) is equivalent to three scalar equations. We have, however, six unknowns, the components of $\mathbf{v}_1'$ and $\mathbf{v}_2'$. The energy equation provides an additional relation between the velocities, as we now show.

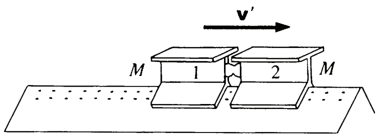


Elastic and Inelastic Collisions

Consider a collision on a linear air track between two riders of equal mass which interact via good coil springs. Suppose that initially rider 1 has speed v as shown and rider 2 is at rest. After the collision, 1 is at rest and 2 moves to the right with speed v. It is clear that momentum has been conserved and that the total kinetic energy of the two bodies, $Mv^2/2$, is the same before and after the collision. A collision in which the total kinetic energy is unchanged is called an *elastic* collision. A collision is elastic if the interaction forces are conservative, like the spring force in our example.



Before



After

As a second experiment, take the same two riders and replace the springs by lumps of sticky putty. Let 2 be initially at rest. After the collision, the riders stick together and move off with speed v'. By conservation of momentum, $Mv = 2Mv'$, so that $v' = v/2$. The initial kinetic energy of the system is $Mv^2/2$, but the final kinetic energy is $(2M)v'^2/2 = Mv^2/4$. Evidently in this collision the kinetic energy is only half as much after the collision as before. The kinetic energy has changed because the interaction forces were nonconservative. Part of the energy of the collective motion was transformed to random heat energy in the putty during the collision. A collision in which the total kinetic energy is not conserved is called an *inelastic* collision.

Although the total energy of the system is always conserved in collisions, part of the kinetic energy may be converted to some other form. To take this into account, we write the conservation of energy equation for collisions as

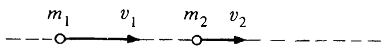
$$K_i = K_f + Q, \qquad 4.30$$

where $Q = K_i - K_f$ is the amount of kinetic energy converted to another form. For a two body collision, Eq. (4.30) becomes

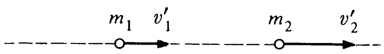
$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2 + Q. \qquad 4.31$$

In most collisions on the everyday scale, kinetic energy is lost and Q is positive. However, Q can be negative if internal energy of the system is converted to kinetic energy in the collision. Such collisions are sometimes called *superelastic,* and they are important in atomic and nuclear physics. Superelastic collisions are rarely encountered in the everyday world, but one example would be the collision of two cocked mousetraps.

Collisions in One Dimension



Before



After

If we have a two body collision in which the particles are constrained to move along a straight line, the conservation laws, Eqs. (4.29) and (4.31), completely determine the final velocities, regardless of the nature of the interaction forces. With the velocities shown in the sketch, the conservation laws give

Momentum:

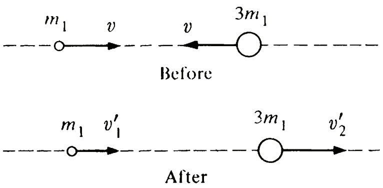
$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'. \qquad 4.32a$$

Energy:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2 + Q. \qquad 4.32b$$

These equations can be solved for v_1' and v_2' in terms of m_1, m_2, v_1, v_2, and Q. The next example illustrates the process.

Example 4.18 **Elastic Collision of Two Balls**



Consider the one dimensional elastic collision of two balls of masses m_1 and m_2, with $m_2 = 3m_1$. Suppose that the balls have equal and opposite velocities **v** before the collision; the problem is to find the final velocities. The conservation laws yield

$$m_1v - 3m_1v = m_1v_1' + 3m_1v_2' \quad 1$$

$$\tfrac{1}{2}m_1v^2 + \tfrac{1}{2}(3m_1)v^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}(3m_1)v_2'^2. \quad 2$$

We can eliminate v_1' using Eq. (1):

$$v_1' = -2v - 3v_2'. \quad 3$$

Inserting this in Eq. (2) gives

$$\begin{aligned} 4v^2 &= (-2v - 3v_2')^2 + 3v_2'^2 \\ &= 4v^2 + 12vv_2' + 12v_2'^2 \end{aligned}$$

or

$$0 = 12vv_2' + 12v_2'^2. \quad 4$$

Equation (4) has two solutions: $v_2' = -v$ and $v_2' = 0$. The corresponding values of v_1' can be found from Eq. (3).

Solution 1:

$v_1' = v$

$v_2' = -v.$

Solution 2:

$v_1' = -2v$

$v_2' = 0.$

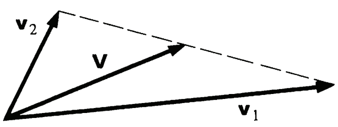
We recognize that solution 1 simply restates the initial conditions: we always obtain such a "solution" in this type of problem because the initial velocities evidently satisfy the conservation law equations.

Solution 2 is the interesting one. It shows that after the collision, m_1 is moving to the left with twice its original speed and the heavier ball is at rest.

Collisions and Center of Mass Coordinates

It is almost always simpler to treat three dimensional collision problems in the center of mass (C) coordinate system than in the laboratory (L) system.

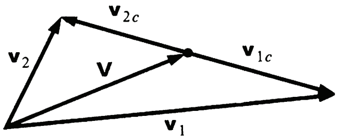
Consider two particles of masses m_1 and m_2, and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. The center of mass velocity is



$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}.$$

As shown in the velocity diagram at left, **V** lies on the line joining $\mathbf{v}_1$ and $\mathbf{v}_2$.

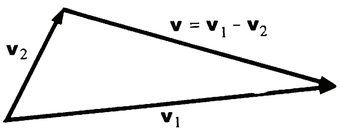
The velocities in the C system are



$$\begin{aligned}\mathbf{v}_{1c} &= \mathbf{v}_1 - \mathbf{V}\\ &= \frac{m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2),\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_{2c} &= \mathbf{v}_2 - \mathbf{V}\\ &= \frac{-m_1}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2).\end{aligned}$$



$\mathbf{v}_{1c}$ and $\mathbf{v}_{2c}$ lie back to back along the relative velocity vector $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.

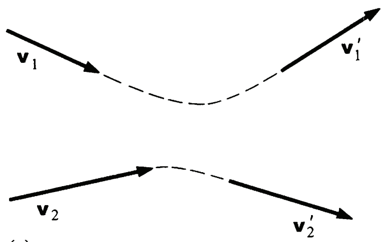
The momenta in the C system are

$$\begin{aligned}\mathbf{p}_{1c} &= m_1\mathbf{v}_{1c}\\ &= \frac{m_1m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2)\\ &= \mu\mathbf{v}\\ \mathbf{p}_{2c} &= m_2\mathbf{v}_{2c}\\ &= \frac{-m_1m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2)\\ &= -\mu\mathbf{v}.\end{aligned}$$

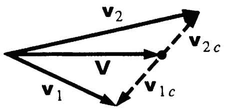
Here $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of the system. We encountered the reduced mass for the first time in Example 4.15. As we shall see in Chap. 9, it is the natural unit of mass in a two particle system. The total momentum in the C system is zero, as we expect.

The total momentum in the L system is

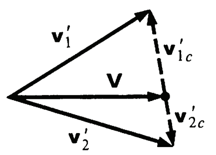
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{V}$$



(a)



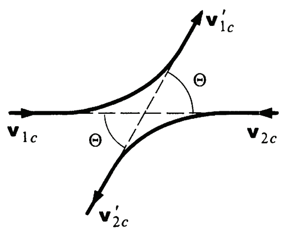
(b)



(c)

and since total momentum is conserved in any collision, $\mathbf{V}$ is constant. We can use this result to help visualize the velocity vectors before and after the collision.

Sketch (a) shows the trajectories and velocities of two colliding particles. In sketch (b) we show the initial velocities in the L and C systems. All the vectors lie in the same plane. $\mathbf{v}_{1c}$ and $\mathbf{v}_{2c}$ must be back to back since the total momentum in the C system is zero. After the collision, sketch (c), the velocities in the C system are again back to back. This sketch also shows the final velocities in the lab system. Note that the plane of sketch c is not necessarily the plane of sketch a. Evidently the geometrical relation between initial and final velocities in the L system is quite complicated. Fortunately, the situation in the C system is much simpler. The initial and final velocities in the C system determine a plane known as the plane of scattering. Each particle is deflected through the same scattering angle Θ in this plane. The interaction force must be known in order to calculate Θ, or conversely, by measuring the deflection we can learn about the interaction force. However, we shall defer these considerations and simply assume that the interaction has caused some deflection in the C system.



An important simplification occurs if the collision is elastic. Conservation of energy applied to the C system gives, for elastic collisions,

$$\tfrac{1}{2}m_1 v_{1c}{}^2 + \tfrac{1}{2}m_2 v_{2c}{}^2 = \tfrac{1}{2}m_1 v_{1c}'^2 + \tfrac{1}{2}m_2 v_{2c}'^2.$$

Since momentum is zero in the C system, we have

$$m_1 v_{1c} - m_2 v_{2c} = 0$$

$$m_1 v_{1c}' - m_2 v_{2c}' = 0.$$

Eliminating v_{2c} and v_{2c}' from the energy equation gives

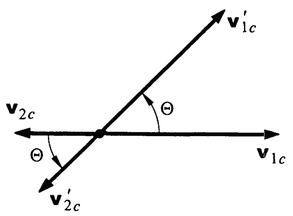
$$\tfrac{1}{2}\left(m_1 + \frac{m_1{}^2}{m_2}\right) v_{1c}{}^2 = \tfrac{1}{2}\left(m_1 + \frac{m_1{}^2}{m_2}\right) v_{1c}'^2$$

or

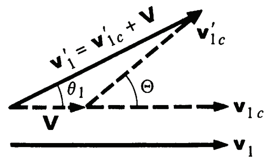
$$v_{1c} = v_{1c}'.$$

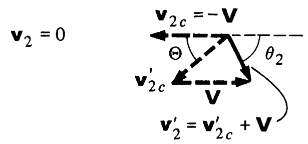
Similarly,

$$v_{2c} = v_{2c}'.$$



In an elastic collision, the speed of each particle in the C system is the same before and after the collision. Thus, the velocity vectors simply rotate in the scattering plane.





In many experiments, one of the particles, say m_2, is initially at rest in the laboratory. In this case

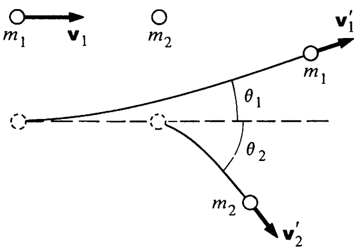
$$\mathbf{V} = \frac{m_1}{m_1 + m_2}\mathbf{v}$$

and

$$\mathbf{v}_{1c} = \mathbf{v}_1 - \mathbf{V}$$
$$= \frac{m_2}{m_1 + m_2}\mathbf{v}_1$$
$$\mathbf{v}_{2c} = -\mathbf{V}$$
$$= -\frac{m_1}{m_1 + m_2}\mathbf{v}_1.$$

The sketches show $\mathbf{v}_1$ and $\mathbf{v}_2$ before and after the collision in the C and L systems. θ_1 and θ_2 are the laboratory angles of the trajectories of the two particles after the collision. The velocity diagrams can be used to relate θ_1 and θ_2 to the scattering angle Θ.

Example 4.19 **Limitations on Laboratory Scattering Angle**



Consider the elastic scattering of a particle of mass m_1 and velocity $\mathbf{v}_1$ from a second particle of mass m_2 at rest. The scattering angle Θ in the C system is unrestricted, but the conservation laws impose limitations on the laboratory angles, as we shall show.

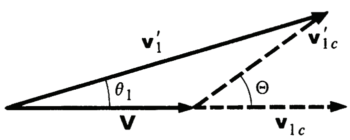
The center of mass velocity has magnitude

$$V = \frac{m_1 v_1}{m_1 + m_2} \qquad 1$$

and is parallel to $\mathbf{v}_1$. The initial velocities in the C system are

$$\mathbf{v}_{1c} = \frac{m_2}{m_1 + m_2}\mathbf{v}_1$$
$$\mathbf{v}_{2c} = -\frac{m_1}{m_1 + m_2}\mathbf{v}_1. \qquad 2$$

Suppose m_1 is scattered through angle Θ in the C system.



From the velocity diagram we see that the laboratory scattering angle of the incident particle is given by

$$\tan\theta_1 = \frac{v_{1c}'\sin\Theta}{V + v_{1c}'\cos\Theta}.$$

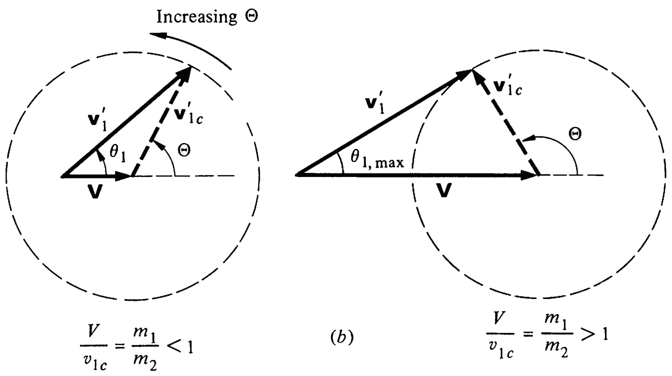
Since the scattering is elastic, $v'_{1c} = v_{1c}$. Hence

$$\tan\theta_1 = \frac{v_{1c}\sin\Theta}{V + v_{1c}\cos\Theta} = \frac{\sin\Theta}{(V/v_{1c}) + \cos\Theta}.$$

From Eqs. (1) and (2), $V/v_{1c} = m_1/m_2$. Therefore

$$\tan\theta_1 = \frac{\sin\Theta}{(m_1/m_2) + \cos\Theta}. \qquad 3$$

The scattering angle Θ depends on the details of the interaction, but in general it can assume any value. If $m_1 < m_2$, it follows from Eq. (3) or the geometric construction in sketch (*a*) that θ_1 is unrestricted. However, the situation is quite different if $m_1 > m_2$. In this case θ_1 is never greater than a certain angle $\theta_{1,\max}$. As sketch (*b*) shows, the maximum value of θ_1 occurs when $\mathbf{v}'_1$ and $\mathbf{v}'_{1c}$ are both perpendicular. In this case $\sin\theta_{1,\max} = v_{1c}/V = m_2/m_1$. If $m_1 \gg m_2$, $\theta_{1,\max} \approx m_2/m_1$ and the maximum scattering angle approaches zero.

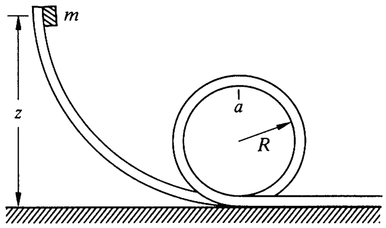


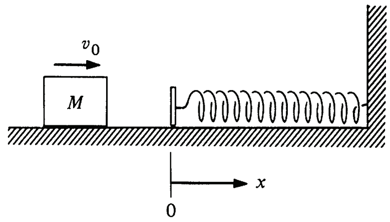
(*a*) (*b*)

Physically, a light particle at rest cannot appreciably deflect a massive particle. The incident particle tends to continue in its forward direction no matter how the light target particle recoils.

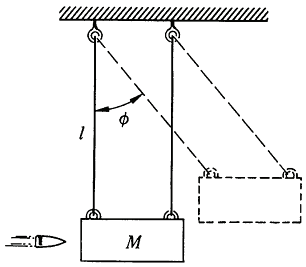
Problems 4.1 A small block of mass m starts from rest and slides along a frictionless loop-the-loop as shown in the left-hand figure on the top of the next page. What should be the initial height z, so that m pushes against the top of the track (at a) with a force equal to its weight?

Ans. $z = 3R$





4.2 A block of mass M slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force (see figure above right). The coefficient of friction is variable and is given by $\mu = bx$, where b is a constant. Find the loss in mechanical energy when the block has first come momentarily to rest.

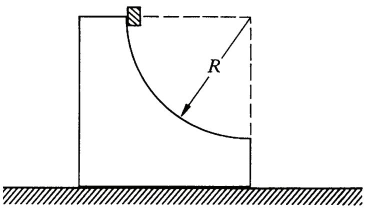


4.3 A simple way to measure the speed of a bullet is with a *ballistic pendulum*. As illustrated, this consists of a wooden block of mass M into which the bullet is shot. The block is suspended from cables of length l, and the impact of the bullet causes it to swing through a maximum angle ϕ, as shown. The initial speed of the bullet is v, and its mass is m.

a. How fast is the block moving immediately after the bullet comes to rest? (Assume that this happens quickly.)

b. Show how to find the velocity of the bullet by measuring m, M, l, and ϕ.

Ans. (*b*) $v = [(m + M)/m]\sqrt{2gl(1 - \cos\phi)}$

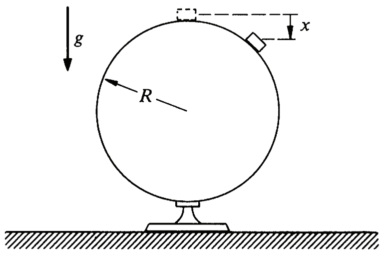


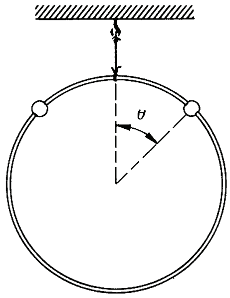
4.4 A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown at left. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path.

Find the velocity v of the cube as it leaves the block.

Ans. clue. If $m = M$, $v = \sqrt{gR}$

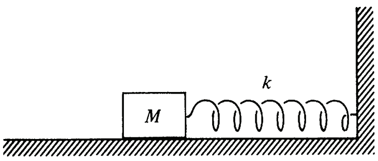
4.5 Mass m whirls on a frictionless table, held to circular motion by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of the circle changes from l_1 to l_2. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.





4.6 A small block slides from rest from the top of a frictionless sphere of radius R (see above left). How far below the top x does it lose contact with the sphere? The sphere does not move. *Ans.* $R/3$

4.7 A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction (see above right). The beads are released simultaneously from the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > 3M/2$, and find the angle at which this occurs. *Ans. clue.* If $M = 0$, $\theta = \arccos \frac{2}{3}$



4.8 The block shown in the drawing is acted on by a spring with spring constant k and a weak friction force of constant magnitude f. The block is pulled distance x_0 from equilibrium and released. It oscillates many times and eventually comes to rest.

a. Show that the decrease of amplitude is the same for each cycle of oscillation.

b. Find the number of cycles n the mass oscillates before coming to rest. *Ans.* $n = \frac{1}{4}[(kx_0/f) - 1] \approx kx_0/4f$

4.9 A simple and very violent chemical reaction is $H + H \rightarrow H_2 + 5$ eV. (1 eV $= 1.6 \times 10^{-19}$ J, a healthy amount of energy on the atomic scale.) However, when hydrogen atoms collide in free space they simply bounce apart! The reason is that it is impossible to satisfy the laws of conservation of momentum and conservation of energy in a simple two body collision which releases energy. Can you prove this? You might start by writing the statements of conservation of momentum and energy. (Be sure to include the energy of reaction in the energy equation, and get the sign right.) By eliminating the final momentum of the molecule from the pair of equations, you should be able to show that the initial momenta would have to satisfy an impossible condition.

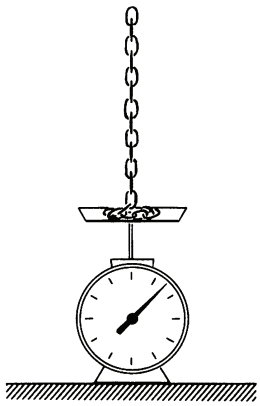
4.10 A block of mass M on a horizontal frictionless table is connected to a spring (spring constant k), as shown.

The block is set in motion so that it oscillates about its equilibrium point with a certain amplitude A_0. The period of motion is $T_0 = 2\pi\sqrt{M/k}$.

a. A lump of sticky putty of mass m is dropped onto the block. The putty sticks without bouncing. The putty hits M at the instant when the velocity of M is zero. Find

(1) The new period
(2) The new amplitude
(3) The change in the mechanical energy of the system

b. Repeat part a, but this time assume that the sticky putty hits M at the instant when M has its maximum velocity.



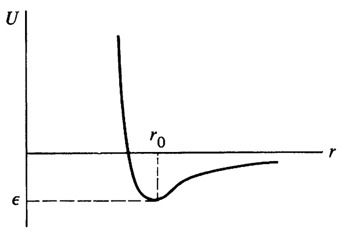
4.11 A chain of mass M and length l is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale.

What is the reading of the scale when a length of chain, x, has fallen? (Neglect the size of individual links.)

Ans. clue. The maximum reading is $3Mg$

4.12 During the Second World War the Russians, lacking sufficient parachutes for airborne operations, occasionally dropped soldiers inside bales of hay onto snow. The human body can survive an average pressure on impact of 30 lb/in^2.

Suppose that the lead plane drops a dummy bale equal in weight to a loaded one from an altitude of 150 ft, and that the pilot observes that it sinks about 2 ft into the snow. If the weight of an average soldier is 144 lb and his effective area is 5 ft^2, is it safe to drop the men?



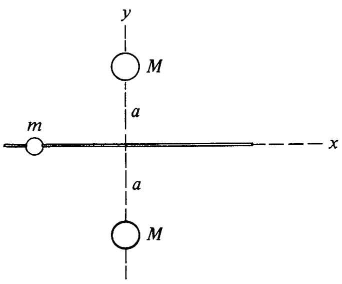
4.13 A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6,12 potential

$$U = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right].$$

a. Show that the radius at the potential minimum is r_0, and that the depth of the potential well is ϵ.

b. Find the frequency of small oscillations about equilibrium for 2 identical atoms of mass m bound to each other by the Lennard-Jones interaction.

Ans. $\omega = 12\sqrt{\epsilon/r_0^2m}$



4.14 A bead of mass m slides without friction on a smooth rod along the x axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0$, $y = \pm a$ as shown, and attract the bead gravitationally.

a. Find the potential energy of the bead.

b. The bead is released at $x = 3a$ with velocity v_0 toward the origin. Find the speed as it passes the origin.

c. Find the frequency of small oscillations of the bead about the origin.

4.15 A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed toward the origin with magnitude B, and an inverse square law repulsive force with magnitude A/x^2.

a. Find the potential energy function $U(x)$.

b. Sketch the energy diagram for the system when the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.

c. Find the equilibrium position, x_0.

d. What is the frequency of small oscillations about x_0?

4.16 An 1,800-lb sportscar accelerates to 60 mi/h in 8 s. What is the average power that the engine delivers to the car's motion during this period?

4.17 A snowmobile climbs a hill at 15 mi/hr. The hill has a grade of 1 ft rise for every 40 ft. The resistive force due to the snow is 5 percent of the vehicle's weight. How fast will the snowmobile move downhill, assuming its engine delivers the same power?

Ans. 45 mi/h

4.18 A 160-lb man leaps into the air from a crouching position. His center of gravity rises 1.5 ft before he leaves the ground, and it then rises 3 ft to the top of his leap. What power does he develop assuming that he pushes the ground with constant force?

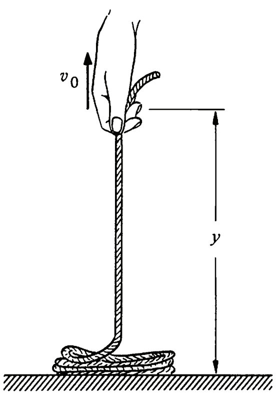
Ans. clue. More than 1 hp, less than 10 hp

4.19 The man in the preceding problem again leaps into the air, but this time the force he applies decreases from a maximum at the beginning of the leap to zero at the moment he leaves the ground. As a reasonable approximation, take the force to be $F = F_0 \cos \omega t$, where F_0 is the peak force, and contact with the ground ends when $\omega t = \pi/2$. Find the peak power the man develops during the jump.

4.20 Sand runs from a hopper at constant rate dm/dt onto a horizontal conveyor belt driven at constant speed V by a motor.

a. Find the power needed to drive the belt.

b. Compare the answer to *a* with the rate of change of kinetic energy of the sand. Can you account for the difference?



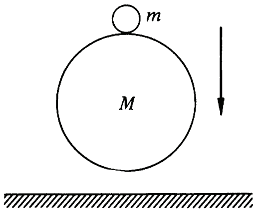
4.21 A uniform rope of mass λ per unit length is coiled on a smooth horizontal table. One end is pulled straight up with constant speed v_0.

a. Find the force exerted on the end of the rope as a function of height y.

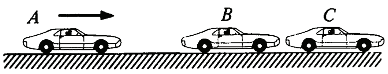
b. Compare the power delivered to the rope with the rate of change of the rope's total mechanical energy.

4.22 A ball drops to the floor and bounces, eventually coming to rest. Collisions between the ball and floor are inelastic; the speed after each collision is e times the speed before the collision where $e < 1$, (e is called the *coefficient of restitution.*) If the speed just before the first bounce is v_0, find the time to come to rest.

Ans. clue. If $v_0 = 5$ m/s, $e = 0.5$, then T $\approx$ 1 s



4.23 A small ball of mass m is placed on top of a "superball" of mass M, and the two balls are dropped to the floor from height h. How high does the small ball rise after the collision? Assume that collisions with the superball are elastic, and that $m \ll M$. To help visualize the problem, assume that the balls are slightly separated when the superball hits the floor. (If you are surprised at the result, try demonstrating the problem with a marble and a superball.)



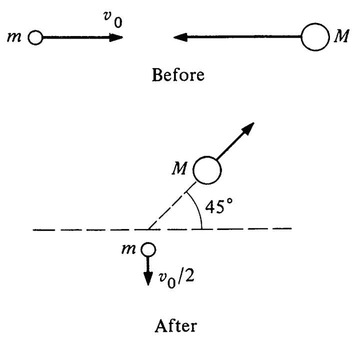
4.24 Cars B and C are at rest with their brakes off. Car A plows into B at high speed, pushing B into C. If the collisions are completely inelastic, what fraction of the initial energy is dissipated in car C? Initially the cars are identical.

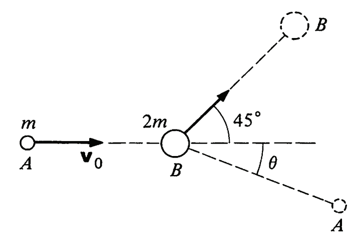
4.25 A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with $\frac{4}{9}$ of its initial kinetic energy.

Find the ratio of the mass of the unknown particle to the mass of the proton, assuming that the collision is elastic.

4.26 A particle of mass m and initial velocity v_0 collides elastically with a particle of unknown mass M coming from the opposite direction as shown at left below. After the collision m has velocity $v_0/2$ at right angles to the incident direction, and M moves off in the direction shown in the sketch. Find the ratio M/m.

4.27 Particle A of mass m has initial velocity v_0. After colliding with particle B of mass $2m$ initially at rest, the particles follow the paths shown in the sketch at right below. Find θ.





4.28 A thin target of lithium is bombarded by helium nuclei of energy E_0. The lithium nuclei are initially at rest in the target but are essentially unbound. When a helium nucleus enters a lithium nucleus, a nuclear reaction can occur in which the compound nucleus splits apart into a boron nucleus and a neutron. The collision is inelastic, and the final kinetic energy is less than E_0 by 2.8 MeV. (1 MeV = 10^6 eV = 1.6×10^{-13} J). The relative masses of the particles are: helium, mass 4; lithium, mass 7; boron, mass 10; neutron, mass 1. The reaction can be symbolized

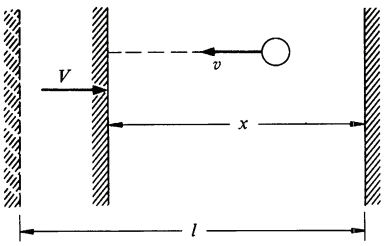
$$^7\text{Li} + {}^4\text{He} \rightarrow {}^{10}\text{B} + {}^1\text{n} - 2.8 \text{ MeV}.$$

a. What is $E_{0,\text{threshold}}$, the minimum value of E_0 for which neutrons can be produced? What is the energy of the neutrons at this threshold?

Ans. Neutron energy = 0.15 MeV

b. Show that if the incident energy falls in the range $E_{0,\text{threshold}} < E_0 < E_{0,\text{threshold}} + 0.27$ MeV, the neutrons ejected in the forward direction do not all have the same energy but must have either one or the other of two possible energies. (You can understand the origin of the two groups by looking at the reaction in the center of mass system.)

4.29 A "superball" of mass m bounces back and forth between two surfaces with speed v_0. Gravity is neglected and the collisions are perfectly elastic.



a. Find the average force F on each wall.

Ans. $F = mv_0^2/l$

b. If one surface is slowly moved toward the other with speed $V \ll v$, the bounce rate will increase due to the shorter distance between collisions, and because the ball's speed increases when it bounces from the moving surface. Find F in terms of the separation of the surfaces, x. (*Hint:* Find the average rate at which the ball's speed increases as the surface moves.)

Ans. $F = (mv_0^2/l)(l/x)^3$

c. Show that the work needed to push the surface from l to x equals the gain in kinetic energy of the ball. (This problem illustrates the mechanism which causes a gas to heat up as it is compressed.)

4.30 A particle of mass m and velocity v_0 collides elastically with a particle of mass M initially at rest and is scattered through angle Θ in the center of mass system.

a. Find the final velocity of m in the laboratory system.

Ans. $v_f = [v_0/(m + M)](m^2 + M^2 + 2mM \cos \Theta)^{\frac{1}{2}}$

b. Find the fractional loss of kinetic energy of m.

Ans. clue. If $m = M$, $(K_0 - K_f)/K_0 = (1 - \cos \Theta)/2$