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AN INTRODUCTION TO MECHANICS

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To our parents

Beatrice and Otto
Katherine and John

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PREFACE

There is good reason for the tradition that students of science and engineering start college physics with the study of mechanics: mechanics is the cornerstone of pure and applied science. The concept of energy, for example, is essential for the study of the evolution of the universe, the properties of elementary particles, and the mechanisms of biochemical reactions. The concept of energy is also essential to the design of a cardiac pacemaker and to the analysis of the limits of growth of industrial society. However, there are difficulties in presenting an introductory course in mechanics which is both exciting and intellectually rewarding. Mechanics is a mature science and a satisfying discussion of its principles is easily lost in a superficial treatment. At the other extreme, attempts to "enrich" the subject by emphasizing advanced topics can produce a false sophistication which emphasizes technique rather than understanding.

This text was developed from a first-year course which we taught for a number of years at the Massachusetts Institute of Technology and, earlier, at Harvard University. We have tried to present mechanics in an engaging form which offers a strong base for future work in pure and applied science. Our approach departs from tradition more in depth and style than in the choice of topics; nevertheless, it reflects a view of mechanics held by twentieth-century physicists.

Our book is written primarily for students who come to the course knowing some calculus, enough to differentiate and integrate simple functions.[1] It has also been used successfully in courses requiring only concurrent registration in calculus. (For a course of this nature, Chapter 1 should be treated as a resource chapter, deferring the detailed discussion of vector kinematics for a time. Other suggestions are listed in To The Teacher.) Our experience has been that the principal source of difficulty for most students is in learning how to apply mathematics to physical problems, not with mathematical techniques as such. The elements of calculus can be mastered relatively easily, but the development of problem-solving ability requires careful guidance. We have provided numerous worked examples throughout the text to help supply this guidance. Some of the examples, particularly in the early chapters, are essentially pedagogical. Many examples, however, illustrate principles and techniques by application to problems of real physical interest.

The first chapter is a mathematical introduction, chiefly on vectors and kinematics. The concept of rate of change of a vector,

[1] The background provided in "Quick Calculus" by Daniel Kleppner and Norman Ramsey, John Wiley & Sons, New York, 1965, is adequate.

probably the most difficult mathematical concept in the text, plays an important role throughout mechanics. Consequently, this topic is developed with care, both analytically and geometrically. The geometrical approach, in particular, later proves to be invaluable for visualizing the dynamics of angular momentum.

Chapter 2 discusses inertial systems, Newton's laws, and some common forces. Much of the discussion centers on applying Newton's laws, since analyzing even simple problems according to general principles can be a challenging task at first. Visualizing a complex system in terms of its essentials, selecting suitable inertial coordinates, and distinguishing between forces and accelerations are all acquired skills. The numerous illustrative examples in the text have been carefully chosen to help develop these skills.

Momentum and energy are developed in the following two chapters. Chapter 3, on momentum, applies Newton's laws to extended systems. Students frequently become confused when they try to apply momentum considerations to rockets and other systems involving flow of mass. Our approach is to apply a differential method to a system defined so that no mass crosses its boundary during the chosen time interval. This ensures that no contribution to the total momentum is overlooked. The chapter concludes with a discussion of momentum flux. Chapter 4, on energy, develops the work-energy theorem and its application to conservative and nonconservative forces. The conservation laws for momentum and energy are illustrated by a discussion of collision problems.

Chapter 5 deals with some mathematical aspects of conservative forces and potential energy; this material is not needed elsewhere in the text, but it will be of interest to students who want a mathematically complete treatment of the subject.

Students usually find it difficult to grasp the properties of angular momentum and rigid body motion, partly because rotational motion lies so far from their experience that they cannot rely on intuition. As a result, introductory texts often slight these topics, despite their importance. We have found that rotational motion can be made understandable by emphasizing physical reasoning rather than mathematical formalism, by appealing to geometric arguments, and by providing numerous worked examples. In Chapter 6 angular momentum is introduced, and the dynamics of fixed axis rotation is treated. Chapter 7 develops the important features of rigid body motion by applying vector arguments to systems dominated by spin angular momentum. An elementary treatment of general rigid body motion is presented in the last sections of Chapter 7 to show how Euler's equations can be developed from simple physical arguments. This more advanced material is optional however; we do not usually treat it in our own course.

Chapter 8, on noninertial coordinate systems, completes the development of the principles of newtonian mechanics. Up to this point in the text, inertial systems have been used exclusively in order to avoid confusion between forces and accelerations. Our discussion of noninertial systems emphasizes their value as computational tools and their implications for the foundations of mechanics.

Chapters 9 and 10 treat central force motion and the harmonic oscillator, respectively. Although no new physical concepts are involved, these chapters illustrate the application of the principles of mechanics to topics of general interest and importance in physics. Much of the algebraic complexity of the harmonic oscillator is avoided by focusing the discussion on energy, and by using simple approximations.

Chapters 11 through 14 present a discussion of the principles of special relativity and some of its applications. We attempt to emphasize the harmony between relativistic and classical thought, believing, for example, that it is more valuable to show how the classical conservation laws are unified in relativity than to dwell at length on the so-called "paradoxes." Our treatment is concise and minimizes algebraic complexities. Chapter 14 shows how ideas of symmetry play a fundamental role in the formulation of relativity. Although we have kept the beginning students in mind, the concepts here are more subtle than in the previous chapters. Chapter 14 can be omitted if desired; but by illustrating how symmetry bears on the principles of mechanics, it offers an exciting mode of thought and a powerful new tool.

Physics cannot be learned passively; there is absolutely no substitute for tackling challenging problems. Here is where students gain the sense of satisfaction and involvement produced by a genuine understanding of the principles of physics. The collection of problems in this book was developed over many years of classroom use. A few problems are straightforward and intended for drill; most emphasize basic principles and require serious thought and effort. We have tried to choose problems which make this effort worthwhile in the spirit of Piet Hein's aphorism

Problems worthy
  of attack
prove their worth
  by hitting back[1]

[1] From *Grooks I,* by Piet Hein, copyrighted 1966, The M.I.T. Press.

It gives us pleasure to acknowledge the many contributions to this book from our colleagues and from our students. In particular, we thank Professors George B. Benedek and David E. Pritchard for a number of examples and problems. We should also like to thank Lynne Rieck and Mary Pat Fitzgerald for their cheerful fortitude in typing the manuscript.

Daniel Kleppner
Robert J. Kolenkow

TO THE TEACHER

The first eight chapters form a comprehensive introduction to classical mechanics and constitute the heart of a one-semester course. In a 12-week semester, we have generally covered the first 8 chapters and parts of Chapters 9 or 10. However, Chapter 5 and some of the advanced topics in Chapters 7 and 8 are usually omitted, although some students pursue them independently.

Chapters 11, 12, and 13 present a complete introduction to special relativity. Chapter 14, on transformation theory and four-vectors, provides deeper insight into the subject for interested students. We have used the chapters on relativity in a three-week short course and also as part of the second-term course in electricity and magnetism.

The problems at the end of each chapter are generally graded in difficulty. They are also cumulative; concepts and techniques from earlier chapters are repeatedly called upon in later sections of the book. The hope is that by the end of the course the student will have developed a good intuition for tackling new problems, that he will be able to make an intelligent estimate, for instance, about whether to start from the momentum approach or from the energy approach, and that he will know how to set off on a new tack if his first approach is unsuccessful. Many students report a deep sense of satisfaction from acquiring these skills.

Many of the problems require a symbolic rather than a numerical solution. This is not meant to minimize the importance of numerical work but to reinforce the habit of analyzing problems symbolically. Answers are given to some problems; in others, a numerical "answer clue" is provided to allow the student to check his symbolic result. Some of the problems are challenging and require serious thought and discussion. Since too many such problems at once can result in frustration, each assignment should have a mix of easier and harder problems.

Chapter 1 Although we would prefer to start a course in mechanics by discussing physics rather than mathematics, there are real advantages to devoting the first few lectures to the mathematics of motion. The concepts of kinematics are straightforward for the most part, and it is helpful to have them clearly in hand before tackling the much subtler problems presented by newtonian dynamics in Chapter 2. A departure from tradition in this chapter is the discussion of kinematics using polar coordinates. Many students find this topic troublesome at first, requiring serious effort. However, we feel that the effort will be amply rewarded. In the first place, by being able to use polar coordinates freely, the kinematics of rotational motion are much easier to understand;

the mystery of radial acceleration disappears. More important, this topic gives valuable insights into the nature of a time-varying vector, insights which not only simplify the dynamics of particle motion in Chapter 2 but which are invaluable to the discussion of momentum flux in Chapter 3, angular momentum in Chapters 6 and 7, and the use of noninertial coordinates in Chapter 8. Thus, the effort put into understanding the nature of time-varying vectors in Chapter 1 pays important dividends throughout the course.

If the course is intended for students who are concurrently beginning their study of calculus, we recommend that parts of Chapter 1 be deferred. Chapter 2 can be started after having covered only the first six sections of Chapter 1. Starting with Example 2.5, the kinematics of rotational motion are needed; at this point the ideas presented in Section 1.9 should be introduced. Section 1.7, on the integration of vectors, can be postponed until the class has become familiar with integrals. Occasional examples and problems involving integration will have to be omitted until that time. Section 1.8, on the geometric interpretation of vector differentiation, is essential preparation for Chapters 6 and 7 but need not be discussed earlier.

Chapter 2 The material in Chapter 2 often represents the student's first serious attempt to apply abstract principles to concrete situations. Newton's laws of motion are not self-evident; most people unconsciously follow aristotelian thought. We find that after an initial period of uncertainty, students become accustomed to analyzing problems according to principles rather than vague intuition. A common source of difficulty at first is to confuse force and acceleration. We therefore emphasize the use of inertial systems and recommend strongly that noninertial coordinate systems be reserved until Chapter 8, where their correct use is discussed. In particular, the use of centrifugal force in the early chapters can lead to endless confusion between inertial and noninertial systems and, in any case, it is not adequate for the analysis of motion in rotating coordinate systems.

Chapters 3 and 4 There are many different ways to derive the rocket equations. However, rocket problems are not the only ones in which there is a mass flow, so that it is important to adopt a method which is easily generalized. It is also desirable that the method be in harmony with the laws of conservation of momentum or, to put it more crudely, that there is no swindle involved. The differential approach used in Section 3.5 was developed to meet these requirements. The approach may not be elegant, but it is straightforward and quite general.

In Chapter 4, we attempt to emphasize the general nature of the work-energy theorem and the difference between conservative and nonconservative forces. Although the line integral is introduced and explained, only simple line integrals need to be evaluated, and general computational techniques should not be given undue attention.

Chapter 5 This chapter completes the discussion of energy and provides a useful introduction to potential theory and vector calculus. However, it is relatively advanced and will appeal only to students with an appetite for mathematics. The results are not needed elsewhere in the text, and we recommend leaving this chapter for optional use, or as a special topic.

Chapters 6 and 7 Most students find that angular momentum is the most difficult physical concept in elementary mechanics. The major conceptual hurdle is visualizing the vector properties of angular momentum. We therefore emphasize the vector nature of angular momentum repeatedly throughout these chapters. In particular, many features of rigid body motion can be understood intuitively by relying on the understanding of time-varying vectors developed in earlier chapters. It is more profitable to emphasize the qualitative features of rigid body motion than formal aspects such as the tensor of inertia. If desired, these qualitative arguments can be pressed quite far, as in the analysis of gyroscopic nutation in Note 7.2. The elementary discussion of Euler's equations in Section 7.7 is intended as optional reading only. Although Chapters 6 and 7 require hard work, many students develop a physical insight into angular momentum and rigid body motion which is seldom gained at the introductory level and which is often obscured by mathematics in advanced courses.

Chapter 8 The subject of noninertial systems offers a natural springboard to such speculative and interesting topics as transformation theory and the principle of equivalence. From a more practical point of view, the use of noninertial systems is an important technique for solving many physical problems.

Chapters 9 and 10 In these chapters the principles developed earlier are applied to two important problems, central force motion and the harmonic oscillator. Although both topics are generally treated rather formally, we have tried to simplify the mathematical development. The discussion of central force motion relies heavily on the conservation laws and on energy diagrams. The treatment of the harmonic oscillator sidesteps much of the usual algebraic complexity by focusing on the lightly damped oscillator. Applications and examples play an important role in both chapters.

Chapters 11 to 14 Special relativity offers an exciting change of pace to a course in mechanics. Our approach attempts to emphasize the connection of relativity with classical thought. We have used the Michelson-Morley experiment to motivate the discussion. Although the prominence of this experiment in Einstein's thought has been much exaggerated, this approach has the advantage of grounding the discussion on a real experiment.

We have tried to focus on the ideas of events and their transformations without emphasizing computational aids such as diagrammatic methods. This approach allows us to deemphasize many of the so-called paradoxes.

For many students, the real mystery of relativity lies not in the postulates or transformation laws but in why transformation principles should suddenly become the fundamental concept for generating new physical laws. This touches on the deepest and most provocative aspects of Einstein's thought. Chapter 14, on four-vectors, provides an introduction to transformation theory which unifies and summarizes the preceding development. The chapter is intended to be optional.

Daniel Kleppner
Robert J. Kolenkow

AN INTRODUCTION TO MECHANICS

1 VECTORS AND KINEMATICS– A FEW MATHEMATICAL PRELIMINARIES

1.1 Introduction

The goal of this book is to help you acquire a deep understanding of the principles of mechanics. The subject of mechanics is at the very heart of physics; its concepts are essential for understanding the everyday physical world as well as phenomena on the atomic and cosmic scales. The concepts of mechanics, such as momentum, angular momentum, and energy, play a vital role in practically every area of physics.

We shall use mathematics frequently in our discussion of physical principles, since mathematics lets us express complicated ideas quickly and transparently, and it often points the way to new insights. Furthermore, the interplay of theory and experiment in physics is based on quantitative prediction and measurement. For these reasons, we shall devote this chapter to developing some necessary mathematical tools and postpone our discussion of the principles of mechanics until Chap. 2.

1.2 Vectors

The study of vectors provides a good introduction to the role of mathematics in physics. By using vector notation, physical laws can often be written in compact and simple form. (As a matter of fact, modern vector notation was invented by a physicist, Willard Gibbs of Yale University, primarily to simplify the appearance of equations.) For example, here is how Newton's second law (which we shall discuss in the next chapter) appears in nineteenth century notation:

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_z = ma_z.$$

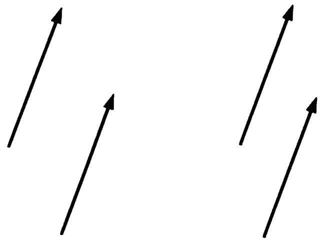
In vector notation, one simply writes

$$\mathbf{F} = m\mathbf{a}.$$

Our principal motivation for introducing vectors is to simplify the form of equations. However, as we shall see in the last chapter of the book, vectors have a much deeper significance. Vectors are closely related to the fundamental ideas of symmetry and their use can lead to valuable insights into the possible forms of unknown laws.

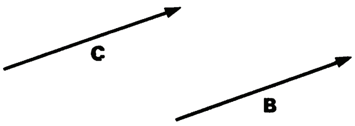
Definition of a Vector

Vectors can be approached from three points of view—geometric, analytic, and axiomatic. Although all three points of view are useful, we shall need only the geometric and analytic approaches in our discussion of mechanics.



From the geometric point of view, a vector is a *directed line segment.* In writing, we can represent a vector by an arrow and label it with a letter capped by a symbolic arrow. In print, boldfaced letters are traditionally used.

In order to describe a vector we must specify both its length and its direction. Unless indicated otherwise, we shall assume that parallel translation does not change a vector. Thus the arrows at left all represent the same vector.



If two vectors have the same length and the same direction they are equal. The vectors **B** and **C** are equal:

$\mathbf{B} = \mathbf{C}$.

The length of a vector is called its *magnitude.* The magnitude of a vector is indicated by vertical bars or, if no confusion will occur, by using italics. For example, the magnitude of **A** is written $|\mathbf{A}|$, or simply A. If the length of **A** is $\sqrt{2}$, then $|\mathbf{A}| = A = \sqrt{2}$.

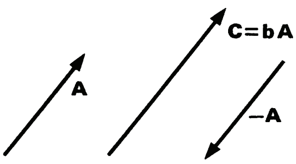
If the length of a vector is one unit, we call it a *unit vector.* A unit vector is labeled by a caret; the vector of unit length parallel to **A** is $\hat{\mathbf{A}}$. It follows that

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|},$$

and conversely

$$\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}.$$

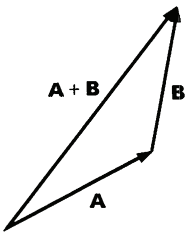
The Algebra of Vectors



Multiplication of a Vector by a Scalar If we multiply **A** by a positive scalar b, the result is a new vector $\mathbf{C} = b\mathbf{A}$. The vector **C** is parallel to **A**, and its length is b times greater. Thus $\hat{\mathbf{C}} = \hat{\mathbf{A}}$, and $|\mathbf{C}| = b|\mathbf{A}|$.

The result of multiplying a vector by -1 is a new vector opposite in direction (antiparallel) to the original vector.

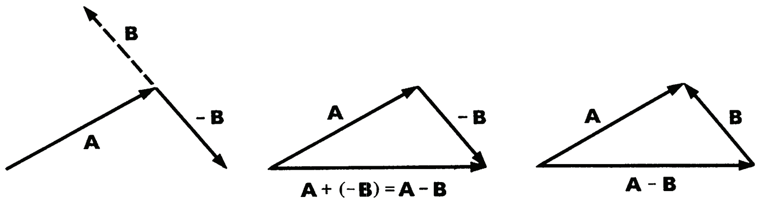
Multiplication of a vector by a negative scalar evidently can change both the magnitude and the direction sense.



Addition of Two Vectors Addition of vectors has the simple geometrical interpretation shown by the drawing.

The rule is: To add **B** to **A**, place the tail of **B** at the head of **A**. The sum is a vector from the tail of **A** to the head of **B**.

Subtraction of Two Vectors Since $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, in order to subtract **B** from **A** we can simply multiply it by -1 and then add. The sketches below show how.

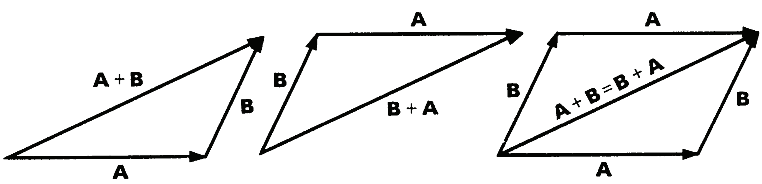


An equivalent way to construct $\mathbf{A} - \mathbf{B}$ is to place the *head* of **B** at the *head* of **A**. Then $\mathbf{A} - \mathbf{B}$ extends from the *tail* of **A** to the *tail* of **B**, as shown in the right hand drawing above.

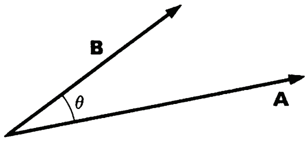
It is not difficult to prove the following laws. We give a geometrical proof of the commutative law; try to cook up your own proofs of the others.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ $c(d\mathbf{A}) = (cd)\mathbf{A}$	Associative law
$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$	Distributive law

Proof of the Commutative law of vector addition



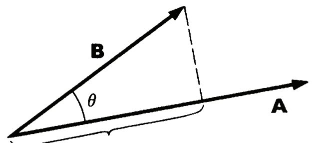
Although there is no great mystery to addition, subtraction, and multiplication of a vector by a scalar, the result of "multiplying" one vector by another is somewhat less apparent. Does multiplication yield a vector, a scalar, or some other quantity? The choice is up to us, and we shall define two types of products which are useful in our applications to physics.



Scalar Product ("Dot" Product) The first type of product is called the *scalar* product, since it represents a way of combining two vectors to form a scalar. The scalar product of **A** and **B** is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is often called the dot product. $\mathbf{A} \cdot \mathbf{B}$ is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}|\,|\mathbf{B}| \cos \theta.$$

Here θ is the angle between **A** and **B** when they are drawn tail to tail.



Projection of **B** on **A**

Since $|\mathbf{B}| \cos \theta$ is the projection of **B** along the direction of **A**, $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \times$ (projection of **B** on **A**).

Similarly,

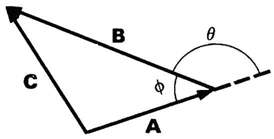
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{B}| \times (\text{projection of } \mathbf{A} \text{ on } \mathbf{B}).$$

If $\mathbf{A} \cdot \mathbf{B} = 0$, then $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, or **A** is perpendicular to **B** (that is, $\cos \theta = 0$). Scalar multiplication is unusual in that the dot product of two nonzero vectors can be 0.

Note that $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$.

By way of demonstrating the usefulness of the dot product, here is an almost trivial proof of the law of cosines.

Example 1.1 **Law of Cosines**



$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$$

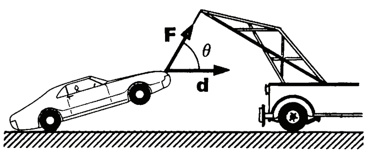
$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2|\mathbf{A}|\,|\mathbf{B}| \cos \theta$$

This result is generally expressed in terms of the angle ϕ:

$$C^2 = A^2 + B^2 - 2AB \cos \phi.$$

(We have used $\cos \theta = \cos (\pi - \phi) = -\cos \phi$.)

Example 1.2 **Work and the Dot Product**



The dot product finds its most important application in the discussion of work and energy in Chap. 4. As you may already know, the work W done by a force F on an object is the displacement d of the object times the component of F along the direction of d. If the force is applied at an angle θ to the displacement,

$$W = (F \cos \theta)d.$$

Granting for the time being that force and displacement are vectors,

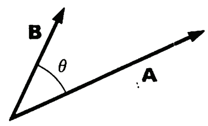
$$W = \mathbf{F} \cdot \mathbf{d}.$$

Vector Product ("Cross" Product) The second type of product we need is the *vector* product. In this case, two vectors **A** and **B** are combined to form a third vector **C**. The symbol for vector product is a cross:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

An alternative name is the *cross product.*

The vector product is more complicated than the scalar product because we have to specify both the magnitude and direction of $\mathbf{A} \times \mathbf{B}$. The magnitude is defined as follows: if

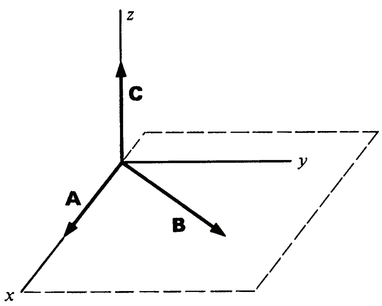


$$\mathbf{C} = \mathbf{A} \times \mathbf{B},$$

then

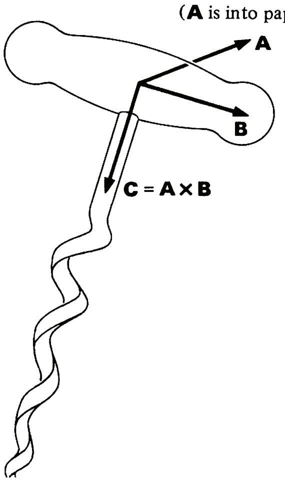
$$|\mathbf{C}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta,$$

where θ is the angle between **A** and **B** when they are drawn tail to tail. (To eliminate ambiguity, θ is always taken as the angle smaller than π.) Note that the vector product is zero when $\theta = 0$ or π, even if $|\mathbf{A}|$ and $|\mathbf{B}|$ are not zero.



When we draw **A** and **B** tail to tail, they determine a plane. We define the direction of **C** to be perpendicular to the plane of **A** and **B**. **A**, **B**, and **C** form what is called a *right hand triple.* Imagine a right hand coordinate system with **A** and **B** in the xy plane as shown in the sketch. **A** lies on the x axis and **B** lies toward the y axis. If **A**, **B**, and **C** form a right hand triple, then **C** lies on the z axis. We shall always use right hand coordinate systems such as the one shown at left. Here is another way to determine the direction of the cross product. Think of a right hand screw with the axis perpendicular to **A** and **B**. Rotate it in the direction which swings **A** into **B**. **C** lies in the direction the screw advances. (Warning: Be sure not to use a left hand screw. Fortunately, they are rare. Hot water faucets are among the chief offenders; your honest everyday wood screw is right handed.)

(**A** is into paper)



A result of our definition of the cross product is that

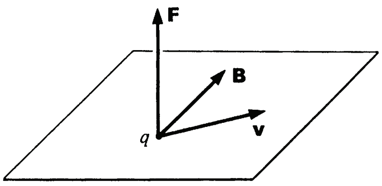
$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}.$$

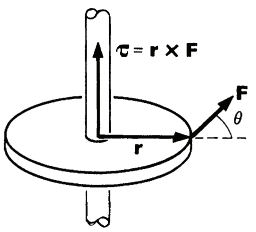
Here we have a case in which the order of multiplication is important. The vector product is *not* commutative. (In fact, since reversing the order reverses the sign, it is anticommutative.) We see that

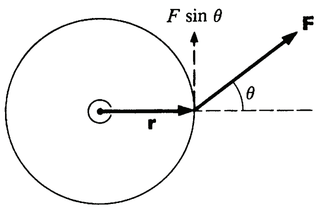
$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

for any vector **A**.

Example 1.3 **Examples of the Vector Product in Physics**







Top view

The vector product has a multitude of applications in physics. For instance, if you have learned about the interaction of a charged particle with a magnetic field, you know that the force is proportional to the charge q, the magnetic field B, and the velocity of the particle v. The force varies as the sine of the angle between v and B, and is perpendicular to the plane formed by v and B, in the direction indicated. A simpler way to give all these rules is

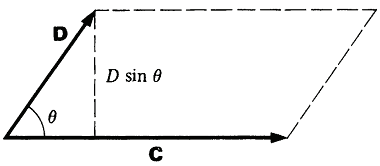
$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

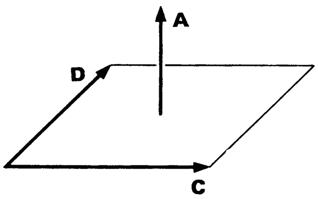
Another application is the definition of torque. We shall develop this idea later. For now we simply mention in passing that the torque $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

where **r** is a vector from the axis about which the torque is evaluated to the point of application of the force **F**. This definition is consistent with the familiar idea that torque is a measure of the ability of an applied force to produce a twist. Note that a large force directed parallel to **r** produces no twist; it merely pulls. Only $F \sin\theta$, the component of force perpendicular to **r**, produces a torque. The torque increases as the lever arm gets larger. As you will see in Chap. 6, it is extremely useful to associate a direction with torque. The natural direction is along the axis of rotation which the torque tends to produce. All these ideas are summarized in a nutshell by the simple equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

Example 1.4 **Area as a Vector**





We can use the cross product to describe an area. Usually one thinks of area in terms of magnitude only. However, many applications in physics require that we also specify the orientation of the area. For example, if we wish to calculate the rate at which water in a stream flows through a wire loop of given area, it obviously makes a difference whether the plane of the loop is perpendicular or parallel to the flow. (In the latter case the flow through the loop is zero.) Here is how the vector product accomplishes this:

Consider the area of a quadrilateral formed by two vectors, **C** and **D**. The area of the parallelogram A is given by

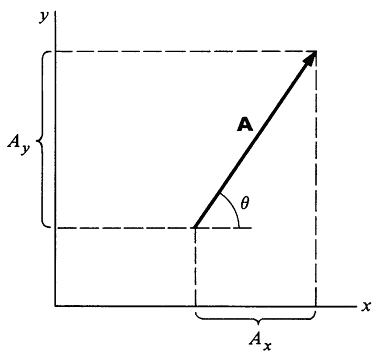
$$\begin{aligned} A &= \text{base} \times \text{height} \\ &= CD \sin\theta \\ &= |\mathbf{C} \times \mathbf{D}|. \end{aligned}$$

If we think of A as a vector, we have

$$\mathbf{A} = \mathbf{C} \times \mathbf{D}.$$

We have already shown that the magnitude of A is the area of the parallelogram, and the vector product defines the convention for assigning a direction to the area. The direction is defined to be perpendicular to the plane of the area; that is, the direction is parallel to a *normal* to the surface. The sign of the direction is to some extent arbitrary; we could just as well have defined the area by $\mathbf{A} = \mathbf{D} \times \mathbf{C}$. However, once the sign is chosen, it is unique.

1.3 Components of a Vector



The fact that we have discussed vectors without introducing a particular coordinate system shows why vectors are so useful; vector operations are defined without reference to coordinate systems. However, eventually we have to translate our results from the abstract to the concrete, and at this point we have to choose a coordinate system in which to work.

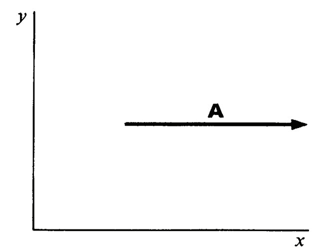
For simplicity, let us restrict ourselves to a two-dimensional system, the familiar xy plane. The diagram shows a vector $\mathbf{A}$ in the xy plane. The projections of $\mathbf{A}$ along the two coordinate axes are called the components of $\mathbf{A}$. The components of $\mathbf{A}$ along the x and y axes are, respectively, A_x and A_y. The magnitude of $\mathbf{A}$ is $|\mathbf{A}| = (A_x^2 + A_y^2)^{\frac{1}{2}}$, and the direction of $\mathbf{A}$ is such that it makes an angle $\theta = \arctan (A_y/A_x)$ with the x axis.

Since the components of a vector define it, we can specify a vector entirely by its components. Thus

$\mathbf{A} = (A_x,A_y)$

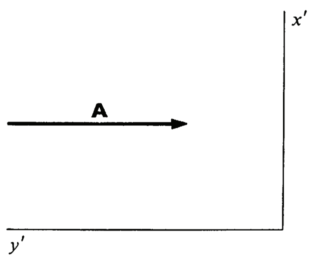
or, more generally, in three dimensions,

$\mathbf{A} = (A_x,A_y,A_z)$.



Prove for yourself that $|\mathbf{A}| = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$. The vector $\mathbf{A}$ has a meaning independent of any coordinate system. However, the components of $\mathbf{A}$ depend on the coordinate system being used. To illustrate this, here is a vector $\mathbf{A}$ drawn in two different coordinate systems. In the first case,

$\mathbf{A} = (A,0)$ (x,y system),



while in the second

$\mathbf{A} = (0,-A)$ (x',y' system).

Unless noted otherwise, we shall restrict ourselves to a single coordinate system, so that if

$\mathbf{A} = \mathbf{B}$,

then

$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z.$$

The single vector equation $\mathbf{A} = \mathbf{B}$ symbolically represents three scalar equations.

All vector operations can be written as equations for components. For instance, multiplication by a scalar gives

$$c\mathbf{A} = (cA_x, cA_y).$$

The law for vector addition is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x, A_y + B_y, A_z + B_z).$$

By writing **A** and **B** as the sums of vectors along each of the coordinate axes, you can verify that

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

We shall defer evaluating the cross product until the next section.

Example 1.5 **Vector Algebra**

Let

$$\mathbf{A} = (3,5,-7)$$
$$\mathbf{B} = (2,7,1).$$

Find $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A} \cdot \mathbf{B}$, and the cosine of the angle between **A** and **B**.

$$\begin{aligned}
\mathbf{A} + \mathbf{B} &= (3 + 2,\ 5 + 7,\ -7 + 1) \\
&= (5,12,-6) \\
\mathbf{A} - \mathbf{B} &= (3 - 2,\ 5 - 7,\ -7 - 1) \\
&= (1,-2,-8) \\
|\mathbf{A}| &= (3^2 + 5^2 + 7^2)^{\frac{1}{2}} \\
&= \sqrt{83} \\
&= 9.11 \\
|\mathbf{B}| &= (2^2 + 7^2 + 1^2)^{\frac{1}{2}} \\
&= \sqrt{54} \\
&= 7.35 \\
\mathbf{A} \cdot \mathbf{B} &= 3 \times 2 + 5 \times 7 - 7 \times 1 \\
&= 34
\end{aligned}$$

$$\cos(\mathbf{A},\mathbf{B}) = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\ |\mathbf{B}|} = \frac{34}{(9.11)(7.35)} = 0.507$$

Example 1.6 Construction of a Perpendicular Vector

Find a unit vector in the xy plane which is perpendicular to $\mathbf{A} = (3,5,1)$.

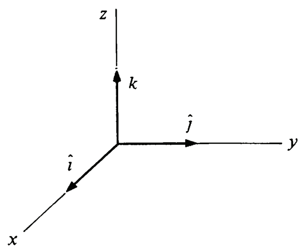
We denote the vector by $\mathbf{B} = (B_x,B_y,B_z)$. Since $\mathbf{B}$ is in the xy plane, $B_z = 0$. For $\mathbf{B}$ to be perpendicular to $\mathbf{A}$, we have $\mathbf{A} \cdot \mathbf{B} = 0$.

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= 3B_x + 5B_y \\ &= 0\end{aligned}$$

Hence $B_y = -\frac{3}{5}B_x$. However, $\mathbf{B}$ is a unit vector, which means that $B_x^2 + B_y^2 = 1$. Combining these gives $B_x^2 + \frac{9}{25}B_x^2 = 1$, or $B_x = \sqrt{\frac{25}{34}} = \pm 0.857$ and $B_y = -\frac{3}{5}B_x = \mp 0.514$.

The ambiguity in sign of B_x and B_y indicates that $\mathbf{B}$ can point along a line perpendicular to $\mathbf{A}$ in either of two directions.

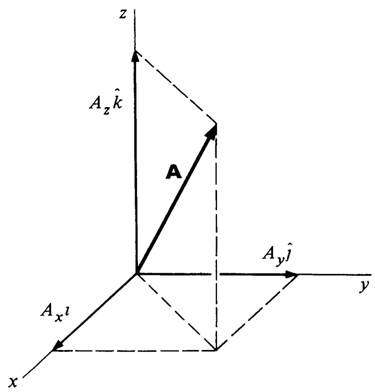
1.4 Base Vectors



Base vectors are a set of orthogonal (perpendicular) unit vectors, one for each dimension. For example, if we are dealing with the familiar cartesian coordinate system of three dimensions, the base vectors lie along the x, y, and z axes. The x unit vector is denoted by $\hat{\mathbf{i}}$, the y unit vector by $\hat{\mathbf{j}}$, and the z unit vector by $\hat{\mathbf{k}}$.

The base vectors have the following properties, as you can readily verify:

$$\begin{aligned}&\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\ &\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0 \\ &\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \\ &\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \\ &\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}.\end{aligned}$$



We can write any vector in terms of the base vectors.

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

The sketch illustrates these two representations of a vector.

To find the component of a vector in any direction, take the dot product with a unit vector in that direction. For instance,

$$A_z = \mathbf{A} \cdot \hat{\mathbf{k}}.$$

It is easy to evaluate the vector product $\mathbf{A} \times \mathbf{B}$ with the aid of the base vectors.

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$

Consider the first term:

$$A_x\hat{\mathbf{i}} \times \mathbf{B} = A_xB_x(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_xB_y(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) + A_xB_z(\hat{\mathbf{i}} \times \hat{\mathbf{k}}).$$

(We have assumed the associative law here.) Since $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0$, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, and $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, we find

$$A_x\hat{\mathbf{i}} \times \mathbf{B} = A_x(B_y\hat{\mathbf{k}} - B_z\hat{\mathbf{j}}).$$

The same argument applied to the y and z components gives

$$A_y\hat{\mathbf{j}} \times \mathbf{B} = A_y(B_z\hat{\mathbf{i}} - B_x\hat{\mathbf{k}})$$
$$A_z\hat{\mathbf{k}} \times \mathbf{B} = A_z(B_x\hat{\mathbf{j}} - B_y\hat{\mathbf{i}}).$$

A quick way to derive these relations is to work out the first and then to obtain the others by cyclically permuting x, y, z, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ (that is, $x \to y$, $y \to z$, $z \to x$, and $\hat{\mathbf{i}} \to \hat{\mathbf{j}}$, $\hat{\mathbf{j}} \to \hat{\mathbf{k}}$, $\hat{\mathbf{k}} \to \hat{\mathbf{i}}$.) A simple way to remember the result is to use the following device: write the base vectors and the components of **A** and **B** as three rows of a determinant,[1] like this

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= \hat{\mathbf{i}}(A_yB_z - A_zB_y) - \hat{\mathbf{j}}(A_xB_z - A_zB_x) + \hat{\mathbf{k}}(A_xB_y - A_yB_x).$$

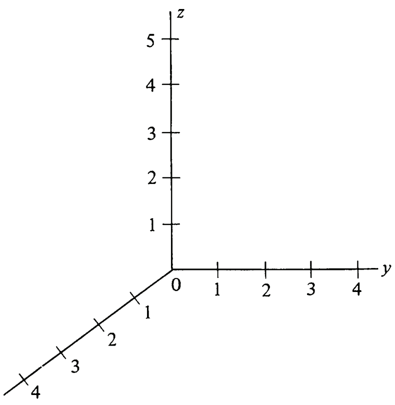
For instance, if $\mathbf{A} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{B} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{vmatrix}$$
$$= 10\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 11\hat{\mathbf{k}}.$$

1.5 Displacement and the Position Vector

So far we have discussed only abstract vectors. However, the reason for introducing vectors here is concrete—they are just right for describing kinematical laws, the laws governing the geometrical properties of motion, which we need to begin our discussion of mechanics. Our first application of vectors will be to the description of position and motion in familiar three dimensional space. Although our first application of vectors is to the motion of a point in space, don't conclude that this is the only

[1] If you are unfamiliar with simple determinants, most of the books listed at the end of the chapter discuss determinants.

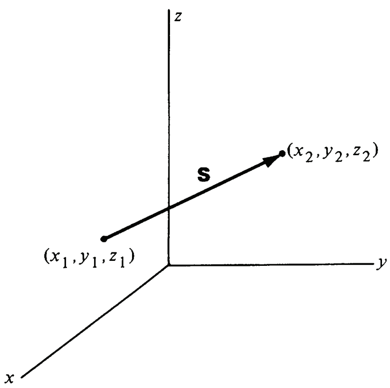


application, or even an unusually important one. Many physical quantities besides displacements are vectors. Among these are velocity, force, momentum, and gravitational and electric fields.

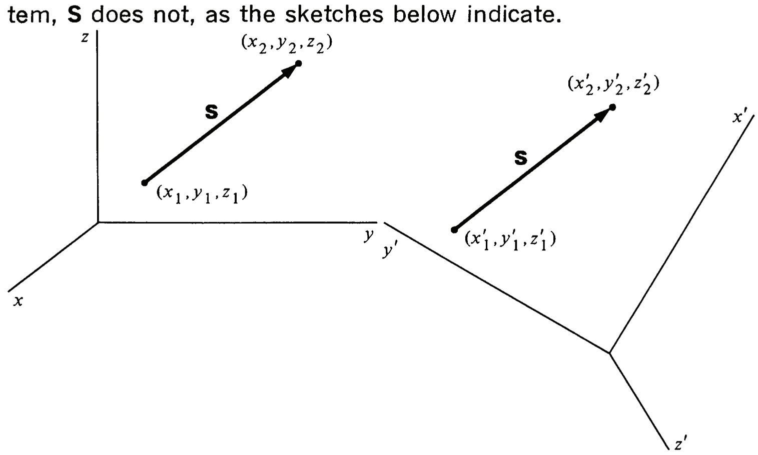
To locate the position of a point in space, we start by setting up a coordinate system. For convenience we choose a three dimensional cartesian system with axes x, y, and z, as shown.

In order to measure position, the axes must be marked off in some convenient unit of length—meters, for instance.

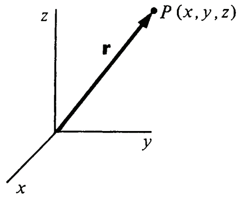
The position of the point of interest is given by listing the values of its three coordinates, x_1, y_1, z_1. These numbers do *not* represent the components of a vector according to our previous discussion. (They specify a position, not a magnitude and direction.) However, if we move the point to some new position, x_2, y_2, z_2, then the *displacement* defines a vector **S** with coordinates $S_x = x_2 - x_1$, $S_y = y_2 - y_1$, $S_z = z_2 - z_1$.



S is a vector from the initial position to the final position—it defines the displacement of a point of interest. Note, however, that **S** contains no information about the initial and final positions separately—only about the *relative* position of each. Thus, $S_z = z_2 - z_1$ depends on the *difference* between the final and initial values of the z coordinates; it does not specify z_2 or z_1 separately. **S** is a true vector; although the values of the coordinates of the initial and final points depend on the coordinate system, **S** does not, as the sketches below indicate.

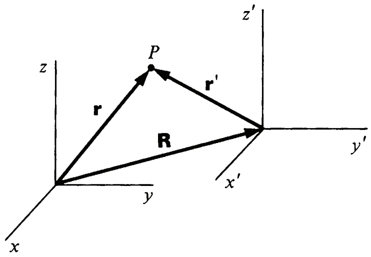


One way in which our displacement vector differs from a mathematician's vector is that his vectors are usually pure quantities, with components given by absolute numbers, whereas **S** has the physical dimension of length associated with it. We will use the convention that the magnitude of a vector has dimensions so that a unit vector is dimensionless. Thus, a displacement of 8 m (8 meters) in the x direction is $\mathbf{S} = (8\text{ m}, 0, 0)$. $|\mathbf{S}| = 8$ m, and $\hat{\mathbf{S}} = \mathbf{S}/|\mathbf{S}| = \hat{\mathbf{i}}$.



Although vectors define displacements rather than positions, it is in fact possible to describe the position of a point with respect to the origin of a given coordinate system by a special vector, known as the *position vector,* which extends from the origin to the point of interest. We shall use the symbol $\mathbf{r}$ to denote the position vector. The position of an arbitrary point P at (x,y,z) is written as

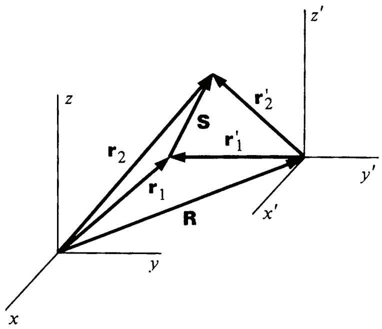
$$\mathbf{r} = (x,y,z) = x\hat{\mathbf{j}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$



Unlike ordinary vectors, $\mathbf{r}$ depends on the coordinate system. The sketch to the left shows position vectors $\mathbf{r}$ and $\mathbf{r}'$ indicating the position of the same point in space but drawn in different coordinate systems. If $\mathbf{R}$ is the vector from the origin of the unprimed coordinate system to the origin of the primed coordinate system, we have

$$\mathbf{r}' = \mathbf{r} - \mathbf{R}.$$

In contrast, a true vector, such as a displacement $\mathbf{S}$, is independent of coordinate system. As the bottom sketch indicates,



$$\begin{aligned}\mathbf{S} &= \mathbf{r}_2 - \mathbf{r}_1\\ &= (\mathbf{r}_2' + \mathbf{R}) - (\mathbf{r}_1' + \mathbf{R})\\ &= \mathbf{r}_2' - \mathbf{r}_1'.\end{aligned}$$

1.6 Velocity and Acceleration

Motion in One Dimension

Before applying vectors to velocity and acceleration in three dimensions, it may be helpful to review briefly the case of one dimension, motion along a straight line.

Let x be the value of the coordinate of a particle moving along a line. x is measured in some convenient unit, such as meters, and we assume that we have a continuous record of position versus time.

The *average velocity* $\bar{v}$ of the point between two times, t_1 and t_2, is defined by

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

(We shall often use a bar to indicate an average of a quantity.)

The *instantaneous velocity* v is the limit of the average velocity as the time interval approaches zero.

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

The limit we have introduced in defining v is precisely that involved in the definition of a derivative. In fact, we have[1]

$$v = \frac{dx}{dt}.$$

In a similar fashion, the *instantaneous acceleration* is

$$a = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$
$$= \frac{dv}{dt}.$$

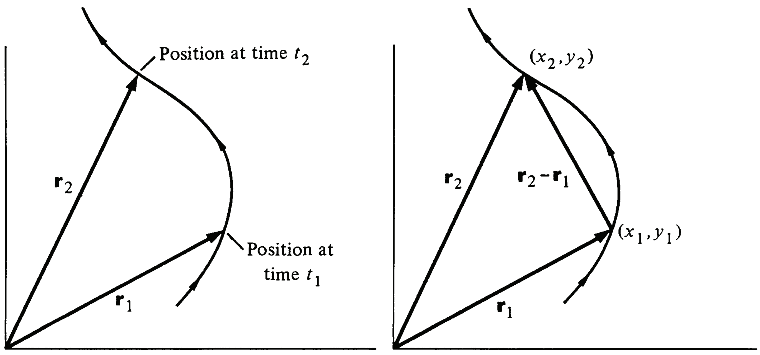
The concept of speed is sometimes useful. Speed s is simply the magnitude of the velocity: $s = |\mathbf{v}|$.

Motion in Several Dimensions

Our task now is to extend the ideas of velocity and acceleration to several dimensions. Consider a particle moving in a plane. As time goes on, the particle traces out a path, and we suppose that we know the particle's coordinates as a function of time. The instantaneous position of the particle at some time t_1 is

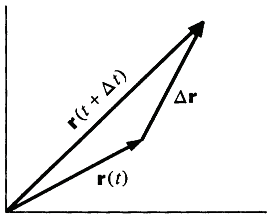
$$\mathbf{r}(t_1) = [x(t_1), y(t_1)] \qquad \text{or} \qquad \mathbf{r}_1 = (x_1, y_1),$$

[1] Physicists generally use the Leibnitz notation dx/dt, since this is a handy form for using differentials (see Note 1.1). Starting in Sec. 1.9 we shall use Newton's notation $\dot{x}$, but only to denote derivatives with respect to time.



where x_1 is the value of x at $t = t_1$, and so forth. At time t_2 the position is

$$\mathbf{r}_2 = (x_2, y_2).$$



The displacement of the particle between times t_1 and t_2 is

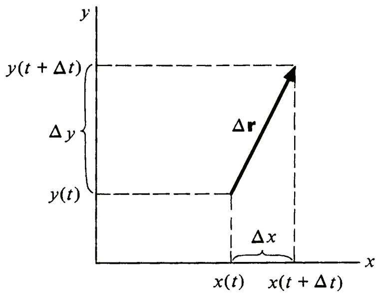
$$\mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1, y_2 - y_1).$$

We can generalize our example by considering the position at some time t, and at some later time $t + \Delta t$.† The displacement of the particle between these times is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

This vector equation is equivalent to the two scalar equations

$$\Delta x = x(t + \Delta t) - x(t)$$
$$\Delta y = y(t + \Delta t) - y(t).$$



The *velocity* $\mathbf{v}$ of the particle as it moves along the path is defined to be

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}.$$

which is equivalent to the two scalar equations

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$
$$v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

Extension of the argument to three dimensions is trivial. The third component of velocity is

$$v_z = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t} = \frac{dz}{dt}.$$

Our definition of velocity as a vector is a straightforward generalization of the familiar concept of motion in a straight line. Vector notation allows us to describe motion in three dimensions with a single equation, a great economy compared with the three equations we would need otherwise. The equation $\mathbf{v} = d\mathbf{r}/dt$ expresses the results we have just found.

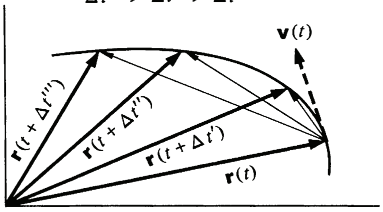
† We will often use the quantity Δ to denote a difference or change, as in the case here of $\Delta\mathbf{r}$ and Δt. However, this implies nothing about the size of the quantity, which may be large or small, as we please.

Alternatively, since $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, we obtain by simple differentiation[1]

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

as before.

Let the particle undergo a displacement $\Delta\mathbf{r}$ in time Δt. In the limit $\Delta t \to 0$, $\Delta\mathbf{r}$ becomes tangent to the trajectory, as the sketch indicates. However, the relation



$$\Delta\mathbf{r} \approx \frac{d\mathbf{r}}{dt}\Delta t$$
$$= \mathbf{v}\,\Delta t,$$

which becomes exact in the limit $\Delta t \to 0$, shows that $\mathbf{v}$ is parallel to $\Delta\mathbf{r}$; the instantaneous velocity $\mathbf{v}$ of a particle is everywhere tangent to the trajectory.

Example 1.7 **Finding v from r**

The position of a particle is given by

$$\mathbf{r} = A(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}),$$

where α is a constant. Find the velocity, and sketch the trajectory.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= A(\alpha e^{\alpha t}\hat{\mathbf{i}} - \alpha e^{-\alpha t}\hat{\mathbf{j}})$$

or

$$v_x = A\alpha e^{\alpha t}$$
$$v_y = -A\alpha e^{-\alpha t}.$$

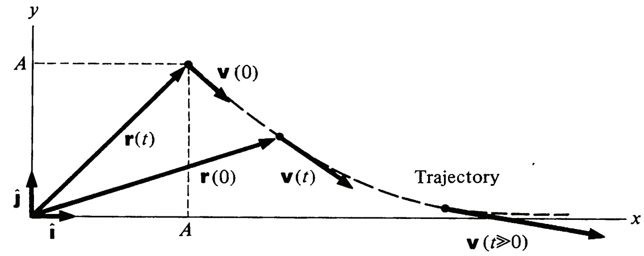
The magnitude of $\mathbf{v}$ is

$$v = (v_x^2 + v_y^2)^{\frac{1}{2}}$$
$$= A\alpha(e^{2\alpha t} + e^{-2\alpha t})^{\frac{1}{2}}.$$

In sketching the motion of a point, it is usually helpful to look at limiting cases. At $t = 0$, we have

$$\mathbf{r}(0) = A(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$
$$v(0) = \alpha A(\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

[1] Caution: We can neglect the cartesian unit vectors when we differentiate, since their directions are fixed. Later we shall encounter unit vectors which can change direction, and then differentiation is more elaborate.

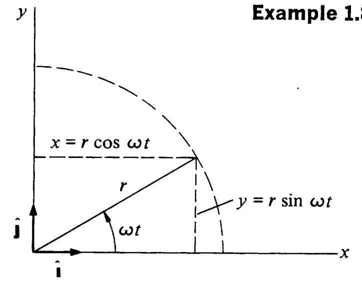


As $t \to \infty$, $e^{\alpha t} \to \infty$ and $e^{-\alpha t} \to 0$. In this limit $\mathbf{r} \to Ae^{\alpha t}\hat{\mathbf{i}}$, which is a vector along the x axis, and $\mathbf{v} \to \alpha A e^{\alpha t}\hat{\mathbf{i}}$; the speed increases without limit.

Similarly, the acceleration **a** is defined by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}$$
$$= \frac{d^2\mathbf{r}}{dt^2}.$$

We could continue to form new vectors by taking higher derivatives of **r**, but we shall see in our study of dynamics that **r**, **v**, and **a** are of chief interest.



Example 1.8 **Uniform Circular Motion**

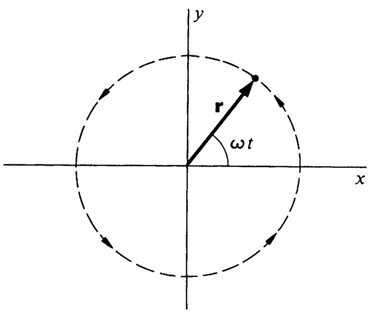
Circular motion plays an important role in physics. Here we look at the simplest and most important case—*uniform* circular motion, which is circular motion at constant speed.

Consider a particle moving in the xy plane according to $\mathbf{r} = r(\cos \omega t\hat{\mathbf{i}} + \sin \omega t\hat{\mathbf{j}})$, where r and ω are constants. Find the trajectory, the velocity, and the acceleration.

$$|\mathbf{r}| = [r^2 \cos^2 \omega t + r^2 \sin^2 \omega t]^{\frac{1}{2}}$$

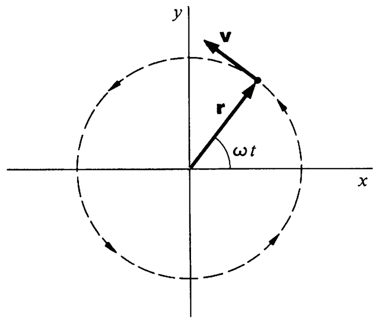
Using the familiar identity $\sin^2 \theta + \cos^2 \theta = 1$,

$$|\mathbf{r}| = [r^2(\cos^2 \omega t + \sin^2 \omega t)]^{\frac{1}{2}}$$
$$= r = \text{constant}.$$



The trajectory is a circle.

The particle moves counterclockwise around the circle, starting from $(r,0)$ at $t = 0$. It traverses the circle in a time T such that $\omega T = 2\pi$. ω is called the *angular velocity* of the motion and is measured in radians



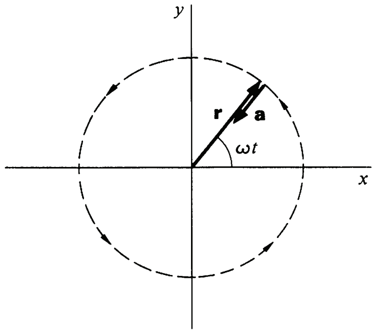
per second. T, the time required to execute one complete cycle, is called the *period.*

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$= r\omega(-\sin \omega t\hat{\mathbf{i}} + \cos \omega t\hat{\mathbf{j}})$$

We can show that $\mathbf{v}$ is tangent to the trajectory by calculating $\mathbf{v} \cdot \mathbf{r}$:

$$\mathbf{v} \cdot \mathbf{r} = r^2\omega(-\sin \omega t \cos \omega t + \cos \omega t \sin \omega t)$$
$$= 0.$$

Since $\mathbf{v}$ is perpendicular to $\mathbf{r}$, it is tangent to the circle as we expect. Incidentally, it is easy to show that $|\mathbf{v}| = r\omega =$ constant.



$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$
$$= r\omega^2[-\cos \omega t\hat{\mathbf{i}} - \sin \omega t\hat{\mathbf{j}}]$$
$$= -\omega^2\mathbf{r}$$

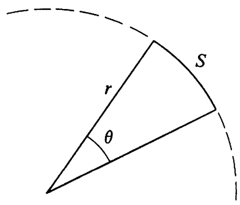
The acceleration is directed radially inward, and is known as the *centripetal acceleration.* We shall have more to say about it shortly.

A Word about Dimension and Units

Physicists call the fundamental physical units in which a quantity is measured the *dimension* of the quantity. For example, the dimension of velocity is distance/time and the dimension of acceleration is velocity/time or (distance/time)/time = distance/time^2. As we shall discuss in Chap. 2, mass, distance, and time are the fundamental physical units used in mechanics.

To introduce a system of units, we specify the standards of measurement for mass, distance, and time. Ordinarily we measure distance in meters and time in seconds. The units of velocity are then meters per second (m/s) and the units of acceleration are meters per second^2 (m/s^2).

The natural unit for measuring angle is the *radian* (rad). The angle θ in radians is S/r, where S is the arc subtended by θ in a circle of radius $\mathbf{r}$:



$$\theta = \frac{S}{r}.$$

2π rad = 360°. We shall always use the radian as the unit of angle, unless otherwise stated. For example, in $\sin \omega t$, ωt is in radians. ω therefore has the dimensions 1/time and the units radians per second. (The radian is dimensionless, since it is the ratio of two lengths.)

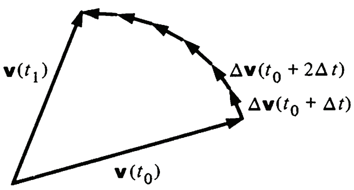
To avoid gross errors, it is a good idea to check to see that both sides of an equation have the same dimensions or units. For example, the equation $v = \alpha r e^{\alpha t}$ is dimensionally correct; since exponentials and their arguments are always dimensionless, α has the units 1/s, and the right hand side has the correct units, meters per second.

1.7 Formal Solution of Kinematical Equations

Dynamics, which we shall take up in the next chapter, enables us to find the acceleration of a body directly. Once we know the acceleration, finding the velocity and position is a simple matter of integration. Here is the formal integration procedure.

If the acceleration is known as a function of time, the velocity can be found from the defining equation

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t)$$



by integration with respect to time. Suppose we want to find $\mathbf{v}(t_1)$ given the initial velocity $\mathbf{v}(t_0)$ and the acceleration $\mathbf{a}(t)$. Dividing the time interval $t_1 - t_0$ into n parts $\Delta t = (t_1 - t_0)/n$,

$$\begin{aligned}\mathbf{v}(t_1) &\approx \mathbf{v}(t_0) + \Delta\mathbf{v}(t_0 + \Delta t) + \Delta\mathbf{v}(t_0 + 2\Delta t) + \cdots + \Delta\mathbf{v}(t_1)\\ &\approx \mathbf{v}(t_0) + \mathbf{a}(t_0 + \Delta t)\,\Delta t + \mathbf{a}(t_0 + 2\Delta t)\,\Delta t + \cdots + \mathbf{a}(t_1)\,\Delta t,\end{aligned}$$

since $\Delta\mathbf{v}(t) \approx \mathbf{a}(t)\,\Delta t$. Taking the x component,

$$v_x(t_1) \approx v_x(t_0) + a_x(t_0 + \Delta t)\,\Delta t + \cdots + a_x(t_1)\,\Delta t.$$

The approximation becomes exact in the limit $n \to \infty\,(\Delta t \to 0)$, and the sum becomes an integral:

$$v_x(t_1) = v_x(t_0) + \int_{t_0}^{t_1} a_x(t)\,dt.$$

The y and z components can be treated similarly. Combining the results,

$$\begin{aligned}v_x(t)_1\hat{\mathbf{i}} + v_y(t_1)\hat{\mathbf{j}} + v_z(t_1)\hat{\mathbf{k}} &= v_x(t_0)\hat{\mathbf{i}} + \int_{t_0}^{t_1} a_x(t)\,dt\,\hat{\mathbf{i}}\\ &\quad + v_y(t_0)\hat{\mathbf{j}} + \int_{t_0}^{t_1} a_y(t)\,dt\,\hat{\mathbf{j}} + v_z(t_0)\hat{\mathbf{k}} + \int_{t_0}^{t_1} a_z(t)\,dt\,\hat{\mathbf{k}}\end{aligned}$$

or

$$\mathbf{v}(t_1) = \mathbf{v}(t_0) + \int_{t_0}^{t_1} \mathbf{a}(t)\,dt.$$

This result is the same as the formal integration of $d\mathbf{v} = \mathbf{a}\,dt$.

$$\int_{t_0}^{t_1} d\mathbf{v} = \int_{t_0}^{t_1} \mathbf{a}(t)\,dt$$

$$\mathbf{v}(t_1) - \mathbf{v}(t_0) = \int_{t_0}^{t_1} \mathbf{a}(t)\,dt$$

Sometimes we need an expression for the velocity at an arbitrary time t, in which case we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')\,dt'.$$

The dummy variable of integration has been changed from t to t' to avoid confusion with the upper limit t. We have designated the initial velocity $\mathbf{v}(t_0)$ by $\mathbf{v}_0$ to make the notation more compact. When $t = t_0$, $\mathbf{v}(t)$ reduces to $\mathbf{v}_0$, as we expect.

Example 1.9 Finding Velocity from Acceleration

A Ping-Pong ball is released near the surface of the moon with velocity $\mathbf{v}_0 = (0,5,-3)$ m/s. It accelerates (downward) with acceleration $\mathbf{a} = (0,0,-2)$ m/s^2. Find its velocity after 5 s.

The equation

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')\,dt'$$

is equivalent to the three component equations

$$v_x(t) = v_{0x} + \int_0^t a_x(t')\,dt'$$

$$v_y(t) = v_{0y} + \int_0^t a_y(t')\,dt'$$

$$v_z(t) = v_{0z} + \int_0^t a_z(t')\,dt'.$$

Taking these equations in turn with the given values of $\mathbf{v}_0$ and $\mathbf{a}$, we obtain at $t = 5$ s:

$$v_x = 0 \text{ m/s}$$

$$v_y = 5 \text{ m/s}$$

$$v_z = -3 + \int_0^5 (-2)\,dt' = -13 \text{ m/s}.$$

Position is found by a second integration. Starting with

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t),$$

we find, by an argument identical to the above,

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t')\,dt'.$$

A particularly important case is that of *uniform acceleration.* If we take $\mathbf{a}$ = constant and $t_0 = 0$, we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t$$

and

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t (\mathbf{v}_0 + \mathbf{a}t')\,dt'$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2.$$

Quite likely you are already familiar with this in its one dimensional form. For instance, the x component of this equation is

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2$$

where v_{0x} is the x component of $\mathbf{v}_0$. This expression is so familiar that you may inadvertently apply it to the general case of varying acceleration. Don't! It only holds for *uniform* acceleration. In general, the full procedure described above must be used.

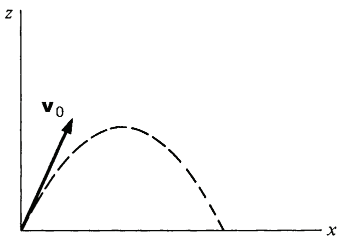
Example 1.10 **Motion in a Uniform Gravitational Field**

Suppose that an object moves freely under the influence of gravity so that it has a constant downward acceleration g. Choosing the z axis vertically upward, we have

$$\mathbf{a} = -g\hat{\mathbf{k}}.$$

If the object is released at $t = 0$ with initial velocity $\mathbf{v}_0$, we have

$$x = x_0 + v_{0x}t$$
$$y = y_0 + v_{0y}t$$
$$z = z_0 + v_{0z}t - \tfrac{1}{2}gt^2.$$



Without loss of generality, we can let $\mathbf{r}_0 = 0$, and assume that $v_{0y} = 0$. (The latter assumption simply means that we choose the coordinate system so that the initial velocity is in the xz plane.) Then

$$x = v_{0x}t$$
$$z = v_{0z}t - \tfrac{1}{2}gt^2.$$

The path of the object is shown in the sketch. We can eliminate time from the two equations for x and z to obtain the *trajectory.*

$$z = \frac{v_{0z}}{v_{0x}}x - \frac{g}{2{v_{0x}}^2}x^2$$

This is the well-known parabola of free fall projectile motion. However, as mentioned above, uniform acceleration is not the most general case.

Example 1.11 Nonuniform Acceleration—The Effect of a Radio Wave on an Ionospheric Electron

The ionosphere is a region of electrically neutral gas, composed of positively charged ions and negatively charged electrons, which surrounds the earth at a height of approximately 200 km (120 mi). If a radio wave passes through the ionosphere, its electric field accelerates the charged particle. Because the electric field oscillates in time, the charged particles tend to jiggle back and forth. The problem is to find the motion of an electron of charge $-e$ and mass m which is initially at rest, and which is suddenly subjected to an electric field $\mathbf{E} = \mathbf{E}_0 \sin \omega t$ (ω is the frequency of oscillation in radians per second).

The law of force for the charge in the electric field is $\mathbf{F} = -e\mathbf{E}$, and by Newton's second law we have $\mathbf{a} = \mathbf{F}/m = -e\mathbf{E}/m$. (If the reasoning behind this is a mystery to you, ignore it for now. It will be clear later. This example is meant to be a mathematical exercise—the physics is an added dividend.) We have

$$\begin{aligned}\mathbf{a} &= \frac{-e\mathbf{E}}{m}\\ &= \frac{-e\mathbf{E}_0}{m}\sin \omega t.\end{aligned}$$

$\mathbf{E}_0$ is a constant vector and we shall choose our coordinate system so that the x axis lies along it. Since there is no acceleration in the y or z directions, we need consider only the x motion. With this understanding, we can drop subscripts and write a for a_x.

$$a(t) = \frac{-eE_0}{m}\sin \omega t = a_0 \sin \omega t$$

where

$$a_0 = \frac{-eE_0}{m}.$$

Then

$$\begin{aligned}v(t) &= v_0 + \int_0^t a(t')\,dt'\\ &= v_0 + \int_0^t a_0 \sin \omega t'\,dt'\\ &= v_0 - \frac{a_0}{\omega}\cos \omega t'\Big|_0^t = v_0 - \frac{a_0}{\omega}(\cos \omega t - 1)\end{aligned}$$

and

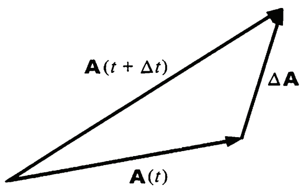
$$\begin{aligned} x &= x_0 + \int_0^t v(t')\,dt' \\ &= x_0 + \int_0^t \left[v_0 - \frac{a_0}{\omega}(\cos \omega t' - 1) \right] dt' \\ &= x_0 + \left(v_0 + \frac{a_0}{\omega} \right) t - \frac{a_0}{\omega^2} \sin \omega t. \end{aligned}$$

We are given that $x_0 = v_0 = 0$, so we have

$$x = \frac{a_0}{\omega} t - \frac{a_0}{\omega^2} \sin \omega t.$$

The result is interesting: the second term oscillates and corresponds to the jiggling motion of the electron, which we predicted. The first term, however, corresponds to motion with uniform velocity, so in addition to the jiggling motion the electron starts to drift away. Can you see why?

1.8 More about the Derivative of a Vector



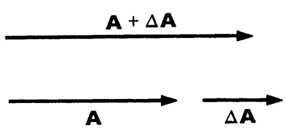
In Sec. 1.6 we demonstrated how to describe velocity and acceleration by vectors. In particular, we showed how to differentiate the vector $\mathbf{r}$ to obtain a new vector $\mathbf{v} = d\mathbf{r}/dt$. We will want to differentiate other vectors with respect to time on occasion, and so it is worthwhile generalizing our discussion.

Consider some vector $\mathbf{A}(t)$ which is a function of time. The change in $\mathbf{A}$ during the interval from t to $t + \Delta t$ is

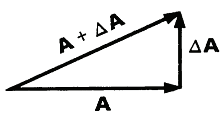
$$\Delta\mathbf{A} = \mathbf{A}(t + \Delta t) - \mathbf{A}(t).$$

In complete analogy to the procedure we followed in differentiating $\mathbf{r}$ in Sec. 1.6, we define the time derivative of $\mathbf{A}$ by

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t}.$$



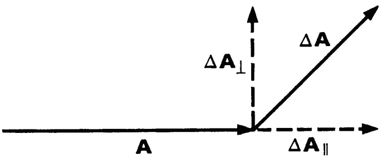
Case 1



Case 2

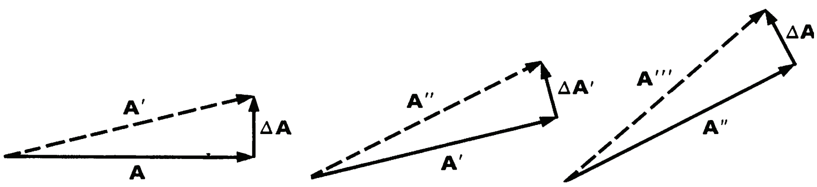
It is important to appreciate that $d\mathbf{A}/dt$ is a new vector which can be large or small, and can point in any direction, depending on the behavior of $\mathbf{A}$.

There is one important respect in which $d\mathbf{A}/dt$ differs from the derivative of a simple scalar function. $\mathbf{A}$ can change in both *magnitude* and *direction*—a scalar function can change only in magnitude. This difference is important. The figure illustrates the addition of a small increment $\Delta\mathbf{A}$ to $\mathbf{A}$. In the first case $\Delta\mathbf{A}$ is parallel to $\mathbf{A}$; this leaves the direction unaltered but changes the magnitude to $|\mathbf{A}| + |\Delta\mathbf{A}|$. In the second, $\Delta\mathbf{A}$ is perpendicular to **A**. This causes a change of *direction* but leaves the magnitude practically unaltered.

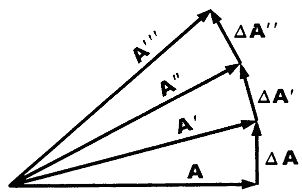


In general, **A** will change in both magnitude and direction. Even so, it is useful to visualize both types of change taking place simultaneously. In the sketch to the left we show a small increment $\Delta\mathbf{A}$ resolved into a component vector $\Delta\mathbf{A}_\parallel$ parallel to **A** and a component vector $\Delta\mathbf{A}_\perp$ perpendicular to **A**. In the limit where we take the derivative, $\Delta\mathbf{A}_\parallel$ changes the magnitude of **A** but not its direction, while $\Delta\mathbf{A}_\perp$ changes the direction of **A** but not its magnitude.

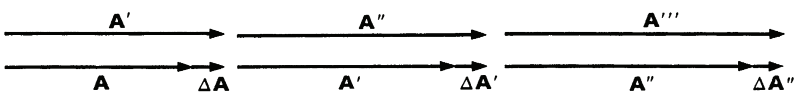
Students who do not have a clear understanding of the two ways a vector can change sometimes make an error by neglecting one of them. For instance, if $d\mathbf{A}/dt$ is always perpendicular to **A**, **A** must *rotate,* since its magnitude cannot change; its time dependence arises solely from change in direction. The illustrations below show how rotation occurs when $\Delta\mathbf{A}$ is always perpendicular to **A**. The rotational motion is made more apparent by drawing



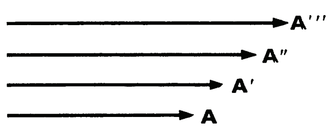
the successive vectors at a common origin.



Contrast this with the case where $\Delta\mathbf{A}$ is always parallel to **A**.



Drawn from a common origin, the vectors look like this:



The following example relates the idea of rotating vectors to circular motion.

Example 1.12 Circular Motion and Rotating Vectors

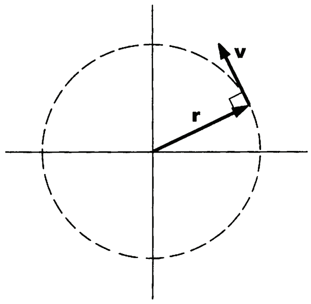
In Example 1.8 we discussed the motion given by

$\mathbf{r} = r(\cos \omega t \hat{\mathbf{i}} + \sin \omega t \hat{\mathbf{j}})$.

The velocity is

$\mathbf{v} = r\omega(-\sin \omega t \hat{\mathbf{i}} + \cos \omega t \hat{\mathbf{j}})$.

Since

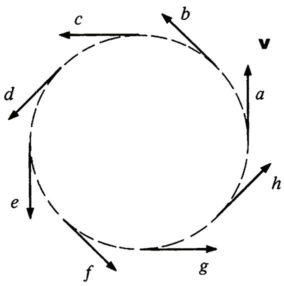


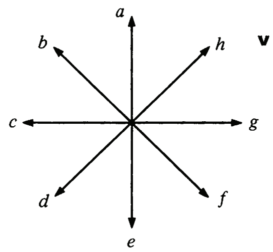
$$\begin{aligned}\mathbf{r} \cdot \mathbf{v} &= r^2\omega(-\cos \omega t \sin \omega t + \sin \omega t \cos \omega t)\\ &= 0,\end{aligned}$$

we see that $d\mathbf{r}/dt$ is perpendicular to $\mathbf{r}$. We conclude that the magnitude of $\mathbf{r}$ is constant, so that the only possible change in $\mathbf{r}$ is due to rotation. Since the trajectory is a circle, this is precisely the case: $\mathbf{r}$ rotates about the origin.

We showed earlier that $\mathbf{a} = -\omega^2\mathbf{r}$. Since $\mathbf{r} \cdot \mathbf{v} = 0$, it follows that $\mathbf{a} \cdot \mathbf{v} = -\omega^2\mathbf{r} \cdot \mathbf{v} = 0$ and $d\mathbf{v}/dt$ is perpendicular to $\mathbf{v}$. This means that the velocity vector has constant magnitude, so that it too must rotate if it is to change in time.

That $\mathbf{v}$ indeed rotates is readily seen from the sketch, which shows $\mathbf{v}$ at various positions along the trajectory. In the second sketch the same

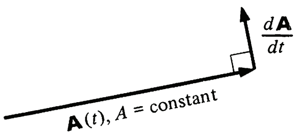




velocity vectors are drawn from a common origin. It is apparent that each time the particle completes a traversal, the velocity vector has swung around through a full circle.

Perhaps you can show that the acceleration vector also undergoes uniform rotation.

Suppose a vector $\mathbf{A}(t)$ has constant magnitude A. The only way $\mathbf{A}(t)$ can change in time is by rotating, and we shall now develop a useful expression for the time derivative $d\mathbf{A}/dt$ of such a



rotating vector. The direction of $d\mathbf{A}/dt$ is always perpendicular to $\mathbf{A}$. The magnitude of $d\mathbf{A}/dt$ can be found by the following geometrical argument.

The change in $\mathbf{A}$ in the time interval t to $t + \Delta t$ is

$$\Delta\mathbf{A} = \mathbf{A}(t + \Delta t) - \mathbf{A}(t).$$

Using the angle $\Delta\theta$ defined in the sketch,

$$|\Delta\mathbf{A}| = 2A \sin\frac{\Delta\theta}{2}.$$

For $\Delta\theta \ll 1$, $\sin \Delta\theta/2 \approx \Delta\theta/2$, as discussed in Note 1.1. We have

$$\begin{aligned}|\Delta\mathbf{A}| &\approx 2A\frac{\Delta\theta}{2}\\ &= A\,\Delta\theta\end{aligned}$$

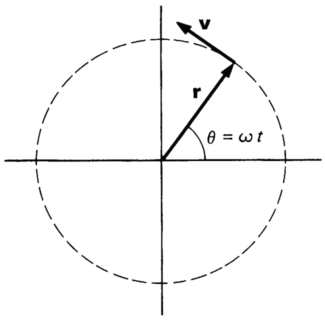
and

$$\left|\frac{\Delta\mathbf{A}}{\Delta t}\right| = A\frac{\Delta\theta}{\Delta t}.$$

Taking the limit $\Delta t \to 0$,

$$\left|\frac{d\mathbf{A}}{dt}\right| = A\frac{d\theta}{dt}.$$

$d\theta/dt$ is called the *angular velocity* of $\mathbf{A}$.



For a simple application of this result, let $\mathbf{A}$ be the rotating vector $\mathbf{r}$ discussed in Examples 1.8 and 1.12. Then $\theta = \omega t$ and

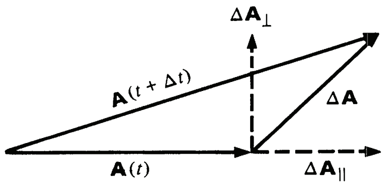
$$\left|\frac{d\mathbf{r}}{dt}\right| = r\frac{d}{dt}(\omega t) = r\omega \qquad \text{or} \qquad v = r\omega.$$

Returning now to the general case, a change in $\mathbf{A}$ is the result of a rotation *and* a change in magnitude.

$$\Delta\mathbf{A} = \Delta\mathbf{A}_\perp + \Delta\mathbf{A}_\parallel.$$

For $\Delta\theta$ sufficiently small,

$$\begin{aligned}|\Delta\mathbf{A}_\perp| &= A\,\Delta\theta\\ |\Delta\mathbf{A}_\parallel| &= \Delta A\end{aligned}$$



and, dividing by Δt and taking the limit,

$$\begin{aligned}\left|\frac{d\mathbf{A}_\perp}{dt}\right| &= A\frac{d\theta}{dt}\\ \left|\frac{d\mathbf{A}_\parallel}{dt}\right| &= \frac{dA}{dt}.\end{aligned}$$

$d\mathbf{A}_\perp/dt$ is zero if **A** does not rotate ($d\theta/dt = 0$), and $d\mathbf{A}_\parallel/dt$ is zero if **A** is constant in magnitude.

We conclude this section by stating some formal identities in vector differentiation. Their proofs are left as exercises. Let the scalar c and the vectors **A** and **B** be functions of time. Then

$$\frac{d}{dt}(c\mathbf{A}) = \frac{dc}{dt}\mathbf{A} + c\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$$

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}.$$

In the second relation, let $\mathbf{A} = \mathbf{B}$. Then

$$\frac{d}{dt}(A^2) = 2\mathbf{A}\cdot\frac{d\mathbf{A}}{dt},$$

and we see again that if $d\mathbf{A}/dt$ is perpendicular to **A**, the magnitude of **A** is constant.

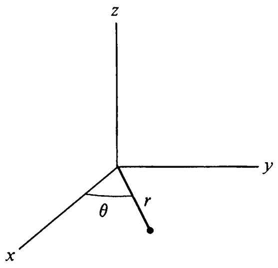
1.9 Motion in Plane Polar Coordinates

Polar Coordinates

Rectangular, or cartesian, coordinates are well suited to describing motion in a straight line. For instance, if we orient the coordinate system so that one axis lies in the direction of motion, then only a single coordinate changes as the point moves. However, rectangular coordinates are not so useful for describing circular motion, and since circular motion plays a prominent role in physics, it is worth introducing a coordinate system more natural to it.

We should mention that although we can use any coordinate system we like, the proper choice of a coordinate system can vastly simplify a problem, so that the material in this section is very much in the spirit of more advanced physics. Quite likely some of this material will be entirely new to you. Be patient if it seems strange or even difficult at first. Once you have studied the examples and worked a few problems, it will seem much more natural.

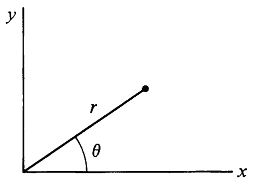
Our new coordinate system is based on the cylindrical coordinate system. The z axis of the cylindrical system is identical to that of the cartesian system. However, position in the xy plane is



described by distance r from the z axis and the angle θ that r makes with the x axis. These coordinates are shown in the sketch. We see that

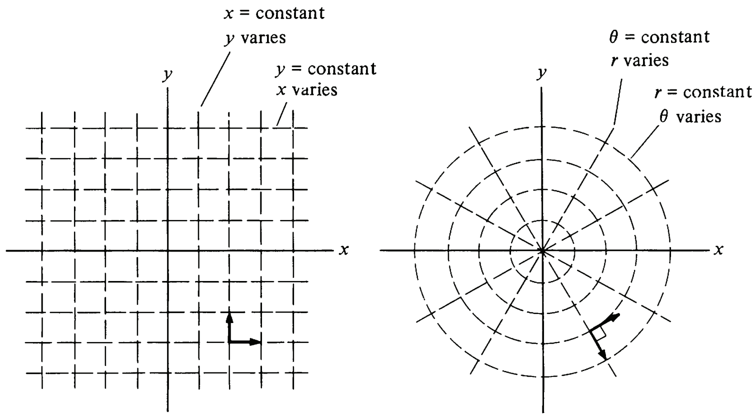
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}.$$



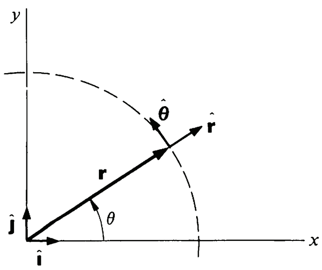
Since we shall be concerned primarily with motion in a plane, we neglect the z axis and restrict our discussion to two dimensions. The coordinates r and θ are called *plane polar* coordinates. In the following sections we shall learn to describe position, velocity, and acceleration in plane polar coordinates.

The contrast between cartesian and plane polar coordinates is readily seen by comparing drawings of constant coordinate lines for the two systems.



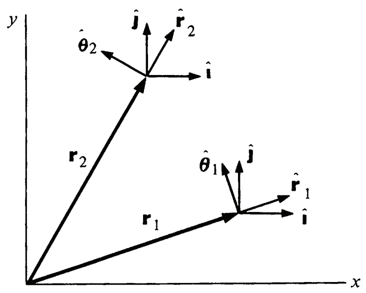
Cartesian

Plane polar



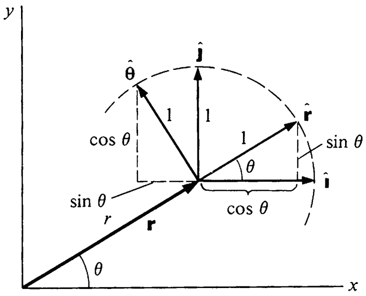
The lines of constant x and of constant y are straight and perpendicular to each other. Lines of constant θ are also straight, directed radially outward from the origin. In contrast, lines of constant r are circles concentric to the origin. Note, however, that the lines of constant θ and constant r are perpendicular wherever they intersect.

In Sec. 1.4 we introduced the base vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ which point in the direction of increasing x and increasing y, respectively. In a similar fashion we now introduce two new unit vectors, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, which point in the direction of increasing r and increasing θ. There is an important difference between these base vectors and the



cartesian base vectors: the directions of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with position, whereas $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ have fixed directions. The drawing shows this by illustrating both sets of base vectors at two points in space. Because $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with position, kinematical formulas can look more complicated in polar coordinates than in the cartesian system. (It is not that polar coordinates are complicated, it is simply that cartesian coordinates are simpler than they have a right to be. Cartesian coordinates are the only coordinates whose base vectors have fixed directions.)

Although $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary with position, note that they depend on θ only, not on r. We can think of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ as being functionally dependent on θ.



The drawing shows the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ at a point in the xy plane. We see that

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \cos \theta + \hat{\mathbf{j}} \sin \theta$$
$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{i}} \sin \theta + \hat{\mathbf{j}} \cos \theta.$$

Before proceeding, convince yourself that these expressions are reasonable by checking them at a few particularly simple points, such as $\theta = 0$, and $\pi/2$. Also verify that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are orthogonal (i.e., perpendicular) by showing that $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$.

It is easy to verify that we indeed have the same vector $\mathbf{r}$ no matter whether we describe it by cartesian or polar coordinates. In cartesian coordinates we have

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},$$

and in polar coordinates we have

$$\mathbf{r} = r\hat{\mathbf{r}}.$$

If we insert the above expression for $\hat{\mathbf{r}}$, we obtain

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r(\hat{\mathbf{i}} \cos \theta + \hat{\mathbf{j}} \sin \theta).$$

We can separately equate the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to obtain

$$x = r \cos \theta \qquad y = r \sin \theta,$$

as we expect.

The relation

$$\mathbf{r} = r\hat{\mathbf{r}}$$

is sometimes confusing, because the equation as written seems to make no reference to the angle θ. We know that two parameters are needed to specify a position in two dimensional space (in cartesian coordinates they are x and y), but the equation $\mathbf{r} = r\hat{\mathbf{r}}$ seems to contain only the quantity r. The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is oriented as well as the value of r to tell how far we are from the origin. Although θ does not occur explicitly in $r\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r} = r\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ. However, by common convention $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

The orthogonality of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ plus the fact that they are unit vectors, $|\hat{\mathbf{r}}| = 1$, $|\hat{\boldsymbol{\theta}}| = 1$, means that we can continue to evaluate scalar products in the simple way we are accustomed to. If

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} \qquad \text{and} \qquad \mathbf{B} = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}},$$

then

$$\mathbf{A}\cdot\mathbf{B} = A_rB_r + A_\theta B_\theta.$$

Of course, the $\hat{\mathbf{r}}$'s and the $\hat{\boldsymbol{\theta}}$'s must refer to the same point in space for this simple rule to hold.

Velocity in Polar Coordinates

Now let us turn our attention to describing velocity with polar coordinates. Recall that in cartesian coordinates we have

$$\begin{aligned}\mathbf{v} &= \frac{d}{dt}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\\ &= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}.\end{aligned}$$

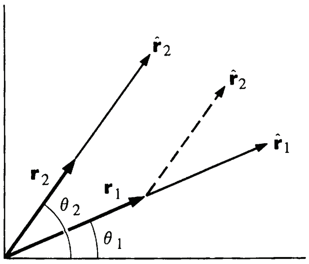
(Remember that $\dot{x}$ stands for dx/dt.)

The same vector, $\mathbf{v}$, expressed in polar coordinates is given by

$$\begin{aligned}\mathbf{v} &= \frac{d}{dt}(r\hat{\mathbf{r}})\\ &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}.\end{aligned}$$

The first term on the right is obviously the component of the velocity directed radially outward. We suspect that the second term is the component of velocity in the tangential ($\hat{\boldsymbol{\theta}}$) direction. This is indeed the case. However to prove it we must evaluate $d\hat{\mathbf{r}}/dt$. Since this step is slightly tricky, we shall do it three different ways. Take your pick!

Evaluating $d\hat{\mathbf{r}}/dt$



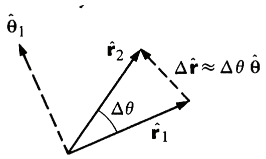
Method 1 We can invoke the ideas of the last section to find $d\hat{\mathbf{r}}/dt$. Since $\hat{\mathbf{r}}$ is a unit vector, its magnitude is constant and $d\hat{\mathbf{r}}/dt$ is perpendicular to $\hat{\mathbf{r}}$; as θ increases, $\hat{\mathbf{r}}$ rotates.

$$|\Delta\hat{\mathbf{r}}| \approx |\hat{\mathbf{r}}|\,\Delta\theta = \Delta\theta,$$

$$\frac{|\Delta\hat{\mathbf{r}}|}{\Delta t} \approx \frac{\Delta\theta}{\Delta t},$$

and, taking the limit, we obtain

$$\left|\frac{d\hat{\mathbf{r}}}{dt}\right| = \frac{d\theta}{dt}.$$



As the sketch shows, as θ increases, $\hat{\mathbf{r}}$ swings in the $\hat{\boldsymbol{\theta}}$ direction, hence

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}.$$

If this method is too casual for your taste, you may find methods 2 or 3 more appealing.

Method 2

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$$

We note that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are fixed unit vectors, and thus cannot vary in time. θ, on the other hand, does vary as $\mathbf{r}$ changes. Using

$$\begin{aligned}\frac{d}{dt}(\cos\theta) &= \left(\frac{d}{d\theta}\cos\theta\right)\frac{d\theta}{dt}\\ &= -\sin\theta\,\dot{\theta}\end{aligned}$$

and

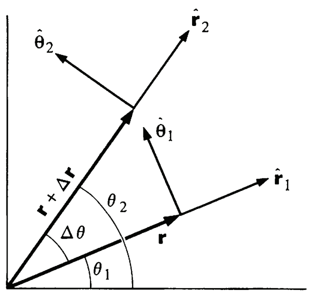
$$\begin{aligned}\frac{d}{dt}(\sin\theta) &= \left(\frac{d}{d\theta}\sin\theta\right)\frac{d\theta}{dt}\\ &= \cos\theta\,\dot{\theta},\end{aligned}$$

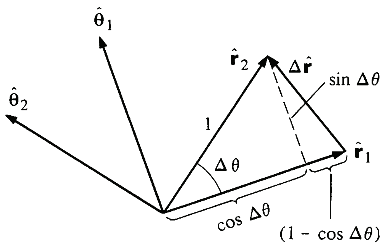
we obtain

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \hat{\mathbf{i}}\frac{d}{dt}(\cos\theta) + \hat{\mathbf{j}}\frac{d}{dt}(\sin\theta)\\ &= -\hat{\mathbf{i}}\sin\theta\,\dot{\theta} + \hat{\mathbf{j}}\cos\theta\,\dot{\theta}\\ &= (-\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta)\,\dot{\theta}.\end{aligned}$$

However, recall that $-\hat{\mathbf{i}} \sin\theta + \hat{\mathbf{j}} \cos\theta = \hat{\boldsymbol{\theta}}$. We obtain

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}.$$





Method 3

The drawing shows $\mathbf{r}$ at two different times, t and $t + \Delta t$. The coordinates are, respectively, (r,θ) and $(r + \Delta r,\ \theta + \Delta\theta)$. Note that the angle between $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ is equal to the angle between $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$; this angle is $\theta_2 - \theta_1 = \Delta\theta$.

The change in $\hat{\mathbf{r}}$ during the time Δt is illustrated by the lower drawing. We see that

$$\Delta\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}_1 \sin\Delta\theta - \hat{\mathbf{r}}_1 (1 - \cos\Delta\theta).$$

Hence

$$\begin{aligned}\frac{\Delta\hat{\mathbf{r}}}{\Delta t} &= \hat{\boldsymbol{\theta}}_1 \frac{\sin\Delta\theta}{\Delta t} - \hat{\mathbf{r}}_1 \frac{(1 - \cos\Delta\theta)}{\Delta t}\\ &= \hat{\boldsymbol{\theta}}_1 \left(\frac{\Delta\theta - \frac{1}{6}(\Delta\theta)^3 + \cdots}{\Delta t}\right) - \hat{\mathbf{r}}_1 \left(\frac{\frac{1}{2}(\Delta\theta)^2 - \frac{1}{24}(\Delta\theta)^4 + \cdots}{\Delta t}\right),\end{aligned}$$

where we have used the series expansions discussed in Note 1.1. We need to evaluate

$$\frac{d\hat{\mathbf{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\hat{\mathbf{r}}}{\Delta t}.$$

In the limit $\Delta t \to 0$, $\Delta\theta$ also approaches zero, but $\Delta\theta/\Delta t$ approaches the limit $d\theta/dt$. Therefore

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} (\Delta\theta)^n = 0 \qquad n > 0.$$

The term in $\hat{\mathbf{r}}$ entirely vanishes in the limit and we are left with

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}},$$

as before. We also need an expression for $d\hat{\boldsymbol{\theta}}/dt$. You can use any, or all, of the arguments above to prove for yourself that

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}.$$

Since you should be familiar with both results, let's summarize them together:

$$\boxed{\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\hat{\boldsymbol{\theta}}\\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\hat{\mathbf{r}}.\end{aligned}}$$

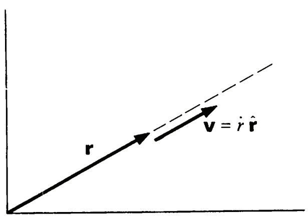
And now, we can return to our problem. On page 30 we showed that

$$\mathbf{v} = \frac{d}{dt}r\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}.$$

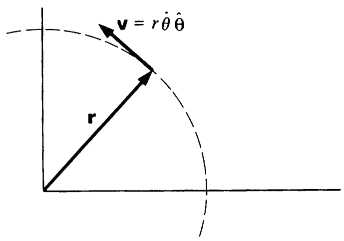
Using the above results, we can write this as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

As we surmised, the second term is indeed in the tangential (that is, $\hat{\boldsymbol{\theta}}$) direction. We can get more insight into the meaning of each term by considering special cases where only one component varies at a time.



Case 1



Case 2

1. θ = constant, velocity is radial. If θ is a constant, $\dot{\theta} = 0$, and $\mathbf{v} = \dot{r}\hat{\mathbf{r}}$. We have one dimensional motion in a fixed radial direction.

2. r = constant, velocity is tangential. In this case $\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}}$. Since r is fixed, the motion lies on the arc of a circle. The speed of the point on the circle is $r\dot{\theta}$, and it follows that $\mathbf{v} = r\dot{\theta}\hat{\boldsymbol{\theta}}$.

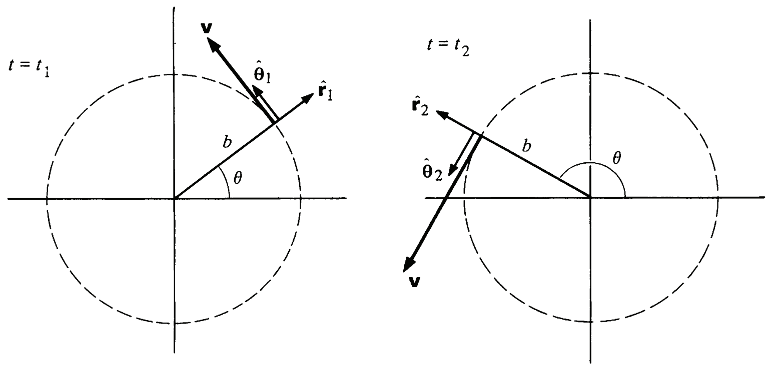
For motion in general, both r and θ change in time.

The next three examples illustrate the use of polar coordinates to describe velocity.

Example 1.13 Circular Motion and Straight Line Motion in Polar Coordinates

A particle moves in a circle of radius b with angular velocity $\dot{\theta} = \alpha t$, where α is a constant. (α has the units radians per second^2.) Describe the particle's velocity in polar coordinates.

Since $r = b$ = constant, $\mathbf{v}$ is purely tangential and $\mathbf{v} = b\alpha t\hat{\boldsymbol{\theta}}$. The sketches show $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\mathbf{v}$ at a time t_1 and at a later time t_2.

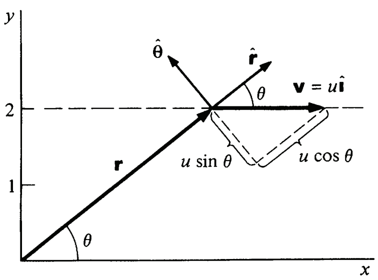


The particle is located at the position

$$r = b \qquad \theta = \theta_0 + \int_0^t \dot{\theta}\,dt = \theta_0 + \tfrac{1}{2}\alpha t^2.$$

If the particle is on the x axis at $t = 0$, $\theta_0 = 0$. The particle's position vector is $\mathbf{r} = b\hat{\mathbf{r}}$, but as the sketches indicate, θ must be given to specify the direction of $\hat{\mathbf{r}}$.

Consider a particle moving with constant velocity $\mathbf{v} = u\hat{\mathbf{i}}$ along the line $y = 2$. Describe $\mathbf{v}$ in polar coordinates.



$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$$

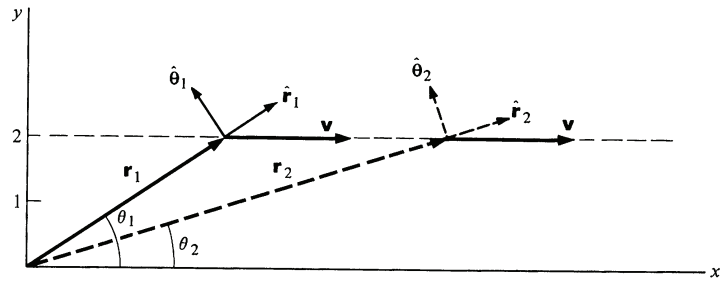
From the sketch,

$$v_r = u\cos\theta$$

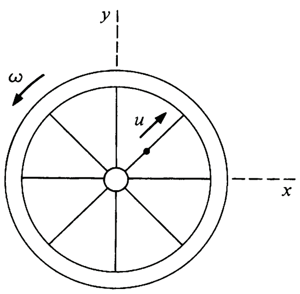
$$v_\theta = -u\sin\theta$$

$$\mathbf{v} = u\cos\theta\,\hat{\mathbf{r}} - u\sin\theta\,\hat{\boldsymbol{\theta}}.$$

As the particle moves to the right, θ decreases and $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ change direction. Ordinarily, of course, we try to use coordinates that make the problem as simple as possible; polar coordinates are not well suited here.



Example 1.14 **Velocity of a Bead on a Spoke**



A bead moves along the spoke of a wheel at constant speed u meters per second. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis, and the bead is at the origin. Find the velocity at time t

a. In polar coordinates

b. In cartesian coordinates.

a. We have $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$. Hence

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

To specify the velocity completely, we need to know the direction of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. This is obtained from $\mathbf{r} = (r,\theta) = (ut,\omega t)$.

b. In cartesian coordinates, we have

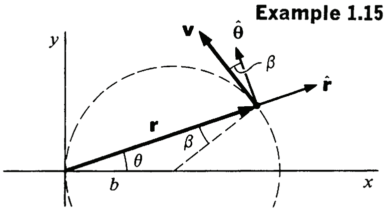
$$v_x = v_r \cos\theta - v_\theta \sin\theta$$
$$v_y = v_r \sin\theta + v_\theta \cos\theta.$$

Since $v_r = u$, $v_\theta = r\omega = ut\omega$, $\theta = \omega t$, we obtain

$$\mathbf{v} = (u \cos\omega t - ut\omega \sin\omega t)\hat{\mathbf{i}} + (u \sin\omega t + ut\omega \cos\omega t)\hat{\mathbf{j}}.$$

Note how much simpler the result is in plane polar coordinates.

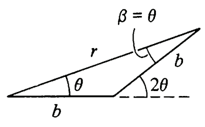
Example 1.15 **Off-center Circle**



A particle moves with constant speed v around a circle of radius b. Find its velocity vector in polar coordinates using an origin lying on the circle.

With this origin, **v** is no longer purely tangential, as the sketch indicates.

$$\begin{aligned}\mathbf{v} &= -v \sin\beta\hat{\mathbf{r}} + v \cos\beta\hat{\boldsymbol{\theta}}\\ &= -v \sin\theta\hat{\mathbf{r}} + v \cos\theta\hat{\boldsymbol{\theta}}.\end{aligned}$$



The last step follows since β and θ are the base angles of an isosceles triangle and are therefore equal. To complete the calculation, we must find θ as a function of time. By geometry, $2\theta = \omega t$ or $\theta = \omega t/2$, where $\omega = v/b$.

Acceleration in Polar Coordinates

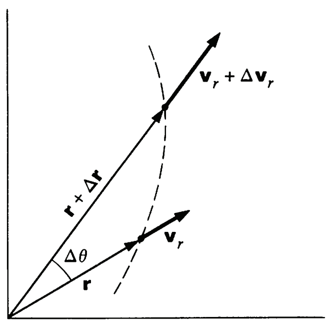
Our final task is to find the acceleration. We differentiate **v** to obtain

$$\begin{aligned}\mathbf{a} &= \frac{d}{dt}\mathbf{v}\\ &= \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})\\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.\end{aligned}$$

If we substitute the results for $d\hat{\mathbf{r}}/dt$ and $d\hat{\boldsymbol{\theta}}/dt$ from page 33, we obtain

$$\begin{aligned}\mathbf{a} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}\\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

The term $\ddot{r}\hat{\mathbf{r}}$ is a linear acceleration in the radial direction due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\boldsymbol{\theta}}$ is a linear acceleration in the tangential direction due to change in the magnitude of the angular velocity.

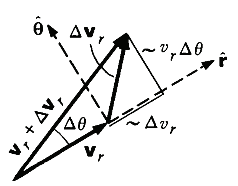


The term $-r\dot{\theta}^2\hat{\mathbf{r}}$ is the centripetal acceleration which we encountered in Example 1.8. Finally, $2\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}}$ is the *Coriolis* acceleration. Perhaps you have heard of the Coriolis force, a fictitious force which appears to act in a rotating coordinate system, and which we shall study in Chap. 8. The Coriolis acceleration that we are discussing here is a real acceleration which is present when r and θ both change with time.

The expression for acceleration in polar coordinates appears complicated. However, by looking at it from the geometric point of view, we can obtain a more intuitive picture.

The instantaneous velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$$



Let us look at the velocity at two different times, treating the radial and tangential terms separately.

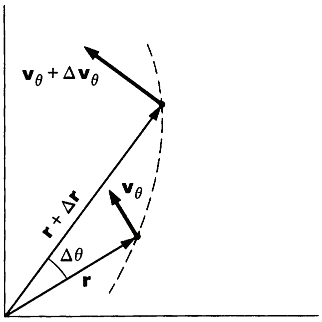
The sketch at left shows the radial velocity $\dot{r}\hat{\mathbf{r}} = v_r\hat{\mathbf{r}}$ at two different instants. The change $\Delta\mathbf{v}_r$ has both a radial and a tangential component. As we can see from the sketch (or from the discussion at the end of Sec. 1.8), the radial component of $\Delta\mathbf{v}_r$ is $\Delta v_r\hat{\mathbf{r}}$ and the tangential component is $v_r\,\Delta\theta\hat{\boldsymbol{\theta}}$. The radial component contributes

$$\lim_{\Delta t\to 0}\left(\frac{\Delta v_r}{\Delta t}\,\hat{\mathbf{r}}\right) = \frac{dv_r}{dt}\,\hat{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}}$$

to the acceleration. The tangential component contributes

$$\lim_{\Delta t\to 0}\left(v_r\,\frac{\Delta\theta}{\Delta t}\,\hat{\boldsymbol{\theta}}\right) = v_r\,\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} = \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}},$$

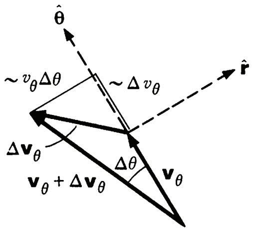
which is one-half the Coriolis acceleration. We see that half the Coriolis acceleration arises from the change of direction of the radial velocity.



The tangential velocity $r\dot{\theta}\hat{\boldsymbol{\theta}} = v_\theta\hat{\boldsymbol{\theta}}$ can be treated similarly. The change in direction of $\hat{\boldsymbol{\theta}}$ gives $\Delta\mathbf{v}_\theta$ an inward radial component $-v_\theta\,\Delta\theta\hat{\mathbf{r}}$. This contributes

$$\lim_{\Delta t\to 0}\left(-v_\theta\,\frac{\Delta\theta}{\Delta t}\,\hat{\mathbf{r}}\right) = -v_\theta\dot{\theta}\hat{\mathbf{r}} = -r\dot{\theta}^2\hat{\mathbf{r}},$$

which we recognize as the centripetal acceleration. Finally, the tangential component of $\Delta\mathbf{v}_\theta$ is $\Delta v_\theta\hat{\boldsymbol{\theta}}$. Since $v_\theta = r\dot{\theta}$, there are two ways the tangential speed can change. If $\dot{\theta}$ increases by $\Delta\dot{\theta}$, v_θ increases by $r\,\Delta\dot{\theta}$. Second, if r increases by Δr, v_θ increases by $\Delta r\dot{\theta}$. Hence $\Delta v_\theta = r\,\Delta\dot{\theta} + \Delta r\;\dot{\theta}$, and the contribution to the acceleration is



$$\lim_{\Delta t\to 0}\left(\frac{\Delta v_\theta}{\Delta t}\,\hat{\boldsymbol{\theta}}\right) = \lim_{\Delta t\to 0}\left(r\,\frac{\Delta\dot{\theta}}{\Delta t} + \frac{\Delta r}{\Delta t}\,\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$
$$= (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

The second term is the remaining half of the Coriolis acceleration; we see that this part arises from the change in tangential speed due to the change in radial distance.

Example 1.16 **Acceleration of a Bead on a Spoke**

A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the velocity and acceleration.

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

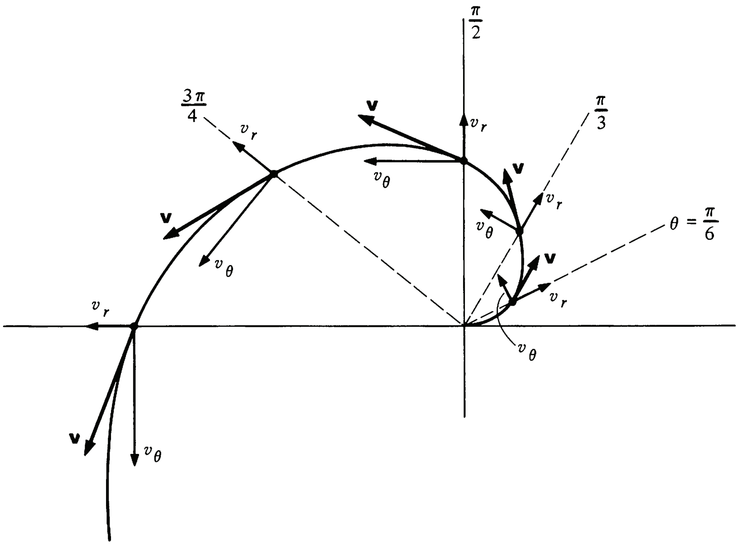
We are given that $\dot{r} = u$ and $\dot{\theta} = \omega$. The radial position is given by $r = ut$, and we have

$$\mathbf{v} = u\hat{\mathbf{r}} + ut\omega\hat{\boldsymbol{\theta}}.$$

The acceleration is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= -ut\omega^2\hat{\mathbf{r}} + 2u\omega\hat{\boldsymbol{\theta}}.\end{aligned}$$

The velocity is shown in the sketch for several different positions of the wheel. Note that the radial velocity is constant. The tangential acceleration is also constant—can you visualize this?



Example 1.17 Radial Motion without Acceleration

A particle moves with $\dot{\theta} = \omega =$ constant and $r = r_0e^{\beta t}$, where r_0 and β are constants. We shall show that for certain values of β, the particle moves with $a_r = 0$.

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= (\beta^2r_0e^{\beta t} - r_0e^{\beta t}\omega^2)\hat{\mathbf{r}} + 2\beta r_0\omega e^{\beta t}\hat{\boldsymbol{\theta}}.\end{aligned}$$

If $\beta = \pm\omega$, the radial part of **a** vanishes.

It is very surprising at first that when $r = r_0e^{\beta t}$ the particle moves with zero radial acceleration. The error is in thinking that $\ddot{r}$ makes the only contribution to a_r; the term $-r\dot{\theta}^2$ is also part of the radial acceleration, and cannot be neglected.

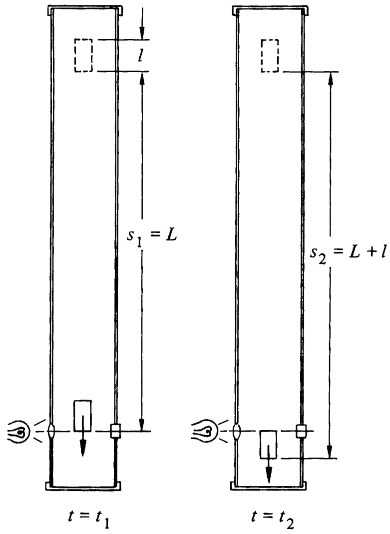
The paradox is that even though $a_r = 0$, the radial velocity $v_r = \dot{r} = r_0\omega e^{\beta t}$ is increasing rapidly with time. The answer is that we can be misled by the special case of cartesian coordinates; in polar coordinates,

$$v_r \neq \int a_r(t)\,dt,$$

because $\int a_r(t)\,dt$ does not take into account the fact that the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are functions of time.

Note 1.1 Mathematical Approximation Methods

Occasionally in the course of solving a problem in physics you may find that you have become so involved with the mathematics that the physics is totally obscured. In such cases, it is worth stepping back for a moment to see if you cannot sidestep the mathematics by using simple approximate expressions instead of exact but complicated formulas. If you have not yet acquired the knack of using approximations, you may feel that there is something essentially wrong with the procedure of substituting inexact results for exact ones. However, this is not really the case, as the following example illustrates.



Suppose that a physicist is studying the free fall of bodies in vacuum, using a tall vertical evacuated tube. The timing apparatus is turned on when the falling body interrupts a thin horizontal ray of light located a distance L below the initial position. By measuring how long the body takes to pass through the light beam, the physicist hopes to determine the local value of g, the acceleration due to gravity. The falling body in the experiment has a height l.

For a freely falling body starting from rest, the distance s traveled in time t is

$$s = \tfrac{1}{2}gt^2,$$

which gives

$$t = \sqrt{\frac{2}{g}}\sqrt{s}.$$

The time interval $t_2 - t_1$ required for the body to fall from $s_1 = L$ centimeters to $s_2 = (L + l)$ centimeters is

$$\begin{aligned} t_2 - t_1 &= \sqrt{\frac{2}{g}}(\sqrt{s_2} - \sqrt{s_1}) \\ &= \sqrt{\frac{2}{g}}(\sqrt{L + l} - \sqrt{L}). \end{aligned}$$

If $t_2 - t_1$ is measured experimentally, g is given by

$$g = 2\left(\frac{\sqrt{L + l} - \sqrt{L}}{(t_2 - t_1)}\right)^2$$

This formula is exact under the stated conditions, but it may not be the most useful expression for our purposes.

Consider the factor

$$\sqrt{L + l} - \sqrt{L}.$$

In practice, L will be large compared with l (typical values might be $L =$ 100 cm, $l = 1$ cm). Our factor is the small difference between two large numbers and is hard to evaluate accurately by using a slide rule or ordinary mathematical tables. Here is a simple approach, known as the method of power series expansion, which enables us to evaluate the factor to any accuracy we please. As we shall discuss formally later in this Note, the quantity $\sqrt{1+x}$ can be written in the series form

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots$$

for $-1 < x < 1$. Furthermore, if we cut off the series at some point, the error we incur by this approximation is of the order of the first neglected term. We can put the factor in a form suitable for expansion by first extracting $\sqrt{L}$:

$$\sqrt{L+l} - \sqrt{L} = \sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right).$$

The dimensionless ratio l/L plays the part of x in our expansion. Expanding $\sqrt{1 + l/L}$ in the series form gives

$$\begin{aligned}\sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right) &= \sqrt{L}\left[1 + \frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \cdots - 1\right] \\ &= \sqrt{L}\left[\frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \cdots\right].\end{aligned}$$

We see that if l/L is much smaller than 1, the successive terms decrease rapidly. The first term in the bracket, $\frac{1}{2}(l/L)$, is the largest term, and extracting it from the bracket yields

$$\begin{aligned}\sqrt{L+l} - \sqrt{L} &= \sqrt{L}\,\frac{1}{2}\left(\frac{l}{L}\right)\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots\right] \\ &= \frac{l}{2\sqrt{L}}\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots\right].\end{aligned}$$

Our expansion is now in its final and most useful form. The first factor, $l/(2\sqrt{L})$, gives the dominant behavior and is a useful first approximation. Furthermore, writing the series as we have, with leading term 1, shows clearly the contributions of the successive powers of l/L. For example, if $l/L = 0.01$, the term $\frac{1}{8}(l/L)^2 = 1.2 \times 10^{-5}$ and we make a fractional error of about 1 part in 10^5 by retaining only the preceding terms. In many cases this accuracy is more than enough. For instance, if the time interval $t_2 - t_1$ in the falling body experiment can be measured to only 1 part in 1,000, we gain nothing by evaluating $\sqrt{L+l} - \sqrt{L}$ to greater accuracy than this. On the other hand, if we require greater accuracy, we can easily tell how many terms of the series should be retained.

Practicing physicists make mathematical approximations freely (when justified) and have no compunctions about discarding negligible terms. The ability to do this often makes the difference between being stymied by impenetrable algebra and arithmetic and successfully solving a problem.

Furthermore, series approximations often allow us to simplify complicated algebraic expressions to bring out the essential physical behavior.

Here are some helpful methods for making mathematical approximations.

1 THE BINOMIAL SERIES

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$$

This series is valid for $-1 < x < 1$, and for any value of n. (If n is an integer, the series terminates, the last term being x^n.) The series is exact; the approximation enters when we truncate it. For $n = \frac{1}{2}$, as in our example,

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots \qquad -1 < x < 1.$$

If we need accuracy only to $O(x^2)$ (order of x^2), we have

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + O(x^3),$$

where the term $O(x^3)$ indicates that terms of order x^3 and higher are not being considered. As a rule of thumb, the error is approximately the size of the first term dropped.

The series can also be applied if $|x| > 1$ as follows:

$$\begin{aligned}(1+x)^n &= x^n\left(1+\frac{1}{x}\right)^n \\ &= x^n\left[1 + n\frac{1}{x} + \frac{n(n-1)}{2!}\left(\frac{1}{x}\right)^2 + \cdots\right].\end{aligned}$$

Examples:

1. $$\begin{aligned}\frac{1}{1+x} &= (1+x)^{-1} \\ &= 1 - x + x^2 - x^3 + \cdots \qquad -1 < x < 1\end{aligned}$$
2. $$\begin{aligned}\frac{1}{1-x} &= (1-x)^{-1} \\ &= 1 + x + x^2 + x^3 + \cdots \qquad -1 < x < 1\end{aligned}$$
3. $$\begin{aligned}(1{,}001)^{\frac{1}{3}} &= (1{,}000 + 1)^{\frac{1}{3}} = 1{,}000^{\frac{1}{3}}(1 + 0.001)^{\frac{1}{3}} \\ &= 10[1 + 0.001(\tfrac{1}{3}) + \cdots] \\ &\approx 10(1.0003) = 10.003\end{aligned}$$
4. $2 - \dfrac{1}{\sqrt{1+x}} - \dfrac{1}{\sqrt{1-x}}$: for small x, this expression is zero to first

approximation. However, this approximation may not be adequate. Using the binomial series, we have

$$2 - \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}} = 2 - (1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots) - (1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots)$$
$$= -\tfrac{3}{4}x^2.$$

Notice that the terms linear in x also cancel. To obtain a nonvanishing result we had to go to a high enough order, in this case to order x^2. It is clear that for a correct result we have to expand all terms to the same order.

2 TAYLOR'S SERIES[1]

Analogous to the binomial series, we can try to represent an arbitrary function $f(x)$ by a power series in x:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\infty} a_kx^k.$$

For $x = 0$ we must have

$$f(0) = a_0.$$

Assuming for the moment that it is permissible to differentiate, we have

$$\frac{df}{dx} = f'(x) = a_1 + 2a_2x + \cdots$$

Evaluating at $x = 0$ we have

$$a_1 = f'(x)\Big|_{x=0}.$$

Continuing this process, we find

$$a_k = \frac{1}{k!} f^{(k)}(x)\Big|_{x=0},$$

where $f^{(k)}(x)$ is the kth derivative of $f(x)$. For the sake of a less cumbersome notation, we often write $f^{(k)}(0)$ to stand for $f^{(k)}(x)\Big|_{x=0}$; but bear in mind that $f^{(k)}(0)$ means that we should differentiate $f(x)$ k times and *then* set x equal to 0.

The power series for $f(x)$, known as a *Taylor series,* can then be expressed formally as

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots.$$

This series, if it converges, allows us to find good approximations to $f(x)$ for small values of x (that is, for values of x near zero). Generalizing,

$$f(a + x) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \cdots$$

[1] Taylor's series is discussed in most elementary calculus texts. See the list at the end of the chapter.

gives us the behavior of the function in the neighborhood of the point a. An alternative form for this expression is

$$f(t) = f(a) + f'(a)(t - a) + f''(a)\frac{(t - a)^2}{2!} + \cdots.$$

Our formal manipulations are valid only if the series converges. The range of convergence of a Taylor series may be $-\infty < x < \infty$ for some functions (such as e^x) but quite limited for other functions. (The binomial series converges only if $-1 < x < 1$.) The range of convergence is hard to find without considering functions of a complex variable, and we shall avoid these questions by simply assuming that we are dealing with simple functions for which the range of convergence is either infinite or is readily apparent. Here are some examples:

a. The Trigonometric Functions

Let $f(x) = \sin x$, and expand about $x = 0$.

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1, \qquad \text{etc.} \end{aligned}$$

Hence

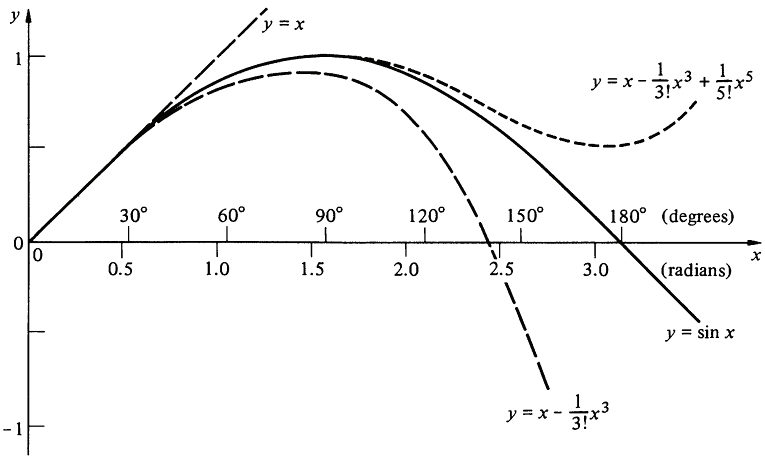
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots.$$

Similarly

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

These expansions converge for all values of x but are particularly useful for small values of x. To $O(x^2)$, $\sin x = x$, $\cos x = 1 - x^2/2$.

The figure below compares the exact value for $\sin x$ with a Taylor series in which successively higher terms are included. Note how each



term increases the range over which the series is accurate. If an infinite number of terms are included, the Taylor series represents the function accurately everywhere.

b. The Binomial Series

We can derive the binomial series introduced in the last section by letting

$$f(x) = (1 + x)^n.$$

Then

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= n(1 + 0)^n = n \\
f''(0) &= n(n - 1) \\
f^{(k)}(0) &= n(n - 1)(n - 2) \cdots (n - k + 1)
\end{aligned}$$

$$(1 + x)^n = 1 + nx + \frac{1}{2!}n(n - 1)x^2 + \cdots + \cdots \frac{n(n - 1) \cdots (n - k + 1)}{k!}x^k + \cdots$$

c. The Exponential Function

If we let $f(x) = e^x$, we have $f'(x) = f(x)$, by the definition of the exponential function. Similarly $f^{(k)}(x) = f(x)$. Since $f(0) = e^0 = 1$, we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

This series converges for all values of x.

A useful result from the theory of the Taylor series is that if the series converges at all, it represents the function so well that we are allowed to differentiate or integrate the series any number of times. For example,

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \\
&= \cos x.
\end{aligned}$$

Furthermore, the Taylor series for the product of two functions is the product of the individual series:

$$\begin{aligned}
\sin x \cos x &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) \\
&= x - \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \left(\frac{1}{4!} + \frac{1}{3!2!} + \frac{1}{5!}\right)x^5 + \cdots
\end{aligned}$$

$$
\begin{aligned}
&= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} + \cdots \\
&= \frac{1}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots\right] \\
&= \frac{1}{2}[\sin(2x)].
\end{aligned}
$$

The Taylor series sometimes comes in handy in the evaluation of integrals. To estimate

$$\int_1^{1.1} \frac{e^z}{z}\,dz,$$

let $z = 1 + x$. We then have

$$
\begin{aligned}
\int_1^{1.1} \frac{e^z}{z}\,dz &= \int_0^{0.1} \frac{e^{(1+x)}}{1+x}\,dx \\
&= (e)\int_0^{0.1} \frac{e^x}{1+x}\,dx \\
&\approx (e)\int_0^{0.1} \frac{(1+x)}{(1+x)}\,dx \\
&\approx 0.1e.
\end{aligned}
$$

The approximation should be better than 1 part in 100 or so, for x always lies in the interval $0 \leq x \leq 0.1$. In this range, $e^x \approx 1 + x$ is a good approximation to two or three significant figures.

3 DIFFERENTIALS

Consider $f(x)$, a function of the independent variable x. Often we need to have a simple approximation for the change in $f(x)$ when x is changed to $x + \Delta x$. Let us denote the change by $\Delta f = f(x + \Delta x) - f(x)$. It is natural to turn to the Taylor series. Expanding the Taylor series for $f(x)$ about the point x gives

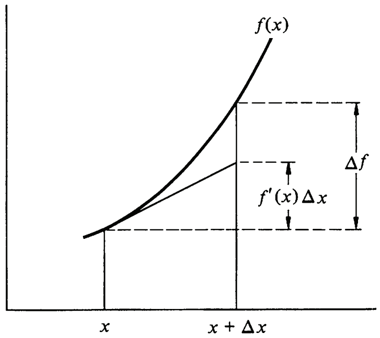
$$f(x + \Delta x) = f(x) + f'(x)\,\Delta x + \frac{1}{2!}f''(x)\,\Delta x^2 + \cdots,$$

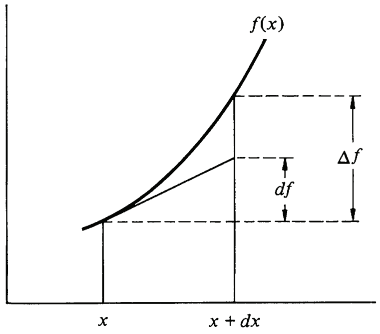
where, for example, $f'(x)$ stands for df/dx evaluated at the point x. Omitting terms of order $(\Delta x)^2$ and higher yields the simple linear approximation

$$\Delta f = f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x.$$

This approximation becomes increasingly accurate the smaller the size of Δx. However, for finite values of Δx, the expression

$$\Delta f \approx f'(x)\,\Delta x$$





has to be considered to be an approximation. The graph at left shows a comparison of $\Delta f \equiv f(x + \Delta x) - f(x)$ with the linear extrapolation $f'(x)\,\Delta x$. It is apparent that Δf, the actual change in $f(x)$ as x is changed, is generally not exactly equal to Δf for finite Δx.

As a matter of notation, we use the symbol dx to stand for Δx, the increment in x. dx is known as the *differential* of x; it can be as large or small as we please. We *define* df, the differential of f, by

$$df \equiv f'(x)\,dx.$$

This notation is illustrated in the lower drawing. Note that dx and Δx are used interchangeably. On the other hand, df and Δf are different quantities. df is a differential defined by $df = f'(x)\,dx$, whereas Δf is the actual change $f(x + dx) - f(x)$. Nevertheless, when the linear approximation is justified in a problem, we often use df to represent Δf. We can always do this when eventually a limit will be taken. Here are some examples.

1. $d(\sin\theta) = \cos\theta\, d\theta$.
2. $d(xe^{x^2}) = (e^{x^2} + 2x^2e^{x^2})\,dx$.
3. Let V be the volume of a sphere of radius r:

$$V = \tfrac{4}{3}\pi r^3$$
$$dV = 4\pi r^2\,dr.$$

4. What is the fractional increase in the volume of the earth if its average radius, 6.4×10^6 m, increases by 1 m?

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3}$$
$$= 3\frac{dr}{r}$$
$$= \frac{3}{6.4 \times 10^6} = 4.7 \times 10^{-7}.$$

One common use of differentials is in changing the variable of integration. For instance, consider the integral

$$\int_a^b xe^{x^2}\,dx.$$

A useful substitution is $t = x^2$. The procedure is first to solve for x in terms of t,

$$x = \sqrt{t},$$

and then to take differentials:

$$dx = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt.$$

This result is exact, since we are effectively taking the limit. The original integral can now be written in terms of t:

$$\int_a^b xe^{x^2}\,dx = \int_{t_1}^{t_2} \sqrt{t}\, e^t \left(\frac{1}{2}\frac{1}{\sqrt{t}}\,dt\right) = \tfrac{1}{2}\int_{t_1}^{t_2} e^t\,dt$$

$$= \tfrac{1}{2}(e^{t_2} - e^{t_1}),$$

where $t_1 = a^2$ and $t_2 = b^2$.

Some References to Calculus Texts

A very popular textbook is G. B. Thomas, Jr., "Calculus and Analytic Geometry," 4th ed., Addison-Wesley Publishing Company, Inc., Reading, Mass.

The following introductory texts in calculus are also widely used:
M. H. Protter and C. B. Morrey, "Calculus with Analytic Geometry," Addison-Wesley Publishing Company, Inc., Reading, Mass.
A. E. Taylor, "Calculus with Analytic Geometry," Prentice-Hall, Inc., Englewood Cliffs, N.J.
R. E. Johnson and E. L. Keokemeister, "Calculus With Analytic Geometry," Allyn and Bacon, Inc., Boston.

A highly regarded advanced calculus text is R. Courant, "Differential and Integral Calculus," Interscience Publishing, Inc., New York.

If you need to review calculus, you may find the following helpful: Daniel Kleppner and Norman Ramsey, "Quick Calculus," John Wiley & Sons, Inc., New York.

Problems

1.1 Given two vectors, $\mathbf{A} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}})$ and $\mathbf{B} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$, find: (*a*) $\mathbf{A} + \mathbf{B}$; (*b*) $\mathbf{A} - \mathbf{B}$; (*c*) $\mathbf{A} \cdot \mathbf{B}$; (*d*) $\mathbf{A} \times \mathbf{B}$.

Ans. (*a*) $7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$; (*c*) 21

1.2 Find the cosine of the angle between

$\mathbf{A} = (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\mathbf{B} = (-2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$.

Ans. -0.805

1.3 The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosine of the angles between the vector and the x, y, and z axes are usually called, in turn α, β, and γ. Prove that $\alpha^2 + \beta^2 + \gamma^2 = 1$, using either geometry or vector algebra.

1.4 Show that if $|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$, then $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

1.5 Prove that the diagonals of an equilateral parallelogram are per pendicular.

1.6 Prove the law of sines using the cross product. It should only take a couple of lines. (*Hint*: Consider the area of a triangle formed by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$.)

1.7 Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the xy plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$, and using vector algebra prove that

$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$

1.8 Find a unit vector perpendicular to

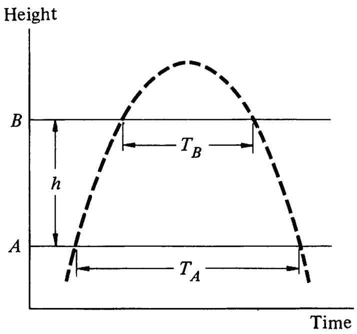
$\mathbf{A} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ and $\mathbf{B} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$.

Ans. $\hat{\mathbf{n}} = \pm(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}})/\sqrt{38}$

1.9 Show that the volume of a parallelepiped with edges **A**, **B**, and **C** is given by $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

1.10 Consider two points located at $\mathbf{r}_1$ and $\mathbf{r}_2$, separated by distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Find a vector **A** from the origin to a point on the line between $\mathbf{r}_1$ and $\mathbf{r}_2$ at distance xr from the point at $\mathbf{r}_1$, where x is some number.

1.11 Let **A** be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$.



1.12 The acceleration of gravity can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions.

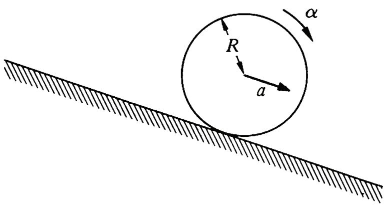
Show that if the time the body takes to pass a horizontal line A in both directions is T_A, and the time to go by a second line B in both directions is T_B, then, assuming that the acceleration is constant, its magnitude is

$$g = \frac{8h}{T_A^2 - T_B^2},$$

where h is the height of line B above line A.

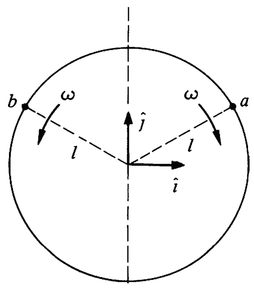
1.13 An elevator ascends from the ground with uniform speed. At time T_1 a boy drops a marble through the floor. The marble falls with uniform acceleration $g = 9.8$ m/s^2, and hits the ground T_2 seconds later. Find the height of the elevator at time T_1.

Ans. clue. If $T_1 = T_2 = 4$ s, $h = 39.2$ m



1.14 A drum of radius R rolls down a slope without slipping. Its axis has acceleration a parallel to the slope. What is the drum's angular acceleration α?

1.15 By *relative velocity* we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.)



a. A point is observed to have velocity $\mathbf{v}_A$ relative to coordinate system A. What is its velocity relative to coordinate system B, which is displaced from system A by distance **R**? (**R** can change in time.)

Ans. $\mathbf{v}_B = \mathbf{v}_A - d\mathbf{R}/dt$

b. Particles a and b move in opposite directions around a circle with angular speed ω, as shown. At $t = 0$ they are both at the point $\mathbf{r} = l\hat{\mathbf{j}}$, where l is the radius of the circle.

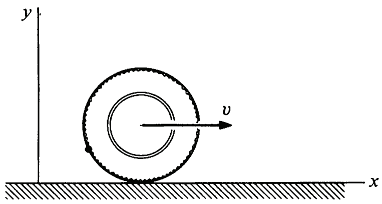
Find the velocity of a relative to b.

1.16 A sportscar, Fiasco I, can accelerate uniformly to 120 mi/h in 30 s. Its *maximum* braking rate cannot exceed $0.7g$. What is the minimum time required to go $\frac{1}{2}$ mi, assuming it begins and ends at rest? (*Hint:* A graph of velocity vs. time can be helpful.)

1.17 A particle moves in a plane with constant radial velocity $\dot{r} = 4$ m/s. The angular velocity is constant and has magnitude $\dot{\theta} = 2$ rad/s. When the particle is 3 m from the origin, find the magnitude of (*a*) the velocity and (*b*) the acceleration.

Ans. (*a*) $v = \sqrt{52}$ m/s

1.18 The rate of change of acceleration is sometimes known as "jerk." Find the direction and magnitude of jerk for a particle moving in a circle of radius R at angular velocity ω. Draw a vector diagram showing the instantaneous position, velocity, acceleration, and jerk.



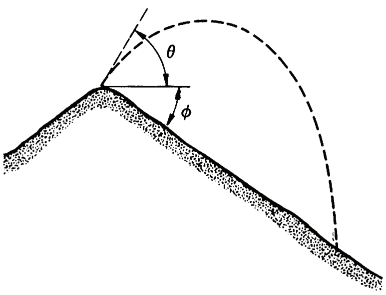
1.19 A tire rolls in a straight line without slipping. Its center moves with constant speed V. A small pebble lodged in the tread of the tire touches the road at $t = 0$. Find the pebble's position, velocity, and acceleration as functions of time.

1.20 A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant. $A = (1/\pi)$ m/rad. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

a. Sketch the motion, and indicate the approximate velocity and acceleration at a few points.

b. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

c. At what angles do the radial and tangential accelerations have equal magnitude?



1.21 A boy stands at the peak of a hill which slopes downward uniformly at angle ϕ. At what angle θ from the horizontal should he throw a rock so that it has the greatest range?

Ans. clue. If $\phi = 60°$, $\theta = 15°$

2 NEWTON'S LAWS–THE FOUNDATIONS OF NEWTONIAN MECHANICS

2.1 Introduction

Our aim in this chapter is to understand Newton's laws of motion. From one point of view this is a modest task: Newton's laws are simple to state and involve little mathematical complexity. Their simplicity is deceptive, however. As we shall see, they combine definitions, observations from nature, partly intuitive concepts, and some unexamined assumptions on the properties of space and time. Newton's statement of the laws of motion left many of these points unclear. It was not until two hundred years after Newton that the foundations of classical mechanics were carefully examined, principally by Ernst Mach,[1] and our treatment is very much in the spirit of Mach.

Newton's laws of motion are by no means self-evident. In Aristotle's system of mechanics, a force was thought to be needed to maintain a body in uniform motion. Aristotelian mechanics was accepted for thousands of years because, superficially, it seemed intuitively correct. Careful reasoning from observation and a real effort of thought was needed to break out of the aristotelian mold. Most of us are still not accustomed to thinking in newtonian terms, and it takes both effort and practice to learn to analyze situations from the newtonian point of view. We shall spend a good deal of time in this chapter looking at applications of Newton's laws, for only in this way can we really come to understand them. However, in addition to deepening our understanding of dynamics, there is an immediate reward—we shall be able to analyze quantitatively physical phenomena which at first sight may seem incomprehensible.

Although Newton's laws provide a direct introduction to classical mechanics, it should be pointed out that there are a number of other approaches. Among these are the formulations of Lagrange and Hamilton, which take energy rather than force as the fundamental concept. However, these methods are physically equivalent to the newtonian approach, and even though we could use one of them as our point of departure, a deep understanding of Newton's laws is an invaluable asset to understanding any systematic treatment of mechanics.

A word about the validity of newtonian mechanics: possibly you already know something about modern physics—the development early in this century of relativity and quantum mechanics. If so,

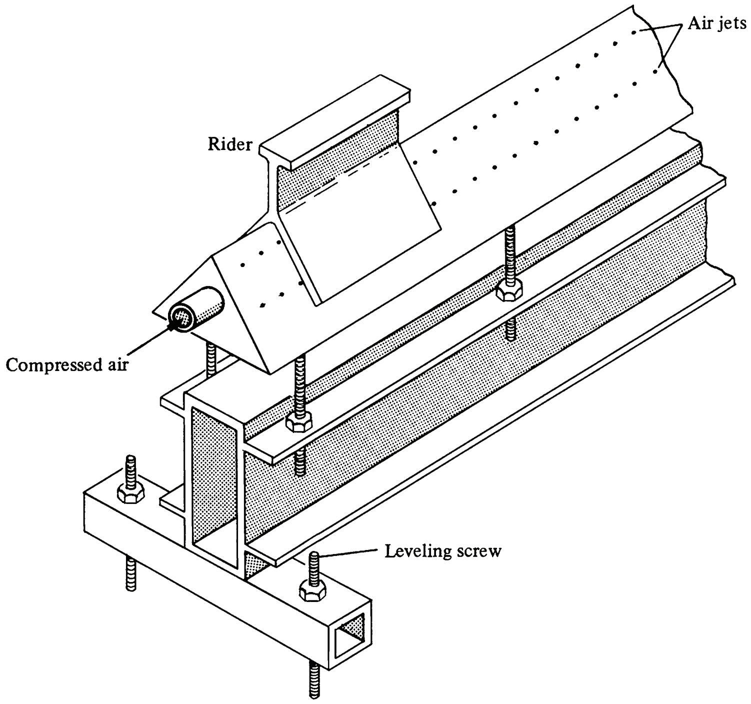
[1] Mach's text, "The Science of Mechanics" (1883), translated the arguments from Newton's "Principia" into a more logically satisfying form. His analysis of the assumptions of newtonian mechanics played a major role in the development of Einstein's special theory of relativity, as we shall see in Chap. 10.

you know that there are important areas of physics in which newtonian mechanics fails, while relativity and quantum mechanics succeed. Briefly, newtonian mechanics breaks down for systems moving with a speed comparable to the speed of light, 3×10^8 m/s, and it also fails for systems of atomic dimensions or smaller where quantum effects are significant. The failure arises because of inadequacies in classical concepts of space, time, and the nature of measurement. A natural impulse might be to throw out classical physics and proceed directly to modern physics. We do not accept this point of view for several reasons. In the first place, although the more advanced theories have shown us where classical physics breaks down, they also show us where the simpler methods of classical physics give accurate results. Rather than make a blanket statement that classical physics is right or wrong, we recognize that newtonian mechanics is exceptionally useful in many areas of physics but of limited applicability in other areas. For instance, newtonian physics enables us to predict eclipses centuries in advance, but is useless for predicting the motions of electrons in atoms. It should also be recognized that because classical physics explains so many everyday phenomena, it is an essential tool for all practicing scientists and engineers. Furthermore, most of the important concepts of classical physics are preserved in modern physics, albeit in altered form.

2.2 Newton's Laws

It is important to understand which parts of Newton's laws are based on experiment and which parts are matters of definition. In discussing the laws we must also learn how to apply them, not only because this is the bread and butter of physics but also because this is essential for a real understanding of the underlying concepts.

We start by appealing directly to experiment. Unfortunately, experiments in mechanics are among the hardest in physics because motion in our everyday surroundings is complicated by forces such as gravity and friction. To see the physical essentials, we would like to eliminate all disturbances and examine very simple systems. One way to accomplish this would be to enroll as astronauts, for in the environment of space most of the everyday disturbances are negligible. However, lacking the resources to put ourselves in orbit, we settle for second best, a device known as a *linear air track,* which approximates ideal conditions, but only in one dimension. (Although it is not clear that we can learn anything about three dimensional motion from studying motion in one dimension, happily this turns out to be the case.)



Linear air track

The linear air track is a hollow triangular beam perhaps 2 m long, pierced by many small holes which emit gentle streams of air. A rider rests on the beam, and when the air is turned on, the rider floats on a thin cushion of air. Because of the air suspension, the rider moves with negligible friction. (The reason for this is that the thin film of air has a viscosity typically 5,000 times less than a film of oil.) If the track is leveled carefully, and if we eliminate stray air currents, the rider behaves as if it were isolated in its motion along the track. The rider moves along the track free of gravity, friction, or any other detectable influences.

Now let's observe how the rider behaves. (Try these experiments yourself if possible.) Suppose that we place the rider on the track and carefully release it from rest. As we might expect, the rider stays at rest, at least until a draft hits it or somebody bumps the apparatus. (This isn't too surprising, since we leveled the track until the rider stayed put when left at rest.) Next, we give the rider a slight shove and then let it move freely. The motion seems uncanny, for the rider continues to move along slowly and evenly, neither gaining nor losing speed. This is contrary to our everyday experience that moving bodies stop moving unless we push them. The reason is that in everyday motion, friction usually plays an important role. For instance, the air track rider comes to a grinding halt if we turn off the air and let sliding friction act. Apparently the friction stops the motion. But we are getting ahead of ourselves; let us return to the properly functioning air track and try to generalize from our experience.

It is possible to make a two dimensional air table analogous to the one dimensional air track. (A smooth sheet of glass with a flat piece of dry ice on it does pretty well. The evaporating dry ice provides the gas cushion.) We find again that the undisturbed rider moves with uniform velocity. Three dimensional isolated motion is hard to observe, short of going into space, but let us for the moment assume that our experience in one and two dimensions also holds in three dimensions. We therefore surmise that an object moves uniformly in space provided there are no external influences.

Newton's First Law

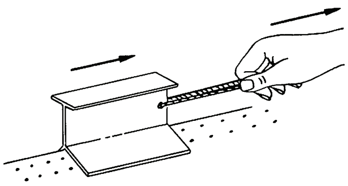
In our discussion of the air track experiments, we glossed over an important point. Motion has meaning only with respect to a particular coordinate system, and in describing motion it is essential to specify the coordinate system we are using. For example, in describing motion along the air track, we implicitly used a coordinate system fixed to the track. However, we are free to choose any coordinate system we please, including systems which are moving with respect to the track. In a coordinate system moving uniformly with respect to the track, the undisturbed rider moves with constant velocity. Such a coordinate system is called an *inertial system*. Not all coordinate systems are inertial; in a coordinate system accelerating with respect to the track, the undisturbed rider does not have constant velocity. However, it is always possible to find a coordinate system with respect to which isolated bodies move uniformly. This is the essence of Newton's first law of motion.

Newton's first law of motion is the assertion that inertial systems exist.

Newton's first law is part definition and part experimental fact. Isolated bodies move uniformly in inertial systems by virtue of the definition of an inertial system. In constrast, that inertial systems exist is a statement about the physical world.

Newton's first law raises a number of questions, such as what we mean by an "isolated body," but we will defer these temporarily and go on.

Newton's Second Law



We now turn to how the rider on the air track behaves when it is no longer isolated. Suppose that we pull the rider with a rubber band. Nothing happens while the rubber band is loose, but as soon as we pull hard enough to stretch the rubber band, the rider starts to move. If we move our hand ahead of the rider so that the rubber band is always stretched to the same standard length, we find that the rider moves in a wonderfully simple way; its velocity increases uniformly with time. The rider moves with constant acceleration.

Now suppose that we try the same experiment with a different rider, perhaps one a good deal larger than the first. Again, the same rubber band stretched to the standard length produces a constant acceleration, but the acceleration is different from that in the first case. Apparently the acceleration depends not only on what we do to the object, since presumably we do the same thing in each case, but also on some property of the object, which we call *mass*.

We can use our rubber band experiment to *define* what we mean by mass. We start by arbitrarily saying that the first body has a mass m_1. (m_1 could be one unit of mass or x units of mass, where x is any number we choose.) We then *define* the mass of the second body to be

$$m_2 = m_1 \frac{a_1}{a_2},$$

where a_1 is the acceleration of the first body in our rubber band experiment and a_2 is the acceleration of the second body.

Continuing this procedure, we can assign masses to other objects by measuring their accelerations with the standard stretched rubber band. Thus

$$m_3 = m_1 \frac{a_1}{a_3}$$

$$m_4 = m_1 \frac{a_1}{a_4} \qquad \text{etc.}$$

Although this procedure is straightforward, there is no obvious reason why the quantity we define this way is particularly important. For instance, why not consider instead some other property, call it property Z, such that $Z_2 = Z_1(a_1/a_2)^2$? The reason is that mass is useful, whereas property Z (or most other quantities you try) is not. By making further experiments with the air track, for instance by using springs or magnets instead of a rubber band, we find that the ratios of accelerations, hence the mass ratios, are the same no matter how we produce the uniform accelerations, provided that we do the same thing to each body. Thus, mass so defined turns out to be independent of the source of acceleration and appears to be an inherent property of a body. Of course, the actual mass value of an individual body depends on our choice of mass unit. The important thing is that two bodies have a unique mass ratio.

Our definition of mass is an example of an *operational* definition. By operational we mean that the definition is dominantly in terms of experiments we perform and not in terms of abstract concepts, such as "mass is a measure of the resistance of bodies to a change in motion." Of course, there can be many abstract concepts hidden in apparently simple operations. For instance, when we measure acceleration, we tacitly assume that we have a clear understanding of distance and time. Although our intuitive ideas are adequate for our purposes here, we shall see when we discuss relativity that the behavior of measuring rods and clocks is itself a matter for experiment.

A second troublesome aspect of operational definitions is that they are limited to situations in which the operations can actually be performed. In practice this is usually not a problem; physics proceeds by constructing a chain of theory and experiment which allows us to employ convenient methods of measurement ultimately based on the operational definitions. For instance, the most practical way to measure the mass of a mountain is to observe its gravitational pull on a test body, such as a hanging plumb bob. According to the operational definition, we should apply a standard force and measure the mountain's acceleration. Nevertheless, the two methods are directly related conceptually.

We defined mass by experiments on laboratory obiects; we cannot say a priori whether the results are consistent on a much larger or smaller scale. In fact, one of the major goals of physics is to find the limitations of such definitions, for the limitations normally reveal new physical laws. Nevertheless, if an operational definition is to be at all useful, it must have very wide applicability. For instance, our definition of mass holds not only for everyday objects on the earth but also, to a very high degree, for planetary motion, motion on an enormously larger scale. It should not surprise us, however, if eventually we find situations in which the operations are no longer useful.

Now that we have defined mass, let us turn our attention to *force*.

We describe the operation of acting on the test mass with a stretched rubber band as "applying" a force. (Note that we have sidestepped the question of what a force is and have limited ourselves to describing how to produce it—namely, by stretching a rubber band by a given amount.) When we apply the force, the test mass accelerates at some rate, a. If we apply two standard stretched rubber bands, side by side, we find that the mass accelerates at the rate $2a$, and if we apply them in opposite directions, the acceleration is zero. The effects of the rubber bands add algebraically for the case of motion in a straight line.

We can establish a force scale by defining the unit force as the force which produces unit acceleration when applied to the unit mass. It follows from our experiments that F units of force accelerate the unit mass by F units of acceleration and, from our definition of mass, it will produce $F \times (1/m)$ units of acceleration in mass m. Hence, the acceleration produced by force F acting on mass m is $a = F/m$ or, in a more familiar order, $F = ma$. In the International System of units (SI), the unit of force is the *newton* (N), the unit of mass is the *kilogram* (kg), and acceleration is in meters per second^2 (m/s^2). Units are discussed further in Sec. 2.3.

So far we have limited our experiments to one dimension. Since acceleration is a vector, and mass, as far as we know, is a scalar, we expect that force is also a vector. It is natural to think of the force as pointing in the direction of the acceleration it produces when acting alone. This assumption appears trivial, but it is not—its justification lies in experiment. We find that forces obey the *principle of superposition:* The acceleration produced by several forces acting on a body is equal to the vector sum of the accelerations produced by each of the forces acting separately. Not only does this confirm the vector nature of force, but it also enables us to analyze problems by considering one force at a time.

Combining all these observations, we conclude that the total force $\mathbf{F}$ on a body of mass m is $\mathbf{F} = \Sigma\mathbf{F}_i$, where $\mathbf{F}_i$ is the ith applied force. If $\mathbf{a}$ is the net acceleration, and $\mathbf{a}_i$ the acceleration due to $\mathbf{F}_i$ alone, then we have

$$\begin{aligned}\mathbf{F} &= \Sigma\mathbf{F}_i\\ &= \Sigma m\mathbf{a}_i\\ &= m\Sigma\mathbf{a}_i\\ &= m\mathbf{a}\end{aligned}$$

or

$$\mathbf{F} = m\mathbf{a}.$$

This is Newton's second law of motion. It will underlie much of our subsequent discussion.

It is important to understand clearly that force is not merely a matter of definition. For instance, if the air track rider starts accelerating, it is not sufficient to claim that there is a force acting defined by $\mathbf{F} = m\mathbf{a}$. Forces always arise from *interactions* between systems, and if we ever found an acceleration without an interaction, we would be in a terrible mess. It is the interaction which is physically significant and which is responsible for the force. For this reason, when we isolate a body sufficiently from its surroundings, we expect the body to move uniformly in an inertial system. Isolation means eliminating interactions. You may question whether it is always possible to isolate a body. Fortunately, as far as we know, the answer is yes. All known interactions decrease with distance. (The forces which extend over the greatest distance are the familiar gravitational and Coulomb forces. They decrease as $1/r^2$, where r is the distance. Most forces decrease much more rapidly. For example, the force between separated atoms decreases as $1/r^7$.) By moving the test body sufficiently far from everything else, the interactions can be reduced as much as desired.

Newton's Third Law

The fact that force is necessarily the result of an interaction between two systems is made explicit by Newton's third law. The third law states that forces always appear in pairs: if body b exerts force $\mathbf{F}_a$ on body a, then there must be a force $\mathbf{F}_b$ acting on body b, due to body a, such that $\mathbf{F}_b = -\mathbf{F}_a$. There is no such thing as a lone force without a partner. As we shall see in the next chapter, the third law leads directly to the powerful law of conservation of momentum.

We have argued that a body can be isolated by removing it sufficiently far from other bodies. However, the following problem arises. Suppose that an isolated body starts to accelerate in defiance of Newton's second law. What prevents us from explaining away the difficulty by attributing the acceleration to carelessness in isolating the system? If this option is open to us, Newton's second law becomes meaningless. We need an independent way of telling whether or not there is a physical interaction on a system. Newton's third law provides such a test. If the acceleration of a body is the result of an outside force, then somewhere in the universe there must be an equal and opposite force acting on another body. If we find such a force, the dilemma is resolved; the body was not completely isolated. The interaction may be new and interesting, but as long as the forces are equal and opposite, Newton's laws are satisfied.

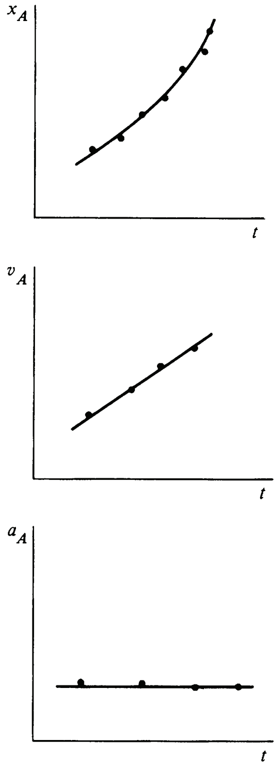
If an isolated body accelerates and we cannot find some external object which suffers an equal and opposite force, then we are in trouble. As far as we know this has never occurred. Thus Newton's third law is not only a vitally important dynamical tool, but it is also an important logical element in making sense of the first two laws.

Newton's second law $\mathbf{F} = m\mathbf{a}$ holds true only in inertial systems. The existence of inertial systems seems almost trivial to us, since the earth provides a reasonably good inertial reference frame for everyday observations. However, there is nothing trivial about the concept of an inertial system, as the following example shows.

Example 2.1 Astronauts in Space—Inertial Systems and Fictitious Forces

Two spaceships are moving in empty space chasing an unidentified flying object, possibly a flying saucer. The captains of the two ships, A and B, must find out if the saucer is flying freely or if it is accelerating. A, B, and the saucer are all moving along a straight line.

The captain of A sets to work and measures the distance to the saucer as a function of time. In principle, he sets up a coordinate system along the line of motion with his ship as origin and notes the position of the saucer, which he calls $x_A(t)$. (In practice he uses his radar set to measure the distance to the saucer.) From $x_A(t)$ he calculates the velocity



$v_A = \dot{x}_A$ and the acceleration $a_A = \ddot{x}_A$. The results are shown in the sketches. The captain of A concludes that the saucer has a positive acceleration $a_A = 1{,}000$ m/s^2. He therefore assumes that its engines are on and that the force on the saucer is

$$\begin{aligned} F_A &= a_A M \\ &= 1{,}000M \text{ newtons,} \end{aligned}$$

where M is the saucer's mass in kilograms.

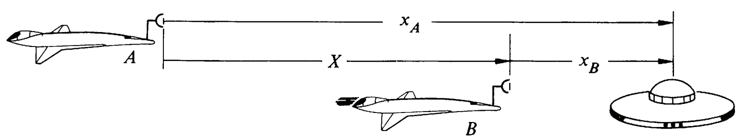
The captain of B goes through the same procedure. He finds that the acceleration is $a_B = 950$ m/s^2 and concludes that the force on the saucer is

$$\begin{aligned} F_B &= a_B M \\ &= 950M \text{ newtons.} \end{aligned}$$

This presents a serious problem. There is nothing arbitrary about force; if different observers obtain different values for the force, at least one of them must be mistaken. The captains of A and B have confidence in the laws of mechanics, so they set about resolving the discrepancy. In particular, they recall that Newton's laws hold only in inertial systems. How can they decide whether or not their systems are inertial?

A's captain sets out by checking to see if all his engines are off. Since they are, he suspects that he is not accelerating and that his spaceship defines an inertial system. To check that this is the case, he undertakes a simple but sensitive experiment. He observes that a pencil, carefully released at rest, floats without motion. He concludes that the pencil's acceleration is negligible and that he is in an inertial system. The reasoning is as follows: as long as he holds the pencil it must have the same instantaneous velocity and acceleration as the spaceship. However, there are no forces acting on the pencil after it is released, assuming that we can neglect gravitational or electrical interactions with the spaceship, air currents, etc. The pencil, then, can be presumed to represent an isolated body. If the spaceship is itself accelerating, it will catch up with the pencil—the pencil will appear to accelerate relative to the cabin. Otherwise, the spaceship must itself define an inertial system.

The determination of the force on the saucer by the captain of A must be correct because A is in an inertial system. But what can we say about the observations made by the captain of B? To answer this problem, we look at the relation of x_A and x_B. From the sketch,



$$x_A(t) = x_B(t) + X(t),$$

where $X(t)$ is the position of B relative to A. Differentiating twice with respect to time, we have

$$\ddot{x}_A = \ddot{x}_B + \ddot{X}. \qquad 1$$

Since system A is inertial, Newton's second law for the saucer is

$$F_{\text{tiue}} = M\ddot{x}_A \qquad 2$$

where F_{true} is the true force on the saucer.

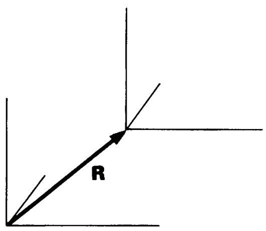
What about the observations made by the captain of B? The apparent force observed by B is

$$F_{B,\text{apparent}} = M\ddot{x}_B. \qquad 3$$

Using the results of (1) and (2), we have

$$\begin{aligned} F_{B,\text{apparent}} &= M\ddot{x}_A - M\ddot{X} \\ &= F_{\text{true}} - M\ddot{X}. \qquad 4 \end{aligned}$$

B will not measure the true force unless $\ddot{X} = 0$. However, $\ddot{X} = 0$ only when B moves uniformly with respect to A. As we suspect, this is not the case here. The captain of B has accidently left on a rocket engine, and he is accelerating away from A at 50 m/s^2. After shutting off the engine, he obtains the same value for the force on the saucer as does A.



Although we considered only motion along a line in Example 2.1, it is easy to generalize the result to three dimensions. If $\mathbf{R}$ is the vector from the origin of an inertial system to the origin of another coordinate system, we have

$$\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} - M\ddot{\mathbf{R}}.$$

If $\ddot{\mathbf{R}} = 0$, then $\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}}$, which means that the second coordinate system is also inertial. In fact, we have merely proven what we asserted earlier, namely, that any system moving uniformly with respect to an inertial system is also inertial.

Sometimes we would like to carry out measurements in noninertial systems. What can we do to get the correct equations of motion? The answer lies in the relation $\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} - M\ddot{\mathbf{R}}$. We can think of the last term as an additional force, which we call a *fictitious force*. (The term fictitious indicates that there is no real interaction involved.) We then write

$$\mathbf{F}_{\text{apparent}} = \mathbf{F}_{\text{true}} + \mathbf{F}_{\text{fictitious}},$$

where $\mathbf{F}_{\text{fictitious}} = -M\ddot{\mathbf{R}}$. Here M is the mass of the particle and $\ddot{\mathbf{R}}$ is the acceleration of the noninertial system with respect to any inertial system.

Fictitious forces are useful in solving certain problems, but they must be treated with care. They generally cause more confusion then they are worth at this stage of your studies, and for that reason we shall avoid them for the present and agree to use inertial systems only. Later on, in Chap. 8, we shall examine fictitious forces in detail and learn how to deal with them.

Although Newton's laws can be stated in a reasonably clear and consistent fashion, it should be realized that there are fundamental difficulties which cannot be argued away. We shall return to these in later chapters after we have had a chance to become better acquainted with the concepts of newtonian physics. Some points, however, are well to bear in mind now.

1. You have had to take our word that the experiments we used to define mass and to develop the second law of motion really give the results claimed. It should come as no surprise (although it was a considerable shock when it was first discovered) that this is not always so. For instance, the mass scale we have set up is no longer consistent when the particles are moving at high speeds. It turns out that instead of the mass we defined, called the rest mass m_0, a more useful quantity is $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the speed of light and v is the speed of the particle. For the case $v \ll c$, m and m_0 differ negligibly. The reason that our tabletop experiments did not lead us to the more general expression for mass is that even for the largest everyday velocities, say the velocity of a spacecraft going around the earth, $v/c \approx 3 \times 10^{-5}$, and m and m_0 differ by only a few parts in 10^{10}.

2. Newton's laws describe the behavior of point masses. In the case where the size of the body is small compared with the interaction distance, this offers no problem. For instance, the earth and sun are so small compared with the distance between them that for many purposes their motion can be adequately described by considering the motion of point masses located at the center of each. However, the approximation that we are dealing with point masses is fortunately not essential, and if we wish to describe the motion of large bodies, we can readily generalize Newton's laws, as we shall do in the next chapter. It turns out to be not much more difficult to discuss the motion of a rigid body composed of 10^{24} atoms than the motion of a single point mass.

3. Newton's laws deal with particles and are poorly suited for describing a continuous system such as a fluid. We cannot directly apply $\mathbf{F} = m\mathbf{a}$ to a fluid, for both the force and the mass are continuously distributed. However, newtonian mechanics can be extended to deal with fluids and provides the underlying principles of fluid mechanics.

One system which is particularly troublesome for our present formulation of newtonian mechanics is the electromagnetic field. Paradoxes can arise when such a field is present. For instance, two charged bodies which interact electrically actually interact via the electric fields they create. The interaction is not instantaneously transmitted from one particle to the other but propagates at the velocity of light. During the propagation time there is an apparent breakdown of Newton's third law; the forces on the particles are not equal and opposite. Similar problems arise in considering gravitational and other interactions. However, the problem lies not so much with newtonian mechanics as with its misapplication. Simply put, fields possess mechanical properties like momentum and energy which must not be overlooked. From this point of view there is no such thing as a simple two particle system. However, for many systems the fields can be taken into account and the paradoxes can be resolved within the newtonian framework.

2.3 Standards and Units

Length, time, and mass play a fundamental role in every branch of physics. These quantities are defined in terms of certain fundamental physical standards which are agreed to by the scientific community. Since a particular standard generally does not have a convenient size for every application, a number of systems of units have come into use. For example, the centimeter, the angstrom, and the yard are all units of length, but each is defined in terms of the standard meter. There are a number of systems of units in widespread use, the choice being chiefly a matter of custom and convenience. This section presents a brief description of the current standards and summarizes the units which we shall encounter.

The Fundamental Standards

The fundamental standards play two vital roles. In the first place, the precision with which these standards can be defined and reproduced limits the ultimate accuracy of experiments. In some cases the precision is almost unbelievably high—time, for instance, can be measured to a few parts in 10^{12}. In addition, agreeing to a standard for a physical quantity simultaneously provides an operational definition for that quantity. For example, the modern view is that time is what is measured by clocks, and that the properties of time can be understood only by observing the properties of clocks. This is not a trivial point; the rates of all clocks are affected by motion and by gravity (as we shall discuss in Chaps. 8 and 12), and unless we are willing to accept the fact that time itself is altered by motion and gravity, we are led into contradictions.

Once a physical quantity has been defined in terms of a measurement procedure, we must appeal to experiment, not to preconceived notions, to understand its properties. To contrast this viewpoint with a nonoperational approach, consider, for example, Newton's definition of time: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equally without relation to anything external." This may be intuitively and philosophically appealing, but it is hard to see how such a definition can be applied. The idea is metaphysical and not of much use in physics.

Once we have agreed on the operation underlying a particular physical quantity, the problem is to construct the most precise practical standard. Until recently, physical standards were man-made, in the sense that they consisted of particular objects to which all other measurements had to be referred. Thus, the unit length, the meter, was defined to be the distance between two scratches on a platinum bar. Such man-made standards have a number of disadvantages. Since the standard must be carefully preserved, actual measurements are often done with secondary standards, which causes a loss of accuracy. Furthermore, the precision of a man-made standard is intrinsically limited. In the case of the standard meter, precision was found to be limited by fuzziness in the engraved lines which defined the meter interval. When more accurate optical techniques for locating position were developed in the latter part of the nineteenth century, it was realized that the standard meter bar was no longer adequate.

Length is now defined by a natural, rather than man-made, standard. The meter is defined to be a given multiple of the wavelength of a particular spectral line. The advantage of such a unit is that anyone who has the required optical equipment can reproduce it. Also, as the instrumentation improves, the accuracy of the standard will correspondingly increase. Most of the standards of physics are now natural.

Here is a brief account of the current status of the standards of length, time, and mass.

Length The meter was intended to be one ten-millionth of the distance from the equator to the pole of the earth along the Dunkirk-Barcelona line. This cannot be measured accurately (in fact it changes due to distortions of the earth), and in 1889 it was agreed to define the meter as the separation between two scratches in a platinum-iridium bar which is preserved at the International Bureau of Weights and Measures, Sèvres, France. In 1960 the meter was redefined to be 1,650,763.73 wavelengths of the orange-red line of krypton 86. The accuracy of this standard is a few parts in 10^8.

Recent advances in laser techniques provide methods which should allow the velocity of light to be measured to better than 1 part in 10^8. It is likely that the velocity of light will replace length as a fundamental quantity. In this case the unit of length would be derived from velocity and time.

Time Time has traditionally been measured in terms of rotation of the earth. Until 1956 the basic unit, the second, was defined as 1/86,400 of the mean solar day. Unfortunately, the period of rotation of the earth is not very uniform. Variations of up to one part in 10^7 per day occur due to atmospheric tides and changes in the earth's core. The motion of the earth around the sun is not influenced by these perturbations, and until recently the mean solar year was used to define the second. Here the accuracy was a few parts in 10^9. Fortunately, time can now be measured in terms of a natural atomic frequency. In 1967 the second was defined to be the time required to execute 9,192,631,770 cycles of a hyperfine transition in cesium 133. This transition frequency can be reliably measured to a few parts in 10^{12}, which means that time is by far the most accurately determined fundamental quantity.

Mass Of the three fundamental units, only mass is defined in terms of a man-made standard. Originally, the kilogram was defined to be the mass of 1,000 cubic centimeters of water at a temperature of 4 degrees Centigrade. The definition is difficult to apply, and in 1889 the kilogram was defined to be the mass of a platinum-iridium cylinder which is maintained at the International Bureau of Weights and Measures. Secondary standards can be compared with it to an accuracy of one part in 10^9. Perhaps someday we will learn how to define the kilogram in terms of a natural unit, such as the mass of an atom. However, at present nobody knows how to count reliably the large number of atoms needed to constitute a useful sample. Perhaps you can discover a method.

Systems of Units

Although the standards for mass, length, and time are accepted by the entire scientific community, there are a variety of systems of units which differ in the scaling factors. The most widely used system of units is the International System, abbreviated SI (for Système International d'Unités). It is the legal system in most countries. The SI units are *meter*, *kilogram*, and *second;* SI replaces the former mks system. The related cgs system, based on the centimeter, gram, and second, is also commonly used. A third system, the English system of units, is used for non-scientific measurements in Britain and North America, although Britain is in the process of switching to the metric system. It is essential to know how to work problems in any system of units. We shall work chiefly with SI units, with occasional use of the cgs system and one or two lapses into the English system.

Here is a table listing the names of units in the SI, cgs, and English systems.

	SI	CGS	ENGLISH
Length	1 meter (m)	1 centimeter (cm)	1 foot (ft)
Mass	1 kilogram (kg)	1 gram (g)	1 slug
Time	1 second (s)	1 second (s)	1 second (s)
Acceleration	1 m/s^2	1 cm/s^2	1 ft/s^2
Force	1 newton (N) $= 1\ kg{\cdot}m/s^2$	1 dyne $= 1\ g{\cdot}cm/s^2$	1 pound (lb) $= 1\ slug{\cdot}ft/s^2$

Some useful relations between these units systems are:

$1\ m = 100\ cm$	$1\ in = \frac{1}{12}\ ft \approx 2.54\ cm$
$1\ kg = 1000\ g$	$1\ slug \approx 14.6\ kg$
$1\ N = 10^5\ dyne$	$1\ N \approx 0.224\ lb$

The word pound sometimes refers to a unit of mass. In this context it stands for the mass which experiences a gravitational force of one pound at the surface of the earth, approximately 0.454 kg. We shall avoid this confusing usage.

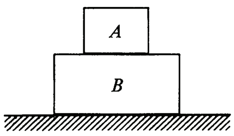
2.4 Some Applications of Newton's Laws

Newton's laws are meaningless equations until we know how to apply them. A number of steps are involved which, once learned, are so natural that the procedure becomes intuitive. Our aim in this section is to outline a method of analyzing physical problems and to illustrate it by examples. A note of reassurance lest you feel that matters are presented too dogmatically: There are many ways of attacking most problems, and the procedure we suggest is certainly not the only one. In fact, no cut-and-dried procedure can ever substitute for intelligent analytical thinking. However, the systematic method suggested here will be helpful in getting started, and we urge you to master it even if you should later resort to shortcuts or a different approach.

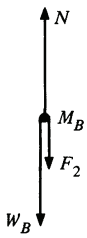
Here are the steps:

1. Mentally divide the system[1] into smaller systems, each of which can be treated as a point mass.
2. Draw a force diagram for each mass as follows:
   a. Represent the body by a point or simple symbol, and label it.
   b. Draw a force vector on the mass for each force acting on it.

Point 2*b* can be tricky. Draw only forces acting *on* the body, not forces exerted *by* the body. The body may be attached to strings, pushed by other bodies, etc. We replace all these physical interactions with other bodies by a system of forces; according to Newton's laws, only forces acting *on* the body influence its motion.



As an example, here are two blocks at rest on a table top. The force diagram for A is shown at left. F_1 is the force exerted on block A by block B, and W_A is the force of gravity on A, called the *weight.*



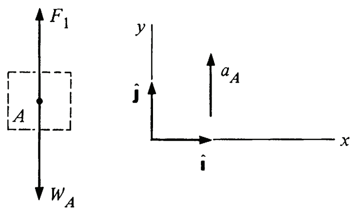
Similarly, we can draw the force diagram for block B. W_B is the force of gravity on B, N is the normal (perpendicular) force exerted by the table top on B, and F_2 is the force exerted by A on B. There are no other physical interactions that would produce a force on B.

It is important not to confuse a force with an acceleration; draw only *real* forces. Since we are using only inertial systems for the present, all the forces are associated with physical interactions. For every force you should be able to answer the question, "What

[1] We use "system" here to mean a collection of physical objects rather than a coordinate system. The meaning should be clear from the context.

exerts this force on the body?" (We shall see how to use so-called fictitious forces in Chap. 8.[1])

3. Introduce a coordinate system. The coordinate system must be inertial—that is, it must be fixed to an inertial frame. With the force diagram as a guide, write separately the component equations of motion for each body. By equation of motion we mean an equation of the form $F_{1x} + F_{2x} + \cdots = Ma_x$, where the x component of each force on the body is represented by a term on the left hand side of the equation. The algebraic sign of each component must be consistent with the force diagram and with the choice of coordinate system.



For instance, returning to the force diagram for block A, Newton's second law gives

$$\mathbf{F}_1 + \mathbf{W}_A = m_A\mathbf{a}_A.$$

Since $\mathbf{F}_1 = F_1\hat{\mathbf{j}}$, $\mathbf{W}_A = -W_A\hat{\mathbf{j}}$, we have

$$0 = m_A(\mathbf{a}_A)_x$$

and

$$F_1 - W_A = m_A(\mathbf{a}_A)_y.$$

The x equation of motion is trivial and normally we omit it, writing simply

$$F_1 - W_A = m_A a_A.$$

The equation of motion for B is

$$N - F_2 - W_B = m_B a_B.$$

4. If two bodies in the same system interact, the forces between them must be equal and opposite by Newton's third law. These relations should be written explicitly.

For example, in the case of the two blocks on the tabletop, $\mathbf{F}_1 = -\mathbf{F}_2$. Hence

$$F_1 = F_2.$$

Note that Newton's third law never relates two forces acting on the *same* body; forces on two different bodies must be involved.

[1] The most notorious fictitious force is the centrifugal force. Long experience has shown that using this force before one has a really solid grasp of Newton's laws invariably causes confusion. Besides, it is only one of several fictitious forces which play a role in rotating systems. For both these reasons, we shall strictly avoid centrifugal forces for the present.

5. In many problems, bodies are constrained to move along certain paths. A pendulum bob, for instance, moves in a circle, and a block sliding on a tabletop is constrained to move in a plane. Each constraint can be described by a kinematical equation known as a *constraint* equation. Write each constraint equation.

Sometimes the constraints are implicit in the statement of the problem. For the two blocks on the tabletop, there is no vertical acceleration, and the constraint equations are

$$(\mathbf{a}_A)_y = 0 \qquad (\mathbf{a}_B)_y = 0.$$

6. Keep track of which variables are known and which are unknown. The force equations and the constraint equations should provide enough relations to allow every unknown to be found. If an equation is overlooked, there will be too few equations for the unknowns.

Completing the problem of the two blocks on the table, we have

$$\left.\begin{aligned} F_1 - W_A &= m_A a_A \\ N - F_2 - W_B &= m_B a_B \end{aligned}\right\}\text{Equations of motion}$$

$$F_1 = F_2 \qquad \text{From Newton's third law}$$

$$\left.\begin{aligned} a_A &= 0 \\ a_B &= 0 \end{aligned}\right\}\text{Constraint equations}$$

All that remains is the mathematical task of solving the equations. We find

$$F_1 = F_2 = W_A$$
$$N = W_A + W_B.$$

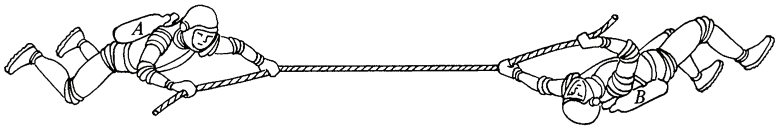
Here are a few examples which illustrate the application of Newton's laws.

The main point of the first example is to help us distinguish between the force we apply to an object and the force it exerts on us. Physiologically, these forces are often confused. If you push a book across a table, the force you feel is not the force that makes the book move; it is the force the book exerts on you. According to Newton's third law, these two forces are always equal and opposite. If one force is limited, so is the other.

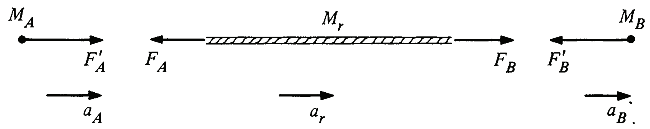
Example 2.2 The Astronauts' Tug-of-war

Two astronauts, initially at rest in free space, pull on either end of a rope. Astronaut Alex played football in high school and is stronger than astronaut Bob, whose hobby was chess. The maximum force with which

Alex can pull, F_A, is larger than the maximum force with which Bob can pull, F_B. Their masses are M_A and M_B, and the mass of the rope, M_r, is negligible. Find their motion if each pulls on the rope as hard as he can.



Here are the force diagrams. For clarity, we show the rope as a line.



Note that the forces F_A and F_B exerted by the astronauts act on the *rope,* not on the astronauts. The forces exerted by the rope on the astronauts are $F_A{}'$ and $F_B{}'$. The diagram shows the directions of the forces and the coordinate system we have adopted; acceleration to the right is positive.

By Newton's third law,

$$\begin{aligned} F'_A &= F_A \\ F'_B &= F_B. \end{aligned} \qquad 1$$

The equation of motion for the rope is

$$F_B - F_A = M_r a_r. \qquad 2$$

Only motion along the line of the rope is of interest, and we omit the equations of motion in the remaining two directions. There are no constraints, and we proceed to the solution.

Since the mass of the rope, M_r, is negligible, we take $M_r = 0$ in Eq. (2). This gives $F_B - F_A = 0$ or

$$F_B = F_A.$$

The total force on the rope is F_B to the right and F_A to the left. These forces are equal in magnitude, and the total force on the rope is zero. In general, the total force on any body of negligible mass must be effectively zero; a finite force acting on zero mass would produce an infinite acceleration.

Since $F_B = F_A$, Eq. (1) gives $F'_A = F_A = F_B = F'_B$. Hence

$$F'_A = F'_B.$$

The astronauts each pull with the *same* force. Physically, there is a limit to how hard Bob can grip the rope; if Alex tries to pull too hard, the rope slips through Bob's fingers. The force Alex can exert is limited by the strength of Bob's grip. If the rope were tied to Bob, Alex could exert his maximum pull.

The accelerations of the two astronauts are

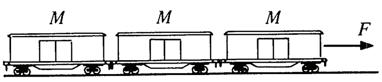
$$a_A = \frac{F'_A}{M_A}$$

$$a_B = \frac{-F'_B}{M_B} = \frac{-F'_A}{M_B}.$$

The negative sign means that a_B is to the left. In many problems the directions of some acceleration or force components are initially unknown. In writing the equations of motion, any choice is valid, provided we are consistent with the convention assumed in the force diagram. If the solution yields a negative sign, the acceleration or force is opposite to the direction assumed.

The next example shows that in order for a compound system to accelerate, there must be a net force on each part of the system.

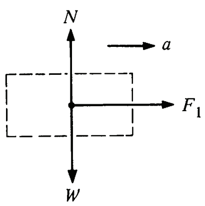
Example 2.3 Freight Train



Three freight cars of mass M are pulled with force F by a locomotive. Friction is negligible. Find the forces on each car.

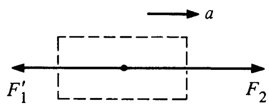
Before drawing the force diagram, it is worth thinking about the system as a whole. Since the cars are joined, they are constrained to have the same acceleration. Since the total mass is $3M$, the acceleration is

$$a = \frac{F}{3M}.$$



A force diagram for the last car is shown at the left. W is the weight and N is the upward force exerted by the track. The vertical acceleration is zero, so that $N = W$. F_1 is the force exerted by the next car. We have

$$F_1 = Ma = M\left(\frac{F}{3M}\right) = \frac{F}{3}.$$

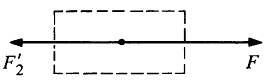


Now let us consider the middle car. The vertical forces are as before, and we omit them. F_1' is the force exerted by the last car, and F_2 is the force exerted by the first car. The equation of motion is

$$F_2 - F_1' = Ma.$$

By Newton's third law, $F_1' = F_1 = F/3$. Since $a = F/3M$, we have

$$\begin{aligned} F_2 &= M\left(\frac{F}{3M}\right) + \frac{F}{3} \\ &= \frac{2F}{3}. \end{aligned}$$

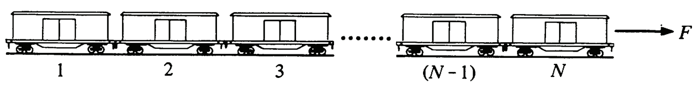


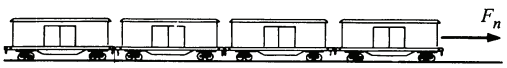
The horizontal forces on the first car are F, to the right, and

$$F_2' = F_2 = \frac{2F}{3},$$

to the left. Each car experiences a total force $F/3$ to the right.

Here is a slightly more general way to look at the problem. Consider a string of N cars, each of mass M, pulled by a force F. The accelera-





tion is $a = F/(NM)$. To find the force F_n pulling the last n cars, note that F_n must give the mass nM an acceleration $F/(NM)$. Hence

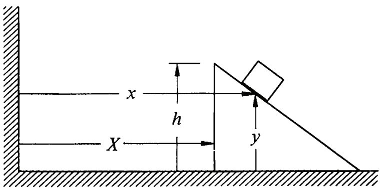
$$\begin{aligned} F_n &= nM\frac{F}{NM} \\ &= \frac{n}{N}F. \end{aligned}$$

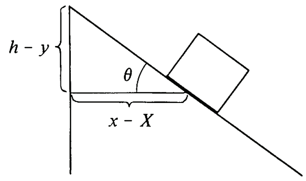
The force is proportional to the number of cars pulled.

In systems composed of several bodies, the accelerations are often related by constraints. The equations of constraint can sometimes be found by simple inspection, but the most general approach is to start with the coordinate geometry, as shown in the next example.

Example 2.4 Constraints

a. WEDGE AND BLOCK





A block moves on a wedge which in turns moves on a horizontal table, as shown in the sketch. The wedge angle is θ. How are the accelerations of the block and the wedge related?

As long as the wedge is in contact with the table, we have the trivial constraint that the vertical acceleration of the wedge is zero. To find the less obvious constraint, let X be the horizontal coordinate of the end of the wedge and let x and y be the horizontal and vertical coordinates of the block, as shown. Let h be the height of the wedge.

From the geometry, we see that

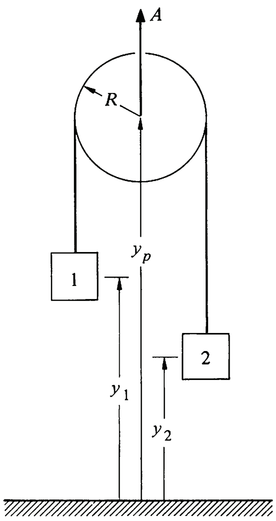
$$(x - X) = (h - y) \cot \theta.$$

Differentiating twice with respect to time, we obtain the equation of constraint

$$\ddot{x} - \ddot{X} = -\ddot{y} \cot \theta. \qquad 1$$

A few comments are in order. Note that the coordinates are inertial. We would have trouble using Newton's second law if we measured the position of the block with respect to the wedge; the wedge is accelerating and cannot specify an inertial system. Second, unimportant parameters, like the height of the wedge, disappear when we take time derivatives, but they can be useful in setting up the geometry. Finally, constraint equations are independent of applied forces. For example, even if friction between the block and wedge affects their accelerations, Eq. (1) is valid as long as the bodies are in contact.

b. MASSES AND PULLEY



Two masses are connected by a string which passes over a pulley accelerating upward at rate A, as shown. Find how the accelerations of the bodies are related. Assume that there is no horizontal motion.

We shall use the coordinates shown in the drawing. The length of the string, l, is constant. Hence, if y_p is measured to the center of the pulley of radius R,

$$l = \pi R + (y_p - y_1) + (y_p - y_2). \qquad 2$$

Differentiating twice with respect to time, we find the constraint condition

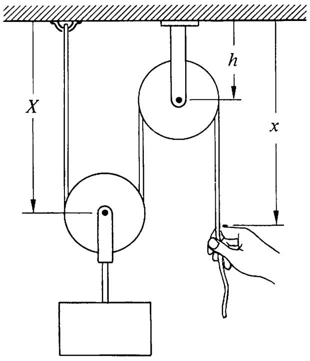
$$0 = 2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2.$$

Using $A = \ddot{y}_p$, we have

$$A = \tfrac{1}{2}(\ddot{y}_1 + \ddot{y}_2).$$

c. PULLEY SYSTEM

The pulley system shown on the opposite page is used to hoist the block. How does the acceleration of the end of the rope compare with the



acceleration of the block? Using the coordinates indicated, the length of the rope is given by

$$l = X + \pi R + (X - h) + \pi R + (x - h),$$

where R is the radius of the pulleys. Hence

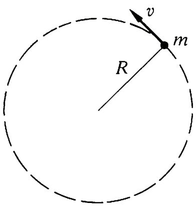
$$\ddot{X} = -\tfrac{1}{2}\ddot{x}.$$

The block accelerates half as fast as the hand, and in the opposite direction.

Our examples so far have involved linear motion only. Let us look at the dynamics of rotational motion.

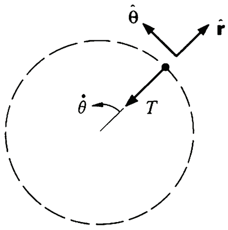
A particle undergoing circular motion must have a radial acceleration. This sometimes causes confusion, since our intuitive idea of acceleration usually relates to change in speed rather than to change in direction of motion. For this reason, we start with as simple an example as possible.

Example 2.5 **Block on String 1**



Mass m whirls with constant speed v at the end of a string of length R. Find the force on m in the absence of gravity or friction.

The only force on m is the string force T, which acts toward the center, as shown in the diagram. It is natural to use polar coordinates. Note that according to the derivation in Sec. 1.9, the radial acceleration is $a_r = \ddot{r} - r\dot{\theta}^2$, where $\dot{\theta}$ is the angular velocity. a_r is positive outward. Since **T** is directed toward the origin, $\mathbf{T} = -T\hat{\mathbf{r}}$ and the radial equation of motion is



$$-T = ma_r$$
$$= m(\ddot{r} - r\dot{\theta}^2).$$

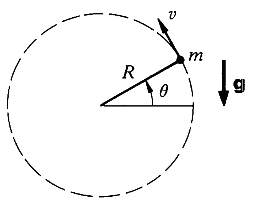
$\ddot{r} = \ddot{R} = 0$ and $\dot{\theta} = v/R$. Hence $a_r = -R(v/R)^2 = -v^2/R$ and

$$T = \frac{mv^2}{R}.$$

Note that T is directed toward the origin; there is no outward force on m. If you whirl a pebble at the end of a string, *you* feel an outward force. However, the force you feel does not act on the pebble, it acts on you. This force is equal in magnitude and opposite in direction to the force with which you pull the pebble, assuming the string's mass to be negligible.

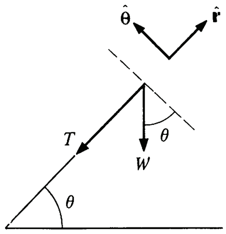
In the following example both radial and tangential acceleration play a role in circular motion.

Example 2.6 **Block on String 2**



Mass m is whirled on the end of a string length R. The motion is in a vertical plane in the gravitational field of the earth. The forces on m are the weight W down, and the string force T toward the center. The instantaneous speed is v, and the string makes angle θ with the horizontal. Find T and the tangential acceleration at this instant.

The lower diagram shows the forces and unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. The radial force is $-T - W \sin \theta$, so the radial equation of motion is



$$\begin{aligned} -(T + W \sin \theta) &= ma_r \\ &= m(\ddot{r} - r\dot{\theta}^2). \end{aligned} \qquad 1$$

The tangential force is $-W \cos \theta$. Hence

$$\begin{aligned} -W \cos \theta &= ma_\theta \\ &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \qquad 2$$

Since $r = R =$ constant, $a_r = -R(\dot{\theta}^2) = -v^2/R$, and Eq. (1) gives

$$T = \frac{mv^2}{R} - W \sin \theta.$$

The string can pull but not push, so that T cannot be negative. This requires that $mv^2/R \geq W \sin \theta$. The maximum value of $W \sin \theta$ occurs when the mass is vertically up; in this case $mv^2/R > W$. If this condition is not satisfied, the mass does not follow a circular path but starts to fall; $\ddot{r}$ is no longer zero.

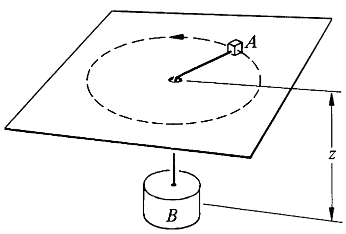
The tangential acceleration is given by Eq. (2). Since $\dot{r} = 0$ we have

$$\begin{aligned} a_\theta &= R\ddot{\theta} \\ &= -\frac{W \cos \theta}{m}. \end{aligned}$$

The mass does not move with constant speed; it accelerates tangentially. On the downswing the tangential speed increases, on the upswing it decreases.

The next example involves rotational motion, translational motion, and constraints.

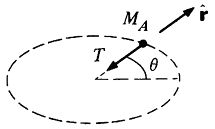
Example 2.7 **The Whirling Block**

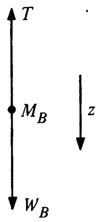


A horizontal frictionless table has a small hole in its center. Block A on the table is connected to block B hanging beneath by a string of negligible mass which passes through the hole.

Initially, B is held stationary and A rotates at constant radius r_0 with steady angular velocity ω_0. If B is released at $t = 0$, what is its acceleration immediately afterward?

The force diagrams for A and B after the moment of release are shown in the sketches.





The vertical forces acting on A are in balance and we need not consider them. The only horizontal force acting on A is the string force T. The forces on B are the string force T and the weight W_B.

It is natural to use polar coordinates r, θ for A, and a single linear coordinate z for B, as shown in the force diagrams. As usual, the unit vector $\hat{\mathbf{r}}$ is radially outward. The equations of motion are

$$-T = M_A(\ddot{r} - r\dot{\theta}^2) \qquad \text{Radial} \tag{1}$$

$$0 = M_A(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad \text{Tangential} \tag{2}$$

$$W_B - T = M_B\ddot{z} \qquad \text{Vertical.} \tag{3}$$

Since the length of the string, l, is constant, we have

$$r + z = l. \tag{4}$$

Differentiating Eq. (4) twice with respect to time gives us the constraint equation

$$\ddot{r} = -\ddot{z}. \tag{5}$$

The negative sign means that if A moves inward, B falls. Combining Eqs. (1), (3), and (5), we find

$$\ddot{z} = \frac{W_B - M_A r\dot{\theta}^2}{M_A + M_B}.$$

It is important to realize that although acceleration can change instantaneously, velocity and position cannot. Thus immediately after B is released, $r = r_0$ and $\dot{\theta} = \omega_0$. Hence

$$\ddot{z}(0) = \frac{W_B - M_A r_0\omega_0^2}{M_A + M_B}. \tag{6}$$

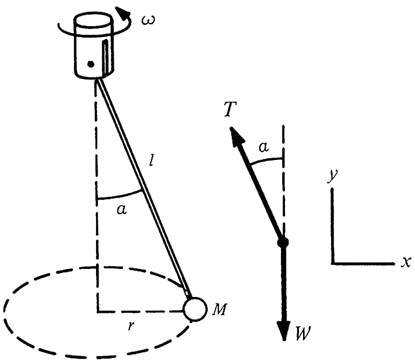
$z(0)$ can be positive, negative, or zero depending on the value of the numerator in Eq. (6); if ω_0 is large enough, block B will begin to rise after release.

The apparently simple problem in the next example has some unexpected subtleties.

Example 2.8 The Conical Pendulum

Mass M hangs by a massless rod of length l which rotates at constant angular frequency ω, as shown in the drawing on the next page. The mass moves with steady speed in a circular path of constant radius. Find α, the angle the string makes with the vertical.

We start with the force diagram. T is the string force and W is the weight of the bob. (Note that there are no other forces on the bob. If this is not clear, you are most likely confusing an acceleration with a



force—a serious error.) The vertical equation of motion is

$$T \cos \alpha - W = 0$$

because y is constant and $\ddot{y}$ is therefore zero.

To find the horizontal equation of motion note that the bob is accelerating in the $\hat{\mathbf{r}}$ direction at rate $a_r = -\omega^2 r$. Then

$$-T \sin \alpha = -Mr\omega^2. \qquad 2$$

Since $r = l \sin \alpha$ we have

$$T \sin \alpha = Ml\omega^2 \sin \alpha \qquad 3$$

or

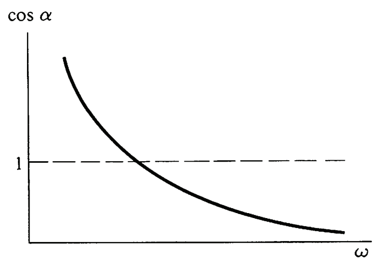
$$T = Ml\omega^2. \qquad 4$$

Combining Eqs. (1) and (3) gives

$$Ml\omega^2 \cos \alpha = W.$$

As we shall discuss in Sec. 2.5, $W = Mg$, where M is the mass and g is known as the acceleration due to gravity. We obtain

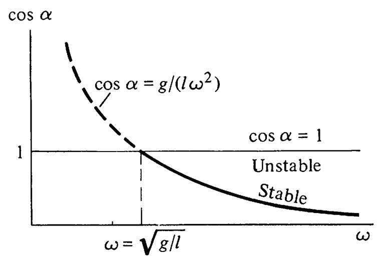
$$\cos \alpha = \frac{g}{l\omega^2}.$$



This appears to be the desired solution. For $\omega \to \infty$, $\cos \alpha \to 0$ and $\alpha \to \pi/2$. At high speeds the bob flies out until it is almost horizontal. However, at low speeds the solution does not make sense. As $\omega \to 0$, our solution predicts $\cos \alpha \to \infty$, which is nonsense since $\cos \alpha \leq 1$. Something has gone wrong. Here is the trouble.

Our solution predicts $\cos \alpha > 1$ for $\omega < \sqrt{g/l}$. When $\omega = \sqrt{g/l}$, $\cos \alpha = 1$ and $\sin \alpha = 0$; the bob simply hangs vertically. In going from Eq. (2) to Eq. (3) we divided both sides of Eq. (2) by $\sin \alpha$ and, in this case we divided by 0, which is not permissible. However, we see that we have overlooked a second possible solution, namely, $\sin \alpha = 0$, $T = W$, which is true for all values of ω. The solution corresponds to the pendulum hanging straight down. Here is a plot of the complete solution.

Physically, for $\omega \leq \sqrt{g/l}$ the only acceptable solution is $\alpha = 0$, $\cos \alpha = 1$. For $\omega > \sqrt{g/l}$ there are two acceptable solutions:



1. $\cos \alpha = 1$
2. $\cos \alpha = \dfrac{g}{l\omega^2}$.

Solution 1 corresponds to the bob rotating rapidly but hanging vertically. Solution 2 corresponds to the bob flying around at an angle with the vertical. For $\omega > \sqrt{g/l}$, solution 1 is unstable—if the system is in that state and is slightly perturbed, it will jump outward. Can you see why this is so?

The moral of this example is that you have to be sure that the mathematics makes good physical sense.

2.5 The Everyday Forces of Physics

When a physicist sets out to design an accelerator, he uses the laws of mechanics and his knowledge of electric and magnetic forces to determine the paths that the particles will follow. Predicting motion from known forces is an important part of physics and underlies most of its applications. Equally important, however, is the converse process of deducing the physical interaction by observing the motion; this is how new laws are discovered. A classic example is Newton's deduction of the law of gravitation from Kepler's laws of planetary motion. The current attempt to understand the interactions between elementary particles from high energy scattering experiments provides a more contemporary illustration.

Unscrambling experimental observations to find the force can be difficult. In a facetious mood, Eddington once said that force is the mathematical expression we put into the left hand side of Newton's second law to obtain results that agree with observed motions. Fortunately, force has a more concrete physical reality.

Much of our effort in the following chapters will be to learn how systems behave under applied forces. If every pair of particles in the universe had its own special interaction, the task would be impossible. Fortunately, nature is kinder than this. As far as we know, there are only four fundamentally different types of interactions in the universe: gravity, electromagnetic interactions, the so-called weak interaction, and the strong interaction.

Gravity and the electromagnetic interactions can act over a long range because they decrease only as the inverse square of the distance. However, the gravitational force always attracts, whereas electrical forces can either attract or repel. In large systems, electrical attraction and repulsion cancel to a high degree, and gravity alone is left. For this reason, gravitational forces dominate the cosmic scale of our universe. In contrast, the world immediately around us is dominated by the electrical forces, since they are far stronger than gravity on the atomic scale. Electrical forces are responsible for the structure of atoms, molecules, and more complex forms of matter, as well as the existence of light.

The weak and strong interactions have such short ranges that they are important only at nuclear distances, typically 10^{-15} m.

They are negligible even at atomic distances, 10^{-10} m. As its name implies, the strong interaction is very strong, much stronger than the electromagnetic force at nuclear distances. It is the "glue" that binds the atomic nucleus, but aside from this it has little effect in the everyday world. The weak interaction plays a less dramatic role; it mediates in the creation and destruction of neutrinos—particles of no mass and no charge which are essential to our understanding of matter but which can be detected only by the most arduous experiments.

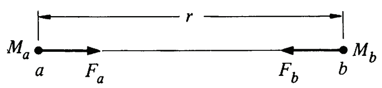
Our object in the remainder of the chapter is to become familiar with the forces which are important in everyday mechanics. Two of these, the forces of gravity and electricity, are fundamental and cannot be explained in simpler terms. The other forces we shall discuss, friction, the contact force, and the viscous force, can be understood as the macroscopic manifestation of interatomic forces.

Gravity, Weight, and the Gravitational Field

Gravity is the most familiar of the fundamental forces. It has close historical ties to the development of mechanics; Newton discovered the law of universal gravitation in 1666, the same year that he formulated his laws of motion. By calculating the motion of two gravitating particles, he was able to derive Kepler's empirical laws of planetary motion. (By accomplishing all this by age 26, Newton established a tradition which still maintains—that great advances are often made by young physicists.)

According to Newton's law of gravitation, two particles attract each other with a force directed along their line of centers. The magnitude of the force is proportional to the product of the masses and decreases as the inverse square of the distance between the particles.

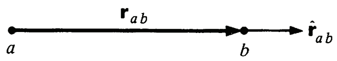
In verbal form the law is bulky and hard to use. However. we can reduce it to a simple mathematical expression.



Consider two particles, a and b, with masses M_a and M_b, respectively, separated by distance r. Let $\mathbf{F}_b$ be the force exerted on particle b by particle a. Our verbal description of the magnitude of the force is summarized by

$$|\mathbf{F}_b| = \frac{GM_aM_b}{r^2}.$$

G is a constant of proportionality called the *gravitational constant.* Its value is found by measuring the force between masses in a known geometry. The first measurements were performed by Henry Cavendish in 1771 using a torsion balance. The modern value of G is 6.67×10^{-11} N·m²/kg². (G is the least accurately known of the fundamental constants. Perhaps you can devise a new way to measure it more precisely.) Experimentally, G is the same for all materials—aluminum, lead, neutrons, or what have you. For this reason, the law is called the universal law of gravitation.



The gravitational force between two particles is *central* (along the line of centers) and attractive. The simplest way to describe these properties is to use vectors. By convention, we introduce a vector $\mathbf{r}_{ab}$ *from* the particle exerting the force, particle a in this case, *to* the particle experiencing the force, particle b. Note that $|\mathbf{r}_{ab}| = r$. Using the unit vector $\hat{\mathbf{r}}_{ab} = \mathbf{r}_{ab}/r$, we have

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}.$$

The negative sign indicates that the force is attractive. The force on a due to b is

$$\mathbf{F}_a = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ba} = +\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab} = -\mathbf{F}_b,$$

since $\hat{\mathbf{r}}_{ba} = -\hat{\mathbf{r}}_{ab}$. The forces are equal and opposite, and Newton's third law is automatically satisfied.

The gravitational force has a unique and mysterious property. Consider the equation of motion of particle b under the gravitational attraction of particle a.

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}$$

$$= M_b\mathbf{a}_b$$

or

$$\mathbf{a}_b = -\frac{GM_a}{r^2}\hat{\mathbf{r}}_{ab}.$$

The acceleration of a particle under gravity is independent of its mass! There is a subtle point connected with our cancelation of M_b, however. The "mass" (*gravitational* mass) in the law of gravitation, which measures the strength of gravitational interaction, is operationally distinct from the "mass" (*inertial* mass) which characterizes inertia in Newton's second law. Why gravitational mass is proportional to inertial mass for all matter is one of the great mysteries of physics. However, the proportionality has been