

3.1.2021

תרגול 11 - תורת גלואה

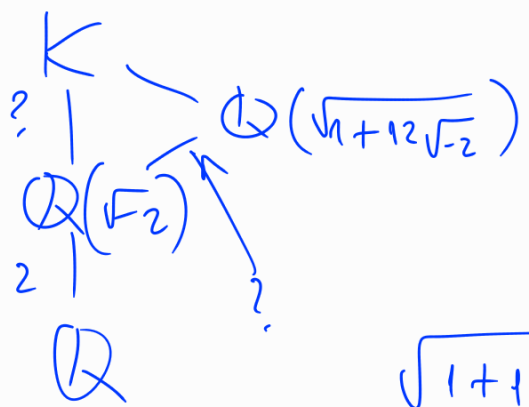
? $x^4 - 2x^2 + 289$

① מהו מספר השורשים הרציונליים? (סעיף 1)

$\pm \sqrt{1 \pm 12\sqrt{-2}}$

$\frac{2 \pm \sqrt{4 - 4 \cdot 289}}{2} =$

$= 1 \pm 12\sqrt{-2}$



$\sqrt{1+12\sqrt{-2}} \stackrel{?}{\in} \mathbb{Q}(\sqrt{-2})$

ראו

$1 + 12\sqrt{-2} = (\alpha + \beta\sqrt{-2})^2, \alpha, \beta \in \mathbb{Q}$

$N_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\ast)$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 1^2 + 144 & = & (\alpha^2 + 2\beta^2)^2 \\ \underline{289} & & \end{array}$$

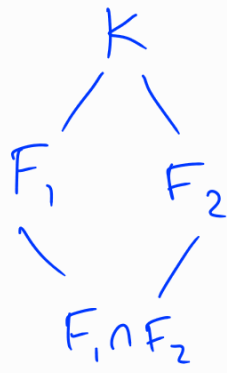
$\Rightarrow \alpha^2 + 2\beta^2 = 17 \Rightarrow 3 + 2\sqrt{-2}$

$(3 + 2\sqrt{-2})^2 = 9 - 8 + 12\sqrt{-2} = 1 + 12\sqrt{-2}$

$K = \mathbb{Q}(\sqrt{-2})$ $\Leftarrow \sqrt{1+12\sqrt{-2}} \in \mathbb{Q}(\sqrt{-2})$ \circledast
[$K : \mathbb{Q}$] = 2

$$[K : F_{1,2}] < \infty$$

המשפט/הערה: (2)



$$\Rightarrow [K : F_1 \cap F_2] < \infty$$

$$\frac{GL_2(\mathbb{C})}{\sim}$$

$$\mathbb{P}(z)$$

הערה:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PGL_2(\mathbb{C}) \rightsquigarrow z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

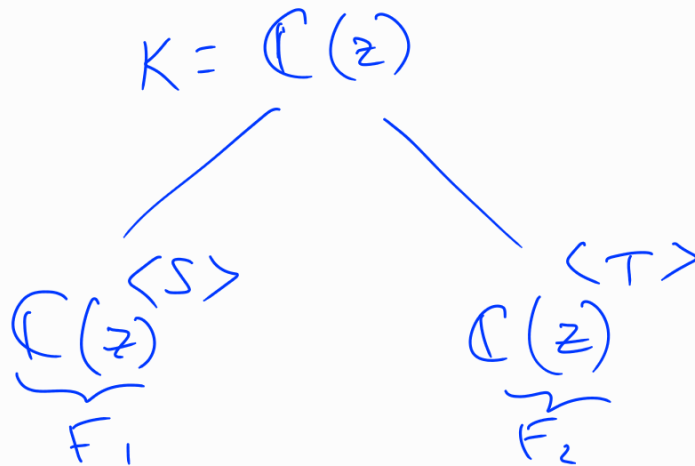
המשפט/הערה: $\mathbb{P}(z)$ הוא המרחב הפרויקטיבי

$$Aut(\mathbb{P}(z)) \cong PGL_2(\mathbb{C})$$

המשפט/הערה: יש

$$S: z \mapsto -\frac{1}{z}$$

$$T: z \mapsto -\frac{1}{z+1}$$



$$[K : F_{1,2}] < \infty$$

המשפט/הערה: המשפט/הערה:

$$F_1 \cap F_2 = \mathbb{C}$$

$\cos 24^\circ$

אלו הפתרונות הם

(3)

$$\cos 24^\circ = \frac{1}{2}(\rho_{15} + \rho_{15}^{-1})$$

$$U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

$\begin{matrix} 1 & 4 & 2 & 7 & 4 & 2 & 4 & 2 \end{matrix}$

$$K = \mathbb{Q}(\rho_{15})$$

$$U_{mn} \cong U_m \times U_n$$

$$K^{\{1,14\}} = K_1 = \mathbb{Q}(\rho + \rho^{-1})$$

$$U_{15} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \leftarrow \dots \leftarrow K^{\{1,4,11,14\}} = K_2 \ni \beta$$

$$\mathbb{Q}(\sqrt{5}) \cong \mathbb{Q} \oplus \mathbb{Q}$$

$$\alpha = \rho + \rho^{-1} \quad \left(\cos \frac{2\pi}{15} = \frac{1}{2} \alpha \right)$$

הערך α הוא שורש של $x^2 - 3x + 1 = 0$ והוא שייך ל K_1 .
 הערך β הוא שורש של $x^2 - 5x + 5 = 0$ והוא שייך ל K_2 .

$$\beta = \text{Tr}_{K_1/K_2}(\alpha)$$

$$\gamma = N_{K_1/K_2}(\alpha)$$

$$\rho + \rho^4 + \rho^{11} + \rho^{14}$$

הערך β הוא שורש של $x^2 - 5x + 5 = 0$ והוא שייך ל K_2 .
 הערך γ הוא שורש של $x^2 - 5x + 5 = 0$ והוא שייך ל K_2 .

• $\text{Tr}_{K_2/\mathbb{Q}}(\beta) = \rho + \rho^4 + \rho^{11} + \rho^{14} + \rho^2 + \rho^8 + \rho^7 + \rho^{13} =$
 $= -1 - \rho^3 - \rho^5 - \rho^6 - \rho^9 - \rho^{10} - \rho^{12} = 1$

$\rho_{15}^5 = \rho_3$ $\rho_{15}^3 = \rho_5$

$1 + \rho_5 + \rho_5^2 + \rho_5^3 + \rho_5^4 = 1 + \rho_{15}^3 + \rho_{15}^6 + \rho_{15}^9 + \rho_{15}^{12} = 0$

• $N_{K_2/\mathbb{Q}}(\beta) = (\rho + \rho^4 + \rho^{11} + \rho^{14})(\rho^2 + \rho^8 + \rho^7 + \rho^{13}) =$

$= \left\{ \begin{array}{cc} \begin{matrix} \cancel{3} & \cancel{9} & 8 & 14 \\ \cancel{6} & \cancel{12} & 11 & 2 \\ 13 & 4 & 3 & 9 \\ \cancel{1} & 7 & 6 & 12 \end{matrix} & \begin{matrix} -1 + \\ (-1 - \rho^5 - \rho^{10}) \\ \rho_{15}^5 = \rho_3 \end{matrix} \end{array} \right\} = -1$

$\sum_{i=0}^{14} \rho_{15}^i = 0$

$\rho + \rho^3 + \rho^6 + \rho^9 + \rho^{12} = -1$

$\beta^2 - \beta - 1 = 0$

$\beta = \frac{1 + \sqrt{5}}{2}$

• $\gamma = N_{K_1/K_2}(\alpha) = (\rho + \rho^{14})(\rho^4 + \rho^{11}) = \rho^5 + \rho^{12} + \rho^3 + \rho^{10} =$
 $= -1 + \rho^3 + \rho^{12}$

$$\begin{aligned} \text{Tr}_{K_2/\mathbb{Q}}(\gamma) &= (-1 + \rho^3 + \rho^{12}) + (-1 + \rho^6 + \rho^9) = \\ &= \underbrace{\rho^3 + \rho^6 + \rho^9 + \rho^{12}}_{\substack{\text{Sum of } \rho^j \text{ for } j=0,3,6,9,12 \\ \text{in } \mathbb{F}_{25} \rightarrow -1}} - 2 = \underline{-3} \end{aligned}$$

$$N_{K_2/\mathbb{Q}}(\gamma) = (-1 + \rho^3 + \rho^{12}) \cdot (-1 + \rho^6 + \rho^9) = (\dots) = \underline{1}$$

$$\gamma = \frac{-3 + \sqrt{5}}{2} \quad \Leftrightarrow \gamma^2 + 3\gamma + 1 = 0$$

$$\alpha^2 - \left(\frac{1 + \sqrt{5}}{2}\right)\alpha + \left(\frac{-3 + \sqrt{5}}{2}\right) \quad : \text{ähn}$$

$$\cos 24^\circ = \frac{1}{2} \alpha = \frac{1}{2} \left(\frac{\frac{1 + \sqrt{5}}{2} + \sqrt{\left(\frac{1 + \sqrt{5}}{2}\right)^2 - 2(-3 + \sqrt{5})}}{2} \right) =$$

$$= \frac{1}{4} \left(\frac{1 + \sqrt{5}}{2} + \sqrt{\frac{3 + \sqrt{5}}{2} + 6 - 2\sqrt{5}} \right) =$$

$$\frac{1}{8}(1 + \sqrt{5}) + \frac{1}{4} \sqrt{\frac{3}{2}(5 - \sqrt{5})}$$