# Integration Bee

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## I. PROBLEM

Calculate the following integral

$$I = \int_0^{\pi} e^{\cos x} \cos(\sin x) \cos nx \mathrm{d}x \tag{1}$$

## II. SOLUTION

### A. First Method

By noticing that the integrand is even, one can extend the integration domain to be symmetric

$$I = \frac{1}{2} \int_{-\pi}^{\pi} e^{\cos x} \cos(\sin x) \cos nx \mathrm{d}x \tag{2}$$

The next step is to shift the integration domain by substituting $u = x + \pi$

$$I = \frac{(-1)^n}{2} \int_0^{2\pi} e^{\cos(u-\pi)} \cos(\sin(u-\pi)) \cos nu \mathrm{d}u \tag{3}$$

Using Euler's identity $e^{it} = \cos t + i \sin t$ we can further simplify this integral

$$I = \frac{(-1)^n}{2} \int_0^{2\pi} e^{\cos(u-\pi)} \mathcal{R}\left[e^{i\sin(u-\pi)}\right] \cos nu \mathrm{d}u = \mathcal{R}\left[\frac{(-1)^n}{2} \int_0^{2\pi} e^{e^{i(u-\pi)}} \cos nu \mathrm{d}u\right] \tag{4}$$

Now we are getting to the interesting part. Recall that

$$\int_0^{2\pi} e^{imx} \mathrm{d}x = 2\pi \delta_{m,0} \tag{5}$$

We will use this integral, together with the following series expansion

$$e^{e^{i(u-\pi)}} = \sum_{m=0}^{\infty} \frac{(-1)^m e^{imu}}{m!} \tag{6}$$

Substituting Eq. (6) in Eq. (4) yields

$$I = \mathcal{R}\left[\frac{(-1)^n}{2}\int_0^{2\pi}\sum_{m=0}^{\infty}\frac{(-1)^m e^{imu}}{m!}\cos nu du\right] \tag{7}$$

Now using Eq. (5)

$$\int_0^{2\pi} e^{imu}\cos nu du = \int_0^{2\pi} e^{imu}\left[\frac{e^{inu}+e^{-inu}}{2}\right]\mathrm{d}u = \pi\left[\delta_{m,-n}+\delta_{m,n}\right] \tag{8}$$

and inserting back into Eq. (7)

$$I = \mathcal{R}\left[\frac{(-1)^n}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\pi\left[\delta_{m,-n}+\delta_{m,n}\right]\right] \tag{9}$$

If $n = 0$, then

$$I\left(n=0\right) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}2\pi\delta_{m,0} = \pi \tag{10}$$

and if $n \neq 0$

$$I\left(n\neq 0\right) = \frac{(-1)^n}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\pi\delta_{m,|n|} = \frac{\pi}{2\left|n\right|!} \tag{11}$$

Combining results, we finally have

$$I\left(n\right) = \begin{cases} \pi & n = 0 \\ \dfrac{\pi}{2\left|n\right|!} & n \neq 0 \end{cases} \tag{12}$$

### B. Second Method

Another way is to use complex integration methods. Starting again with Eq. (4) and substituting $z = e^{iu}$ we arrive at

$$I = \mathcal{I}\left[\frac{(-1)^n}{4}\int_{C(0,1)} e^{-z}\left(z^{n-1}+z^{-n-1}\right)\mathrm{d}z\right] \tag{13}$$

where $C\left(0,1\right)$ is the unit circle in the complex plane. This integral can be easily solved by means of Cauchy's residue theorem to give

$$I = \mathcal{I}\left[\frac{(-1)^n}{4}\cdot 2\pi i\mathrm{Res}\left(e^{-z}\left(z^{n-1}+z^{-n-1}\right), z=0\right)\right] \tag{14}$$

Calculating the residues is pretty straight-forward

$$\operatorname{Res}\left(e^{-z}\left(z^{n-1}+z^{-n-1}\right), z=0\right)=\frac{1}{(-n)!}\left.\frac{\mathrm{d}^{-n} e^{-z}}{\mathrm{d} z^{-n}}\right|_{z=0}+\frac{1}{n!}\left.\frac{\mathrm{d}^{n} e^{-z}}{\mathrm{d} z^{n}}\right|_{z=0} \tag{15}$$

In Eq. (15) we have used the formula for the residue of a function with a pole of order $m$, given by $\operatorname{Res}\left(f(z), z=z_0\right)=\frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d} z^{m-1}}\left.\left(\left(z-z_0\right)^m f(z)\right)\right|_{z=z_0}$. In doing so, we follow the convention that a derivative of negative order is zero. This gives

$$\operatorname{Res}\left(e^{-z}\left(z^{n-1}+z^{-n-1}\right), z=0\right)=\begin{cases}2 & n=0 \\ \dfrac{(-1)^n}{|n|!} & n \neq 0\end{cases} \tag{16}$$

in turn yielding the same result as before

$$I=\begin{cases}\pi & n=0 \\ \dfrac{\pi}{2\,|n|!} & n \neq 0\end{cases} \tag{17}$$

### C. Third Method

Notice that our integral is in fact the coefficients of the cosine components in the Fourier series representation of the function $f(x)=e^{\cos x}\cos(\sin x)$. In other words, writing

$$e^{\cos x}\cos(\sin x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right] \tag{18}$$

our integral is

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \rightarrow I=\frac{\pi}{2}a_n \tag{19}$$

This expansion is easily achieved by noting that

$$e^{\cos x}\cos(\sin x)=\mathcal{R}\left[e^{e^{ix}}\right]=\mathcal{R}\left[\sum_{n=0}^{\infty}\frac{e^{inx}}{n!}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\cos nx \tag{20}$$

where we have made used of the Taylor expansion of $e^x$. Thus we arrive again at

$$I=\begin{cases}\pi & n=0 \\ \dfrac{\pi}{2\,|n|!} & n \neq 0\end{cases} \tag{21}$$

but this time much faster.