Voter Model with stubborn vertices.(A random walk exercise in disguise)

Hint – remember the graphical representation and time reversal. (there could be other ways)

1. Let G be the segment [1,n]. i.e. G=(V,E), V={1,2,…,n}, E={(1,2), (2,3),…,(n-1,n)}.
   Assume we put two "stubborn agents" at the endpoints. The agent at 1 always thinks "0", and the agent at n always thinks "1" (they never change their opinions). The rest of the vertices start with iid ½ 0-1 opinions. Let $\eta_t(k)$ denote the opinion of vertex k at time t.
   Find $\lim_{t\to\infty} P(\eta_t(k) = 1)$? Does this depend on the initial distribution?
2. Let G=(V,E) be some finite, connected graph. Let O,Z be two distinct vertices in V.
   Put stubborn agents in O (always 1) and in Z (always 0).
   What can you say about $f(v) = \lim_{t\to\infty} P(\eta_t(v) = 1)$?
   What if there were 3 stubborn vertices with 3 different opinions 1,2,3 ?
3. What about G=Z and a (single) stubborn agent at the origin. (and iid ½ everywhere else). What is $\lim_{t\to\infty} P(\eta_t(k) = 1)$?
4. What about when G=Z^3 and there is a stubborn agent at the origin? (and everything else iid ½)

Bootstrap Percolation

In Bootstrap percolation with parameter r, each site has an color in {black,white}, and every time step (discrete time) every vertex that has at least r black neighbors is colored black.

(see http://mathworld.wolfram.com/BootstrapPercolation.html for some examples with r=2 on the square grid)

Let P(G,p,r) be the probability that when starting with an intial configuration where every vertex is black with probability p, iid between vertices , every site will be eventually black.

Let p_c(G,r)=inf {p: P(G,p,r)>0}

0. Prove that every vertex eventually fixates.
1. Find an infinite connected, finite degrees graph G, and some p,r so that 0<P(G,p,r)<1 (i.e. not 0 or 1).
2. (riddle): For an nxn board (e.g. uncolored chessboard) , what is the minimal number of squares that can be colored black so that eventually the whole board is black.
3. Prove that p_c(Z^2,2)=0 (Hint show that a large box of blacks has a positive probability to take over the world)
   Remark: Actually, much more precise statements are known. For an LxL grid, the right critical probability is of order c/log L , c is known, and even more, see https://www.math.ubc.ca/~holroyd/boot/
4. Prove that p_c(Z^2,3)=1.
5. Prove that p_c(T_3,2)=1/2 , where T_3 is the 3-regular tree.