

# Example questions, 2016

July 15, 2018

1. (a) The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{c}{r}. \quad (1)$$

- (b) From E.L equation for  $\theta$  it follows that

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{d}{dt} (mr^2\dot{\theta}) = 0, \quad (2)$$

therefore  $L \equiv L_\theta = mr^2\dot{\theta} = \text{const.}$

- (c) Substituting  $L$  in (1) it follows that

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{c}{r}. \quad (3)$$

- (d) Assuming  $r(0) = r_0, \dot{\theta}(0) = q, \dot{r}(0) = 0$  yields  $E = mr_0^2q^2 - c/r_0$  and  $L = mr_0^2q$ .

- (e) As the energy is conserved, we can get the distances by solving  $U_{eff} = E$ . This yields

$$r_{\min, \max} = \frac{c}{2|E|} \pm \sqrt{\frac{c^2}{4E^2} - \frac{L^2}{2m|E|}} \quad (4)$$

2. (a) Since the problem has cylindrical symmetry, it is convenient to work with cylindrical frame. The Lagrangian is given by (assuming the spring has a zero length at rest)

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}m\omega^2(r^2 + z_0^2), \quad (5)$$

- (b) The Hamiltonian  $\mathcal{H} = rp_r + \theta p_\theta - \mathcal{L}$  then reads

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2(r^2 + z_0^2). \quad (6)$$

(c) Hamilton's equations are

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}, \quad (7)$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad (8)$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{mr^3} - m\omega^2 r, \quad (9)$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0. \quad (10)$$

From (10) it follows that the angular momentum  $p_\theta$  is conserved.

(d) From (6) it follows that

$$U_{eff} = \frac{L_0^2}{2mr^2} + \frac{1}{2}m\omega^2(r^2 + z_0^2). \quad (11)$$

(e) Using  $\Omega = \sqrt{U_{eff}''(r_0)/m}$  where  $r_0$  solves  $\dot{p}_r = 0$  (eq. (9)) gives  $\Omega = \omega\sqrt{3}$ .

(f) Performing similar calculation as in (1e) gives  $r_{\min, \max} = \sqrt{a \pm \sqrt{a^2 - b}}$  where  $a = (2mE_0 - m^2\omega^2 z_0^2)/2m^2\omega^2 E_0$ ,  $b = L_0^2/m^2\omega^2$ .

3. (a) Schroedinger's equation for a free particle is

$$-\frac{\hbar^2}{2m}\psi'' = E\psi. \quad (12)$$

Denoting  $k^2 = 2mE/\hbar^2$  the eigenvalues and eigenvectors are  $k^2$  and

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \equiv |k\rangle, \quad (13)$$

respectively.

(b) There are no boundary conditions, therefore the eigenvalues are continuous.

(c) A straightforward calculation gives from  $\langle \psi | \psi \rangle = 1$   $A = \alpha/2$ .

(d) Calculating  $P(k) = |\langle k | \psi \rangle|^2$  where

$$\langle k | \psi \rangle = \frac{\alpha}{2} \int_{-\infty}^{\infty} \frac{e^{-\alpha|x| - ikx}}{\sqrt{2\pi}} dx, \quad (14)$$

gives

$$P(k) = \frac{\alpha^2}{2\pi} \frac{k^2}{(\alpha^2 + k^2)^2}. \quad (15)$$

4. (a) Noticing that  $|\psi|^2$  is a Gaussian we have that  $\langle x \rangle = a$ .  
 (b) Similar considerations imply that since  $\langle p_x \rangle = -i\hbar I$  where  $I$  is a real number, and since clearly  $\langle p_x \rangle$  is a real,  $I$  must be zero and so  $\langle p_x \rangle = 0$ .  
 (c) solving

$$\hat{p}|\psi_k\rangle = p|\psi_k\rangle, \quad (16)$$

for  $\hat{p} = -i\hbar \frac{\partial}{\partial x}$  and  $|\psi_k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$  gives  $p = \hbar k$ . This means that (16) is an eigenvalues problem with eigenvectors and eigenvalues  $|\psi_k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$  and  $p = \hbar k$ , respectively.

- (d) An integral expression for  $|\langle \psi_k | \psi \rangle|^2$  for  $k = p_0/\hbar$  is

$$|\langle \psi_k | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left\{ \frac{(x-2)^2}{2} - i \frac{p_0}{\hbar} x \right\} dx \right|^2. \quad (17)$$

5. Suppose the road has a deflection angle  $\alpha$  with respect to the horizon. The rider "fills" the normal force  $N$  operated by the road, his weight  $mg$  and the centrifugal force  $m\omega^2 R$  which balances the normal force in the horizontal directions. Solving

$$N \cos \alpha = mg \quad (18)$$

$$N \sin \alpha = m\omega^2 R \quad (19)$$

for  $\alpha$  gives  $\tan \alpha = \omega^2 R/g$ .

6. (a) Denoting the eigenstates of  $A$  corresponding to eigenvalues  $\lambda_1 = 0$  and  $\lambda_2 = 2$  by  $|1\rangle$  and  $|2\rangle$ , respectively, it follows that

$$|\psi\rangle = |x\rangle = \langle 1|x\rangle|1\rangle + \langle 2|x\rangle|2\rangle. \quad (20)$$

- (b) The probabilities are given by

$$P(\lambda = 0) = |\langle 1|x\rangle|^2, \quad (21)$$

$$P(\lambda = 2) = |\langle 2|x\rangle|^2. \quad (22)$$

- (c) Decomposing  $A$  using the spectral theorem yields  $A = 2|2\rangle\langle 2|$ . Clearly

$$\langle A \rangle = 2P(\lambda = 2) = 2|\langle 2|x\rangle|^2. \quad (23)$$

On the other hand

$$\langle \psi | A | \psi \rangle = (\langle x|1\rangle\langle 1| + \langle x|2\rangle\langle 2|)2|2\rangle\langle 2|(\langle 1|x\rangle|1\rangle + \langle 2|x\rangle|2\rangle) = 2\langle x|2\rangle\langle 2|x\rangle = 2|\langle 2|x\rangle|^2. \quad (24)$$

- (d) From  $U = e^{iAt}$  operating on  $|\psi\rangle$  we have

$$|\psi(t)\rangle = e^{iAt}|x\rangle = \langle 1|x\rangle|1\rangle + \langle 2|x\rangle e^{2it}|2\rangle. \quad (25)$$

(e) Expressing  $|y\rangle$  using the eigenstates of  $A$  gives

$$|y\rangle = \langle 1|y\rangle|1\rangle + \langle 2|y\rangle|2\rangle. \quad (26)$$

A mixed state occurs with some nonzero probability  $|\langle y|\psi(t)\rangle|^2 > 0$ . Thus, the lifetime of the state  $|\psi(0)\rangle = |x\rangle$  is obtained by solving

$$|\langle y|\psi(t_0)\rangle|^2 = |a + e^{2it_0}b|^2 = |a|^2 + |b|^2 + 2\text{Re}(abe^{2it_0}) = 0 \quad (27)$$

for  $t_0$  where  $a = \langle 1|y\rangle\langle 1|x\rangle$  and  $b = \langle 2|y\rangle\langle 2|x\rangle$

7. (a) In  $x_1, y_1, x_2, y_2$  system the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2). \quad (28)$$

Together with the constraint

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L^2 \quad (29)$$

$\mathcal{L}$  can be written using 3 generalized coordinates.

(b) In CM and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  ( $|\mathbf{r}_1 - \mathbf{r}_2| = L$ ) system the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\omega^2), \quad (30)$$

where  $M = m_1 + m_2$  and  $\mu = m_1m_2/M$  is the reduced mass. Note that the new Lagrangian (30) also depends on 3 generalized coordinates  $(R, r, \theta, \text{ with } \dot{\theta} = \omega)$ .

(c) Since (30) does not depend on  $R$  it follows that  $P_{CM} = M\dot{R}$  is conserved.

(d) Similarly, since (30) does not depend on  $\theta$  it follows that  $L = mr^2\omega$  is conserved.