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1. (a) The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - Ce^{-\alpha r}. \quad (1)$$

- (b) E.L equation for r, θ are

$$m\ddot{r} = mr\dot{\theta}^2 + C\alpha e^{-\alpha r}, \quad (2)$$

$$\frac{d}{dt} (mr^2\dot{\theta}) = 0, \quad (3)$$

- (c) From (3) we deduce conservation of angular momentum.

- (d) Taking an angular momentum L_0 the effective potential is given by

$$U_{eff} = \frac{L_0^2}{2mr^2} + Ce^{-\alpha r}. \quad (4)$$

- (e) Using the initial conditions $r(0) = r_0, \dot{r}(0) = 0, \dot{\theta}(0) = \omega$ the initial energy is

$$E = \frac{1}{2}mr_0^2\omega^2 + Ce^{-\alpha r_0}. \quad (5)$$

Since the effective potential after a very long time is very small, we get from energy conservation

$$\dot{r} \approx \sqrt{r_0^2\omega^2 + \frac{2C}{m}e^{-\alpha r_0}} = Const, \quad (6)$$

and so $r \sim t$ and from conservation of angular momentum $\theta \sim t^{-2}$.

2. (a) The Hamiltonian is given by

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r). \quad (7)$$

(b) Hamilton's equations read

$$\dot{r} = \frac{p_r}{m}, \quad (8)$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \quad (9)$$

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - U', \quad (10)$$

$$\dot{p}_\theta = 0. \quad (11)$$

(c) In Cartesian frame the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(\sqrt{x^2 + y^2}). \quad (12)$$

Making the transformation $x \rightarrow x' = x + \epsilon y$, $y \rightarrow y' = y - \epsilon x$ result in a new Lagrangian

$$\mathcal{L}' = \frac{1}{2}m((\dot{x} + \epsilon\dot{y})^2 + (\dot{y} - \epsilon\dot{x})^2) - U(\sqrt{(x + \epsilon y)^2 + (y - \epsilon x)^2}). \quad (13)$$

The new Lagrangian (13) can be written as

$$\mathcal{L}' = \mathcal{L} + O(\epsilon^2), \quad (14)$$

so indeed the Lagrangian is invariant under infinitesimal rotation.

(d) Eq. (14) implies that Noether's theorem applies, and therefore

$$y \frac{\partial \mathcal{L}}{\partial \dot{x}} - x \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{x}y - m\dot{y}x = L, \quad (15)$$

which is the angular momentum in Cartesian frame, is conserved.

3. (a) Schroedinger equation is

$$-\frac{\hbar^2}{2m}\psi'' = E\psi. \quad (16)$$

The eigenstates solving the boundary problem (16) with the boundary conditions $\psi(0) = \psi(a) = 0$ are given by

$$\varphi_n = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi n}{a}x\right), \quad (17)$$

and the energies are $E_n = \frac{\pi^2 \hbar^2}{2ma}n$.

(b) The normalized ψ is $\psi = 1/\sqrt{a}$.

(c) Writing

$$|\psi\rangle = \sum_n \langle \varphi_n | \psi \rangle |\varphi_n\rangle, \quad (18)$$

with

$$\langle \varphi_n | \psi \rangle = \frac{1}{a} \int_0^a \sin\left(\frac{\pi n}{a}x\right) dx = -\frac{2}{\pi(2n-1)} \quad (19)$$

we have that

$$\psi(x, t) = -\frac{2}{\pi\sqrt{a}} \sum_{n=1}^{\infty} \frac{e^{-iE_n t/\hbar}}{2n-1} \sin\left(\frac{\pi n}{a}x\right). \quad (20)$$

(d) The probability to find the particle at the n th level is

$$P_n = |\langle \varphi_n | \psi \rangle|^2 = \frac{4}{\pi^2(2n-1)^2}. \quad (21)$$

4. (a) It is easily seen that

$$B|\varphi\rangle = 0. \quad (22)$$

(b) A straightforward calculation implies

$$C|\chi\rangle = |\mu\rangle, \quad (23)$$

$$C|\mu\rangle = |\chi\rangle. \quad (24)$$

(c) The two eigenstates of C are $|\varphi\rangle$ and $|\eta\rangle = (|\chi\rangle + |\mu\rangle)/\sqrt{2}$ with eigenvalues 0 and 1, respectively.

(d) Using the spectral theorem we obtain

$$C = |\eta\rangle\langle\eta| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad (25)$$

(e) Expressing $|\psi\rangle$ in terms of $|\varphi\rangle, |\eta\rangle$ results in

$$|\psi\rangle = \frac{1}{2}((1+i)|\varphi\rangle + (1-i)|\eta\rangle), \quad (26)$$

which implies $P(\lambda_C = 0) = P(\lambda_C = 1) = 1/2$.

5. (a) Taking the weight mg to be equal to the centrifugal force $m\omega^2 R$ we obtain $\omega = \sqrt{g/R}$.

(b) $a_c = 2v\omega$ directed right (in a frame where the radial direction is pointing up).

(c) Let α be the deflection angle due to a_c , then

$$\tan \alpha = a_c/g = 2v/\sqrt{Rg}. \quad (27)$$