פתרון תרגול מס' 12

א

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}\,dx$$

נפרק

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

נשווה מקדמים

$$1 = x(A+B) + B - A$$

$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ B-A=1 \end{array}\right\} \Longrightarrow \begin{array}{l} B=\frac{1}{2} \\ A=-\frac{1}{2} \end{array}$$

נחזור לאינטגרל

$$\int \frac{1}{x^2-1}\,dx = \int \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\,dx = -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

ד

$$\int \frac{x}{x^2-4}\,dx = \frac{1}{2}\int \frac{2x}{x^2-4}\,dx = \frac{1}{2}\ln|x^2-4| + C$$

ב

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{x^2+2^2} = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

ג

$$\int \frac{x+4}{x^2-2x+5}\,dx = \int \frac{x-1+1+4}{x^2-2x+5}\,dx$$

$$= \int \frac{x-1}{x^2-2x+5}\,dx + \int \frac{5}{x^2-2x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2(x-1)}{x^2-2x+5}\,dx + \int \frac{5}{(x-1)^2+2^2} = \frac{1}{2}\ln|x^2-2x+5| + \frac{5\cdot 1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

(0) $\int \frac{4}{x-1}\,dx = 4\int \frac{1}{x-1}\,dx = 4\ln|x-1| + C$

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(1) $\int \frac{3x+2}{x-1}\,dx = \int \frac{3x}{x-1}\,dx + \int \frac{2}{x-1}\,dx$

$= \int \frac{3x-3+3}{x-1}\,dx + \int \frac{2}{x-1}\,dx$

$= \int 3 + \frac{3}{x-1}\,dx + \int \frac{2}{x-1}\,dx = 3x + 3\ln|x-1|$

$+ 2\ln|x-1| + C$

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(5) $\int \frac{2x+1}{7x+1}\,dx = \int \frac{2x}{7x+1}\,dx + \underbrace{\int \frac{1}{7x+1}\,dx}$

$= \underbrace{\frac{2}{7}\int \frac{7x}{7x+1}\,dx}_{(*)} + \underbrace{\frac{1}{7}\int \frac{7}{7x+1}\,dx}_{\frac{1}{7}\ln|7x+1|} =$

(*) קח

$\int \frac{7x}{7x+1}\,dx = \int \frac{7x+1-1}{7x+1}\,dx = \int 1 - \frac{1}{7x+1}\,dx$

$= x - \frac{1}{7}\ln|7x+1|$

(נק׳)

$\int \frac{2x+1}{7x+1} = \frac{2}{7}\left(x - \frac{1}{7}\ln|7x+1|\right) + \frac{1}{7}\ln|7x+1| + C$

$= \frac{2}{7}x - \frac{2}{49}\ln|7x+1| + \frac{1}{7}\ln|7x+1| + C$

$= \frac{2}{7}x + \frac{5}{49}\ln|7x+1| + C$

(ב) $\displaystyle\int \frac{x+1}{x^2+3x+2}\,dx = \int \frac{\cancel{x+1}}{\cancel{(x+1)}(x+2)}\,dx = \int \frac{1}{x+2}\,dx$

$\displaystyle = \ln|x+2| + C$

(ג) $\displaystyle\int \frac{x+1}{x^2+4x+4}\,dx = \int \frac{x+1}{(x+2)^2}\,dx$ $\left\{ x = u-2 \Leftarrow u = x+2,\quad du = dx \right.$ (נסמן)

$\displaystyle = \int \frac{u-2+1}{u^2}\,du = \int \frac{u-1}{u^2}\,du = \int \frac{u}{u^2}\,du - \int \frac{1}{u^2}\,du$

$\displaystyle = \int \frac{1}{u}\,du - \int \frac{1}{u^2}\,du = \ln|u| + \frac{1}{u} + C$

$\displaystyle = \ln|x+2| + \frac{1}{x+2} + C$

(ד) $\displaystyle\int \frac{2x+5}{x^2+4x+10}\,dx = \int \frac{2x+4+1}{x^2+4x+10}\,dx$

$\displaystyle = \int \frac{2x+4}{x^2+4x+10}\,dx + \underbrace{\int \frac{1}{x^2+4x+10}\,dx}_{(*)} = \ln|x^2+4x+10| + (*)$

$(*)$ (נחשב)

$\displaystyle\int \frac{1}{x^2+4x+10}\,dx = \int \frac{1}{(x+2)^2+6}\,dx = \frac{1}{\sqrt{6}}\arctan\left(\frac{x+2}{\sqrt{6}}\right)$

$\displaystyle \Rightarrow \int \frac{2x+5}{x^2+4x+10}\,dx = \ln|x^2+4x+10| + \frac{1}{\sqrt{6}}\arctan\left(\frac{x+2}{\sqrt{6}}\right) + C$

(1כ) $\int \frac{1}{2\sin x + 2\cos x + 4} dx$

נשתמש בהצבה אוניברסלית

$t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$

$$= \int \frac{1}{\frac{2\cdot 2t}{1+t^2} + \frac{2\cdot(1-t^2)}{1+t^2} + 4} \cdot \frac{2}{(1+t^2)} dt =$$

$$= \int \frac{1}{2t+1-t^2+2+2t^2} dt = \int \frac{1}{t^2+2t+3} dt = \int \frac{1}{(t+1)^2+3+1} dt$$

$$= \frac{1}{(t+1)^2+2^2} dt = \frac{1}{2}\arctan\left(\frac{t+1}{2}\right) + C$$

(ל1) $\int \frac{1}{2\sin x + 2} dx$

נשתמש בהצבה האוניברסלית

$t = \tan\frac{x}{2}$ $\quad$ $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$

$$= \int \frac{1}{\frac{2\cdot 2t}{1+t^2} + 2} \cdot \frac{2}{(1+t^2)} dt = \int \frac{1}{2t+1+t^2} dt$$

$$= \int \frac{1}{t^2+2t+1} dt = \int \frac{1}{(t+1)^2} dt \underset{\substack{u=t+1 \\ du=dt}}{=} \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + C = -\frac{1}{t+1} + C = -\frac{1}{\tan\frac{x}{2}+1} + C$$

(יד) $\int \frac{1}{\cos x}\,dx$ (עבור הצבה האוניברסלית)

$t=\tan\frac{x}{2}$ $\quad \cos x=\frac{1-t^2}{1+t^2}$, $dx=\frac{2}{1+t^2}\,dt$

$$=\int \frac{1}{\frac{1-t^2}{\cancel{1+t^2}}}\cdot\frac{2}{\cancel{(1+t^2)}}\,dt=\int\frac{2}{1-t^2}\,dt$$

$$\frac{2}{1-t^2}=\frac{2}{(1-t)\cdot(1+t)}=\frac{A}{1-t}+\frac{B}{1+t}=\frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

$$\Rightarrow\quad \frac{2}{(1-t)(1+t)}=\frac{A+B+t(A-B)}{(1-t)(1+t)}$$

$$\Rightarrow\quad \begin{cases}A+B=2\\ A-B=0\end{cases}\Rightarrow A=B=1$$

$$\int\frac{2}{1-t^2}\,dt=\int\frac{1}{1-t}+\frac{1}{1+t}\,dt=\ln|1-t|+\ln|1+t|+C$$

$$=\ln\left|1-\tan\frac{x}{2}\right|+\ln\left|1+\tan\frac{x}{2}\right|+C$$

(טו) $\int\frac{1}{\sin x}\,dx$ (משתמש בהצבה אוניברסלית)

$t=\tan\frac{x}{2}$, $\sin x=\frac{2t}{1+t^2}$, $dx=\frac{2}{1+t^2}\,dt$

$$=\int\frac{1}{\frac{2t}{\cancel{1+t^2}}}\cdot\frac{\cancel{2}}{\cancel{(1+t^2)}}\,dt=\int\frac{1}{t}\,dt=\ln|t|+C$$

$$=\ln\left|\tan\frac{x}{2}\right|+C$$

(א) $\displaystyle\int \frac{1}{\sin x+1}\,dx$

נשתמש בהצבה אוניברסלית

$t=\tan\frac{x}{2}\ ,\ \sin x=\frac{2t}{1+t^2}\ ,\ dx=\frac{2}{1+t^2}\,dt$

$$=\int \frac{1}{\frac{2t}{1+t^2}+1}\cdot\frac{2}{(1+t^2)}\,dt=\int\frac{2}{2t+1+t^2}\,dt$$

$$=\int\frac{2}{(t+1)^2}\,dt=-2\int\frac{-1}{(t+1)^2}\,dt$$

נציב $t+1=u$

$dt=du$

$$=-2\int\frac{-1}{u^2}\,du=-2\cdot\frac{1}{u}+C$$

$$=\frac{-2}{t+1}+C=\frac{-2}{\tan\frac{x}{2}+1}+C$$