פתרון תרגיל 3

.1

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx$$

$$\left\{\begin{matrix} x=t^6 \\ dx=6t^5dt \end{matrix}\right\}$$

$$=\int \frac{t^6+t^4+t}{t^6\left(1+t^2\right)}\cdot 6t^5dt=6\int \frac{t^5+t^3+1}{\left(1+t^2\right)}dt=6\int t^3dt+6\int \frac{dt}{t^2+1}$$

$$=\frac{3}{2}t^4+6\arctan t+c=\frac{3}{2}x^{2/3}+6\arctan x^{1/6}+c$$

.2

$$\int \frac{dx}{x+\sqrt{x^2-x+1}}$$

$$\sqrt{x^2-x+1}=t-x$$

$$x^2-x+1=t^2-2tx+x^2$$

$$t^2-2tx+x-1=0$$

$$x\left(1-2t\right)=1-t^2$$

$$x=\frac{1-t^2}{1-2t}$$

$$dx=\frac{-2t\left(1-2t\right)+2\left(1-t^2\right)}{\left(1-2t\right)^2}dt=\frac{2t^2-2t+2}{\left(1-2t\right)^2}dt$$

$$\Rightarrow \int \frac{2\left(t^2-t+1\right)}{\left(1-2t\right)^2 \cdot t}dt$$

$$\frac{2\left(t^2-t+1\right)}{\left(1-2t\right)^2 \cdot t} = \frac{A}{t} + \frac{B}{1-2t} + \frac{D}{\left(1-2t\right)^2}$$

$$A\left(1-2t\right)^2 + Bt\left(1-2t\right) + Dt = 2t^2 - 2t + 2$$

$$t^2 : 4A - 2B = 2$$

$$t \ : -4A + B + D = -2$$

$$t^0 : A = 2$$

$$A = 2$$

$$B = 3$$

$$D = 3$$

$$\Rightarrow \int \frac{2dt}{t} + \int \frac{3dt}{1-2t} + \int \frac{3dt}{\left(1-2t\right)^2}$$

$$= 2\ln|t| - \frac{3}{2}\ln|1-2t| + \frac{3}{2}\frac{1}{1-2t} + c$$

$$= 2\ln\left|x + \sqrt{x^2-x+1}\right| - \frac{3}{2}\ln\left|1-2x-2\sqrt{x^2-x+1}\right| + \frac{3}{2}\frac{1}{1-2x-2\sqrt{x^2-x+1}} + c$$

.3

$$\int \frac{dx}{\left(1+x\right)\sqrt{1+x-x^2}}$$

$$\sqrt{1+x-x^2} = tx + 1$$

$$1 + x - x^2 = t^2x^2 + 2tx + 1$$

$$1 - x = t^2 x + 2t$$

$$x\left(t^2+1\right) = 1 - 2t$$

$$x = \frac{1-2t}{t^2+1}$$

$$dx = \frac{-2\left(t^2+1\right) - \left(1-2t\right)\cdot 2t}{\left(t^2+1\right)^2}dt = \frac{2t^2-2t-2}{\left(t^2+1\right)^2}dt$$

$$1 + x = 1 + \frac{1-2t}{t^2+1} = \frac{t^2+1+1-2t}{t^2+1} = \frac{t^2-2t+2}{t^2+1}$$

$$\Rightarrow \int \frac{\left(2t^2-2t-2\right)\cdot\left(t^2+1\right)}{\left(t^2+1\right)^2\cdot\left(t^2-2t+2\right)\left[\frac{t(1-2t)}{t^2+1}\right]}dt = \int \frac{2t^2-2t-2}{\left(t^2-2t+2\right)\left(t-2t^2+t^2+1\right)}dt$$

$$= -2\int \frac{\left(t^2-t-1\right)}{\left(t^2-2t+2\right)\left(t^2-t-1\right)}dt = -2\int \frac{dt}{t^2-2t+2} = -2\int \frac{dt}{(t-1)^2+1} = -2\arctan(t-1)+c$$

$$= -2\arctan\left(\frac{\sqrt{1+x-x^2}-1}{x}-1\right)+c$$

.4

$$\int \frac{xdx}{\sqrt{\left(7x-10-x^2\right)^3}}$$

$$-x^2+7x-10=0$$

$$x^2-7x+10=0$$

$$x_1=2 \quad x_2=5$$

$$\sqrt{7x-10-x^2}=t(x-2)$$

$$\sqrt{-(x-2)(x-5)}=t(x-2)$$

$$-(x-2)(x-5)=t^2(x-2)^2$$

$$-(x-5)=t^2(x-2)$$

$$x\left(t^2+1\right)=5+2t^2$$

$$x=\frac{5+2t^2}{t^2+1}$$

$$dx=\frac{4t\left(t^2+1\right)-\left(5+2t^2\right)\cdot 2t}{\left(t^2+1\right)^2}dt=\frac{-6t}{\left(t^2+1\right)^2}dt$$

$$\Rightarrow \int \frac{5+2t^2}{\left(t^2+1\right)t^3\left[\frac{5+2t^2}{t^2+1}-2\right]^3}\cdot\frac{(-6t)}{\left(t^2+1\right)^2}dt$$

$$=-6\int\frac{\left(5+2t^2\right)dt}{t^2\left(5+2t^2-2t^2-2\right)^3}dt=-\frac{2}{9}\int\frac{5+2t^2}{t^2}dt=\frac{10}{9}\frac{1}{t}-\frac{4}{9}t+c$$

$$=\frac{10}{9}\cdot\frac{(x-2)}{\sqrt{7x-10-x^2}}-\frac{4}{9}\frac{\sqrt{7x-10-x^2}}{x-2}+c$$

.5

$$\int\frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}dx$$

$$\frac{2-x}{2+x}=t^3$$

$$2-x=(2+x)t^3$$

$$2-2t^3=x\left(t^3+1\right)$$

$$x=\frac{2-2t^3}{t^3+1}=2\cdot\frac{1-t^3}{t^3+1}$$

$$dx=2\cdot\frac{\left(-3t^2\left(t^3+1\right)-3t^2\left(1-t^3\right)\right)}{\left(t^3+1\right)^2}dt=\frac{-12t^2}{\left(t^3+1\right)^2}dt$$

$$2-x=2-\frac{2\left(1-t^3\right)}{\left(t^3+1\right)}=\frac{4t^3}{t^3+1}$$

$$=\int\frac{2\left(t^3+1\right)^2}{16t^6}\cdot t\cdot\left(\frac{-12t^2}{\left(t^3+1\right)^2}\right)dt$$

$$=\int\frac{-24t^3}{16t^6}dt=-\frac{3}{2}\int\frac{dt}{t^3}=\frac{3}{4}\frac{1}{t^2}+c=\frac{3}{4}\left(\frac{2+x}{2-x}\right)^{2/3}+c$$

.6

$$\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\mathrm{d}x = \frac{1}{4}\int \frac{\cos 2x}{\sin^2 2x}\mathrm{d}x$$

נציב $t = \sin 2x$ ואז

$$\mathrm{d}t = \cos 2x\mathrm{d}x$$

כלומר

$$\frac{1}{4}\int \frac{\cos 2x}{\sin^2 2x}\mathrm{d}x = \frac{1}{4}\int \frac{\mathrm{d}t}{t^2} = -\frac{1}{4t} + C = -\frac{1}{4\sin 2x} + C$$

.7

$$\int \frac{1}{1+\sin x + \cos x}\mathrm{d}x$$

נשתמש בהצבה אוניברסלית ונקבל

$$\int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t = \int \frac{1+t^2}{1+t^2+2t+1-t^2}\frac{2}{1+t^2}\mathrm{d}t$$
$$= \int \frac{2}{2+2t}\mathrm{d}t = \ln(1+t) + C = \ln(\tan\frac{x}{2}+1) + C$$

.8

$$\int \frac{\sin 2x}{1+\sin^2 x}\mathrm{d}x = \int \frac{2\sin x\cos x}{1+\sin^2 x}\mathrm{d}x$$

נציב $t = \sin x$ ונקבל

$$\int \frac{2t}{1+t^2}\mathrm{d}t = \ln(1+t^2) = \ln(1+\sin^2 x)$$