Inner Product Space

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Def.: *Inner Product Space* is a vector space V over $\mathbb{F} = \mathbb{C}\backslash\mathbb{R}$ with an *inner product*, i.e.:
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$
that satisfies

- Linearity in the first argument:
    - $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle$
    - $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_1, v_3 \rangle$
- Conjugate symmetry:
    - $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$
- Positive-definiteness:
    - $\langle v, v \rangle \geq 0$
    - $\langle v, v \rangle = 0 \iff v = 0$

Elementary properties

- Almost Linearity in the second argument:
  - $\langle v_1, \alpha v_2 \rangle = \overline{\alpha} \langle v_1, v_2 \rangle$
  - $\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
- Generalization:
  - $\left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{m} \alpha'_j v'_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \overline{\alpha'_j} \left\langle v_i, v'_j \right\rangle$
- Zero vector
  - $\langle v, 0 \rangle = \langle 0, v \rangle = 0$

Examples

- $V = \mathbb{R}^n$ with
$$\langle x, y\rangle = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots y_n \end{pmatrix} \right\rangle = x^t y = \sum_{i=1}^n x_i y_i$$
- $V = \mathbb{C}^n$ with
$$\langle z, w\rangle = \left\langle \begin{pmatrix} z_1 \\ z_2 \\ \vdots z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ \vdots w_n \end{pmatrix} \right\rangle = z^t \bar{w} = \sum_{i=1}^n z_i \bar{w}_i$$

Ex. $V = \mathbb{C}^3, \langle z, w \rangle = z^t \bar{w}$

Compute inner product of $v = \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}$ with:

- $v_1 = (-i, -4, \sqrt{2})$

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- $v_4 = v_1 + 2v_2 - 3v_3$
  Sol.
  $\langle v, v_4 \rangle = \langle v, v_1 \rangle + 2 \langle v, v_2 \rangle - 3 \langle v, v_2 \rangle = 2\sqrt{2} - 3i - 3 \cdot 6$

Examples (Cont.)

- $V = \mathbb{R}^{n\times n}$ with
  $\langle A, B\rangle = trace(AB^t)$
- $V = \{f : [-1,1] \to \mathbb{C} \mid \text{f is continuous function } \}$ with
  $\langle f, g\rangle := \int\limits_{-1}^{1} f(x)\overline{g(x)}\, dx$

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  - Compute $\langle \sin(x), \cos(x)\rangle$
  - $\langle \sin(x), \cos(x)\rangle = \int\limits_{-1}^{1} \sin(x)\cos(x)\,dx = \int\limits_{-1}^{1} \frac{\sin(2x)}{2}\,dx = 0$

The norm induced by an inner product

Def.: Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$. The *norm* induced by the inner product is the function $|| * || : V \to \mathbb{R}_{\geq 0}$ defined by $||v|| := \sqrt{\langle v, v \rangle}$

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Facts:

- $||v|| \geq 0$ and $v = 0 \iff ||v|| = 0$
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Ex:

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- $||z|| = \sqrt{\sum_{i=1}^{n} z_i \bar{z}_i} = \sqrt{\sum_{i=1}^{n} |z_i|^2}$

CauchySchwarz inequality

Th.: Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|| * ||$. Then:

$$| \langle x, y \rangle | \leq ||x|| \cdot ||y|||$$

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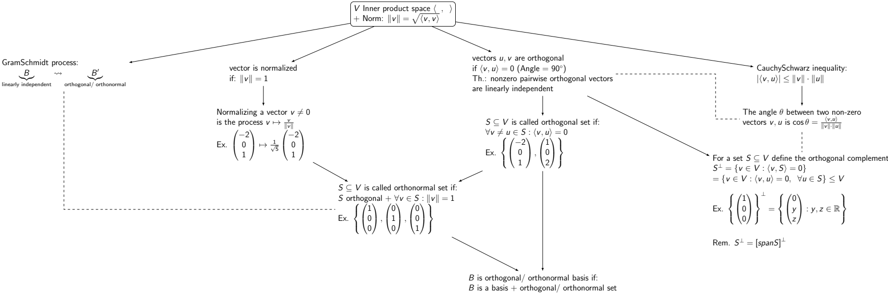

V Inner product space $\langle\ ,\ \rangle$
+ Norm: $\|v\| = \sqrt{\langle v, v\rangle}$

GramSchmidt process:
$\underbrace{B}_{\text{linearly independent}} \rightsquigarrow \underbrace{B'}_{\text{orthogonal/ orthonormal}}$

vector is normalized
if: $\|v\| = 1$

Normalizing a vector $v \neq 0$
is the process $v \mapsto \frac{v}{\|v\|}$
Ex. $\begin{pmatrix}-2\\0\\1\end{pmatrix} \mapsto \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\0\\1\end{pmatrix}$

vectors u, v are orthogonal
if $\langle v, u\rangle = 0$ (Angle $= 90^\circ$)
Th.: nonzero pairwise orthogonal vectors
are linearly independent

$S \subseteq V$ is called orthogonal set if:
$\forall v \neq u \in S : \langle v, u\rangle = 0$
Ex. $\left\{\begin{pmatrix}-2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$

CauchySchwarz inequality:
$|\langle v, u\rangle| \leq \|v\| \cdot \|u\|$

The angle θ between two non-zero
vectors v, u is $\cos\theta = \frac{\langle v,u\rangle}{\|v\|\cdot\|u\|}$

For a set $S \subseteq V$ define the orthogonal complement
$S^\perp = \{v \in V : \langle v, S\rangle = 0\}$
$= \{v \in V : \langle v, u\rangle = 0,\ \ \forall u \in S\} \leq V$

Ex. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}^\perp = \left\{\begin{pmatrix}0\\y\\z\end{pmatrix} : y, z \in \mathbb{R}\right\}$

Rem. $S^\perp = [spanS]^\perp$

$S \subseteq V$ is called orthonormal set if:
S orthogonal + $\forall v \in S : \|v\| = 1$
Ex. $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

B is orthogonal/ orthonormal basis if:
B is a basis + orthogonal/ orthonormal set

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Ex. Let $v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, define $S = \{u, v\}$.

Find basis to $S^{\perp}$.

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Find basis to $S^{\perp}$.

Sol. If $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ then $S^{\perp} = N(A)$. Therefore

$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$N(A) = \{ \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} \} : t \in \mathbb{R}$.

And basis $\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$

Gram-Schmidt process

Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|| * ||$.
GramSchmidt process:

- Input: Linearly independent set $B = \{v_1, \ldots, v_n\}$
- Output: Orthogonal/Orthonormal set $B' = \{w_1, \ldots, w_n\}$ s.t. span(B) = span(B')

Gram-Schmidt process $B \rightsquigarrow B'$

$B = \{v_1, \ldots, v_n\}, B^{'} = \{w_1, \ldots, w_n\}$.
The Algo.

$$w_1 := v_1$$

$$w_2 := v_2 - \frac{<v_2, w_1>}{||w_1||^2} w_1$$

$$w_3 := v_3 - \frac{<v_3, w_1>}{||w_1||^2} w_1 - \frac{<v_3, w_2>}{||w_2||^2} w_2$$

$$\vdots$$

$$w_i := v_i - \sum_{k=1}^{i-1} \frac{<v_i, w_k>}{||w_k||^2} w_k$$

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B' Orthogonal. Normalized each w_i implies B' Orthonormal.

Example

$B = \{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$. Perform GramSchmidt, to obtain an orthogonal set of vectors

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{<v_2, w_1>}{||w_1||^2} w_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Example

$$\begin{aligned}
w_3 &= v_3 - \frac{<v_3, w_1>}{||w_1||^2} w_1 - \frac{<v_3, w_2>}{||w_2||^2} w_2 \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{(\frac{1}{2})}{(\frac{6}{4})} \cdot \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}
\end{aligned}$$

Example

Now, $\{w_1, w_2, w_3\}$ Orthogonal.

and $\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$ Orthonormal

Corollaries and Properties

- It is possible to start with $w_1 = v_2$
- Every $W \leq V$ has an orthonormal basis
- For each $1 \leq t \leq n$:
  $span(\{v_1, \dots, v_t\}) = span(\{w_1, \dots, w_t\})$