

Th.(Cayley-Hamilton) $P_A(A) = 0$

Algebraic multiplicity of λ_0 is
 $\max k$ s.t. $(\lambda - \lambda_0)^k$ divides $P_A(\lambda)$

Geometric multiplicity of λ_0 is $\dim V_{\lambda_0}$

Th.: Geo. \leq Alge.

1) λ is eigenvalue $\iff P_A(\lambda) = |\lambda I - A| = 0$

2) v is eigenvector $\iff v \in V_\lambda = N(A - \lambda I) = \{v : Av = \lambda v\}$

Given $A \in \mathbb{F}^{n \times n}$
 Find $v \neq 0, \lambda : Av = \lambda v$
 λ - eigenvalue, v -eigenvector

A can be diagonalized if:

$$\exists P : P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Note: $P = (v_1, \dots, v_n), v_i$ is eigenvector
 and λ_i its eigenvalue

Th.:

1. eigenvectors of **different** eigenvalues are linearly independent.
2. $A \in \mathbb{F}^{n \times n}$ is diagonalizable \iff
 \exists n linearly independent eigenvectors \iff
 \exists an eigenvectors basis \iff
 $P_A(\lambda) = \prod_i (\lambda - \lambda_i)^{\alpha_i}$ and for all λ_i : Geo.=Alge.

Def: $A \in \mathbb{C}^{n \times n}$ is **normal** if $AA^* = A^*A$.

Def: $P \in \mathbb{C}^{n \times n}$ is **unitary** if $PP^* = I$, i.e. $P^{-1} = P^*$.

Th.: For a normal matrix $A \in \mathbb{C}^{n \times n}$:

1. it is diagonalizable by unitary matrix, i.e.

$$\exists P \text{ unitary: } P^*AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (P^* = P^{-1})$$

2. eigenvectors of **different** eigenvalues are orthogonal.

Th.: For a complex hermitian matrix $A^* = A: \forall \lambda_i \in \mathbb{R}$

Th.: For a real $A \in \mathbb{R}^{n \times n}$ symmetric matrix $A^t = A$:
 P is (also) real matrix and $P^{-1} = P^* = P^t$, i.e.,

$$\exists P : P^tAP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (P^t = P^{-1})$$