THE GAUSS-BONNET THEOREM

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ABSTRACT. We develop some preliminary differential geometry in order to state and prove the Gauss-Bonnet theorem, which relates a compact surface's Gaussian curvature to its Euler characteristic. We show the Euler characteristic is a topological invariant by proving the theorem of the classification of compact surfaces. We use the Gauss-Bonnet theorem to give a geometric proof of the Poincaré-Hopf index theorem, which relates the index of a smooth tangent vector field on a surface to the surface's Euler characteristic.

CONTENTS

1. INTRODUCTION

The Gauss-Bonnet theorem is perhaps one of the deepest theorems of differential geometry. It relates a compact surface's total Gaussian curvature to its Euler characteristic. A surface's Euler characteristic tells us what kind of surface we have up to homeomorphism. For example, under this classification a sphere and an ellipsoid are the same, whereas a sphere and a torus are different. The theorem is surprising because Gaussian curvature at a point is certainly not invariant under homeomorphism. We can homeomorphically deform the unit sphere, which has Gaussian curvature 1 everywhere, into an ellipsoid which will have some flatter parts and some more curved parts. However, the theorem tells us both surfaces have the same total Gaussian curvature.

We can also ask how the global shape of a surface limits the properties of smooth tangent vector fields we can define on it; for instance, we can ask about the number and nature of the vector field's stationary points. The famous Hairy Ball Theorem states that there is no smooth non-vanishing vector field on a sphere. However, there is one on a torus.

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Our goal is to show

$$\int_S K\,dA = 2\pi\chi = 2\pi\sum_{i=1}^{n}\mu(p_i)$$

for a compact surface S in $\mathbb{R}^3$. K is the Gaussian curvature, χ is the Euler characteristic and $\mu(p_i)$ is the multiplicity of the singular point p_i of a tangent vector field on S. These concepts will be formally discussed in the sections that follow. The first equality is the Gauss-Bonnet theorem, the second is the Poincaré-Hopf index theorem.

In Section 2, we introduce basic concepts from differential geometry in order to define Gaussian curvature. In Section 4, we prove the Gauss-Bonnet theorem for compact surfaces by considering triangulations. To do so, we use a result relating the total geodesic curvature of a curve on a surface to the Gaussian curvature of the region it encloses, which we prove in Section 3. In Section 5, we show that all compact surfaces can be classified up to homeomorphism by their Euler characteristics. In Section 6, we define the index of a vector field and use the Gauss-Bonnet theorem to prove the Poincaré-Hopf theorem.

2. Preliminaries

In this section, we define surfaces in $\mathbb{R}^3$. We introduce the first and second fundamental forms, central for the study of the local geometry of surfaces. We then define the Gaussian curvature of a surface.

Definition 2.1. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ containing p and a map $\sigma : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that
i) σ is smooth, meaning it has continuous partial derivatives of all orders.
ii) σ is a homeomorphism.
iii) For each $q \in U$ the differential $d\sigma_q$ is injective, or equivalently, $\sigma_u \times \sigma_v$ is never 0.

We call σ a surface patch or parametrization of $S \cap V$. We call a collection $\{(U_i, \sigma_i)\}$ with $\bigcup \sigma_i(U_i) = S$ an atlas. We will use the term surface to mean smooth, regular surface.

Condition i) allows us to use the tools of differential calculus to study surfaces, condition ii) means that locally we can flatten or straighten out surfaces to look like $\mathbb{R}^2$, and condition iii) allows us to talk about tangent planes to surfaces.

Example 2.2. The unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a surface.

Let

$$\sigma_1(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

and take U_1 to be the open subset of $\mathbb{R}^2$ given by

$$U_1 = \{(u, v) \in \mathbb{R}^2 : -\pi/2 < \theta < \pi/2, 0 < \phi < 2\pi\}.$$

The image of σ_1 does not cover all of S^2; we are missing points of the form $(x, 0, z)$ with $x \geq 0$. So we define another surface patch also on U_1 by

$$\sigma_2(\theta, \phi) = (-\cos\theta\cos\phi, -\sin\theta, -\cos\theta\sin\phi).$$

The reader can check that the surface patches σ_1 and σ_2 satisfy the properties in Definition 2.1, and that these two surface patches give an atlas for S^2.

Example 2.3. The torus obtained by revolving a circle in the xz-plane with center $(a, 0)$ and radius $b < a$ about the z-axis is a surface.

We can parametrize the circle by $\gamma(\theta) = ((a+b\cos\theta), b\sin\theta)$. Then, by revolving this curve about the z-axis, we obtain the surface patch

$$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta).$$

We leave it to the reader to find an atlas for the torus.

Next, we introduce the notion of tangent space.

Definition 2.4. A *tangent vector to a surface* S at a point $p \in S$ is a tangent vector at p to a curve in S passing through p. The *tangent space* T_pS is the set of all tangent vectors to S at p.

Proposition 2.5. *Let* S *be a surface and let* $\sigma(u, v)$ *be a surface patch containing a point* $p \in S$. *Then* $T_pS = \operatorname{span}(\sigma_u, \sigma_v)$.

Proof. Let $\gamma(t) = \sigma(u(t), v(t))$ be a curve on S. Differentiating, we get

$$\dot\gamma = \dot u\sigma_u + \dot v\sigma_v.$$

Conversely, if $v = \lambda_1\sigma_u + \lambda_2\sigma_v$, we can define

$$\gamma(t) = \sigma(u_0 + \lambda_1 t, v_0 + \lambda_2 t).$$

Then $\dot\gamma = v$. □

Remark 2.6. Condition iii) of Definition 2.1 guarantees that σ_u and σ_v are linearly independent, so we see T_pS has dimension 2. We sometimes refer to T_pS as a tangent plane.

To make measurements on a surface, such as lengths of curves or areas of regions, we need to define a metric on the tangent space. This brings us to the first fundamental form.

Definition 2.7. The *first fundamental form* is a symmetric bilinear form on T_pS given by

$$I_p(v, w) = \langle v, w\rangle$$

where the right hand side denotes the usual inner product in $\mathbb{R}^3$.

We will consider the associated quadratic form $I_p(w, w)$. Fixing a basis, we can find a matrix representation of this quadratic form. This means we can write $I_p(w, w) = \langle AX, X\rangle$, where X is a vector in $\mathbb{R}^2$ which gives the coordinates of w with respect to the chosen basis of T_pS.

Take $\{\sigma_u, \sigma_v\}$ as the basis for T_pS. Then we can write $w = \alpha_1\sigma_u + \alpha_2\sigma_v$. Since I_p is a symmetric bilinear form, we have

$$I_p(\alpha_1\sigma_u + \alpha_2\sigma_v) = \alpha_1^2\langle\sigma_u, \sigma_u\rangle + 2\alpha_1\alpha_2\langle\sigma_u, \sigma_v\rangle + \alpha_2^2\langle\sigma_v, \sigma_v\rangle.$$

Now let

$$E = \langle\sigma_u, \sigma_u\rangle,\ F = \langle\sigma_u, \sigma_v\rangle,\ G = \langle\sigma_v, \sigma_v\rangle,$$

where the partial derivatives are evaluated at (u_0, v_0) with $p = \sigma(u_0, v_0)$. Then we have

$$\left\langle \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right\rangle = I_p(\alpha_1\sigma_u + \alpha_2\sigma_v).$$

Since the Gauss-Bonnet theorem involves a surface integral, we briefly discuss areas of regions on surfaces. We define the area $A_\sigma(R)$ of a region $\sigma(R)$ on a surface by $A_\sigma(R) = \int_R \|\sigma_u \times \sigma_v\| \, du \, dv$. The reader can easily verify $\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$. We will write

$$dA = \sqrt{EG - F^2} \, du \, dv,$$

dropping the subscript σ because this quantity is independent of parametrization. This is clear since $EG - F^2 = \det(I_p)$, which is independent of the choice of basis.

In this paper, we are interested in the curvature of surfaces. Curvature of a surface should measure how much a surface deviates from a plane, which has to do with how the tangent plane T_pS changes as p changes. Note that since we are working in three-dimensional space, the tangent plane is completely determined by a unit vector perpendicular to it. One such vector is

$$\mathbf{N}_p = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

(The other option is to take $-\mathbf{N}_p$). We call $\mathbf{N}_p$ the standard unit normal. Note that $\pm\mathbf{N}_p$ does not depend on the parametrization (see [3], Section 4.5), but the parametrization pins down the sign. So we say that σ induces an orientation on the part of S contained in its image.

In this paper, we will restrict ourselves to considering surfaces which are orientable. We take this to mean that there is a smooth choice of unit normal $\mathbf{N}_p$ on all of S. This means $\mathbf{N}_p$ varies smoothly in p, even as we move from one surface patch to another. Intuitively, this means a surface has two separate sides. (A classic example of a non-orientable surface is a Mobius strip.)

Definition 2.8. The *Gauss map* is the map $G : S \to S^2$ which associates to each $p \in S$ its normal vector $\mathbf{N}_p \in S^2$, where $\mathbf{N}_p$ is defined as above.

Remark 2.9. The Gauss map changes sign when the orientation of the surface changes.

Our definition of curvature of a surface should be closely related to how the normal vector changes as p changes, so we will work with the derivative of the Gauss map.

Definition 2.10. The *Weingarten map* W is given by $W_p = -D_pG$.

By Remark 2.9, the sign of W depends on the orientation of the surface.

By the definition of the derivative, W is a map from T_pS to $T_{G(p)}S^2$. The unit normal of $T_{G(p)}S^2$ is $G(p)$. Since $G(p) = \mathbf{N}_p$, we have $T_{G(p)}S^2 = T_pS$. So W is a map from T_pS to itself.

Definition 2.11. The *second fundamental form* $II_p : T_pS \to \mathbb{R}$ is a bilinear form given by

$$II_p(v, w) = \langle W(v), w \rangle.$$

The bilinearity of II_p follows from the bilinearity of the inner product and the linearity of the derivative.

Lemma 2.12. *The second fundamental form is a symmetric bilinear form.*

Proof. First, note

$$W(\sigma_u) = -\frac{d}{du}G(\sigma(u, v_0)) = -\mathbf{N}_u.$$

Similarly, $W(\sigma_v) = -\mathbf{N}_v$. Let $v = \alpha_1\sigma_u + \alpha_2\sigma_v$ and let $w = \beta_1\sigma_u + \beta_2\sigma_v$. Then,

$$\begin{aligned} II_p(v, w) &= \langle -\alpha_1\mathbf{N}_u - \alpha_2\mathbf{N}_v, \beta_1\sigma_u + \beta_2\sigma_v\rangle \\ &= -\alpha_1\beta_1\langle\mathbf{N}_u, \sigma_u\rangle - \alpha_1\beta_2\langle\mathbf{N}_u, \sigma_v\rangle - \alpha_2\beta_1\langle\mathbf{N}_v, \sigma_u\rangle - \alpha_2\beta_2\langle\mathbf{N}_v, \sigma_v\rangle \\ &= \langle -\beta_1\mathbf{N}_u - \beta_2\mathbf{N}_v, \alpha_1\sigma_u + \alpha_2\sigma_v\rangle \\ &= II_p(w, v). \end{aligned}$$

□

Now we can consider the associated quadratic form. We have

$$II_p(v, v) = -\alpha_1^2\langle\mathbf{N}_u, \sigma_u\rangle - 2\alpha_1\alpha_2\langle\mathbf{N}_u, \sigma_v\rangle - \alpha_2^2\langle\mathbf{N}_v, \sigma_v\rangle,$$

where we used the fact that $\langle\mathbf{N}_u, \sigma_v\rangle = \langle\mathbf{N}_v, \sigma_u\rangle$, which can be seen by differentiating $\langle\mathbf{N}, \sigma_u\rangle = \langle\mathbf{N}, \sigma_v\rangle = 0$ with respect to u and v.

So the matrix of the second fundamental form is

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

where $L = -\langle\mathbf{N}_u, \sigma_u\rangle$, $M = -\langle\mathbf{N}_u, \sigma_v\rangle = -\langle\mathbf{N}_v, \sigma_u\rangle$, $N = -\langle\mathbf{N}_v, \sigma_v\rangle$.

The reader can easily check that $L = \sigma_{uu}\cdot\mathbf{N}$, $M = \sigma_{uv}\cdot\mathbf{N}$, and $N = \sigma_{vv}\cdot\mathbf{N}$.

Definition 2.13. The Gaussian curvature K of S at p is given by $K = \det W$.

Example 2.14. The Gaussian curvature of S^2 is 1 everywhere.

To see this, note that the Gauss map of S^2 is the identity, and hence so is the Weingarten map. So

$$K = \det W = 1.$$

It is not usually this simple to determine the Weingarten map for other surfaces. However, by considering $-\mathbf{N}_u$ and $-\mathbf{N}_v$ as linear combinations of σ_u and σ_v, it is easy to see that the matrix of the Weingarten map with respect to the basis $\{\sigma_u, \sigma_v\}$ is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

(For the full derivation, the reader can consult [3], Proposition 8.1.2.) Then we see that

$$K = \frac{LN - M^2}{EG - F^2}.$$

Remark 2.15. The Gaussian curvature K does not depend on orientation. To see this recall W changes sign with a change of orientation, and the determinant of a 2×2 matrix stays the same when we multiply all the entries by -1.

Example 2.16. We can now compute the Gaussian curvature of the torus.

Recalling the parametrization from Example 2.3, we have

$$\begin{aligned} \sigma_\theta &= (-b\sin\theta\cos\phi, -b\sin\theta\sin\phi, b\cos\theta), \\ \sigma_\phi &= (-(a + b\cos\theta)\sin\phi, (a + b\cos\theta)\cos\phi, 0). \end{aligned}$$

This gives $E = b^2$, $F = 0$, $G = (a + b\cos\theta)^2$. We can also compute $\mathbf{N}$, σ_{uu} and σ_{vv} to get the coefficients of the second fundamental form: $L = b$, $M = 0$, $N = (a + b\cos\theta)\cos\theta$. Hence,

$$K = \frac{\cos\theta}{b(a + b\cos\theta)}.$$

So $K \geq 0$ when $-\pi/2 \leq \theta \leq \pi/2$ and $K \leq 0$ when $\pi/2 \leq \theta \leq 3\pi/2$.

Remark 2.17. If we apply a dilation $(x, y, z) \mapsto (ax, ay, az)$ to a surface S, the parametrization σ gets multiplied by a. Thus E, F, G get multiplied by a^2 and L, M, N get multiplied by a. Thus K gets multiplied by a^2.

Example 2.18. Let S be a sphere of radius a. Its Gaussian curvature is $1/a^2$ everywhere by the above remark. Hence,

$$\int_S K\,dA = \left(\frac{1}{a^2}\right) 4\pi a^2 = 4\pi.$$

3. Gauss-Bonnet Theorem for Curves

Our goal is to prove the Gauss-Bonnet theorem for compact surfaces. This theorem relates the local property of curvature to a global topological invariant. It is instructive to first consider how the total curvature of a *curve* is affected by its global properties. For simple closed curves in the plane, we have the following result.

Theorem 3.1. *Let γ be a unit-speed simple closed curve in $\mathbb{R}^2$. Then*

$$\int_0^{l(\gamma)} \kappa_s\,ds = \pm 2\pi.$$

The quantity κ_s denotes the signed curvature, where the sign depends on the orientation of the parametrization. Also note that $\kappa_s = \frac{d\phi}{ds}$, where $\phi(s)$ is the angle the tangent vector $\dot{\gamma}(s)$ makes with some fixed unit vector (see [3], Proposition 2.2.3.). Thus, Theorem 3.1 says that the tangent vector rotates by an angle of 2π when going once around the curve. For a proof, see Section 5-7 of [2].

Next, we consider curves on curved surfaces. A simple closed curve γ on a surface patch $\sigma : U \to \mathbb{R}^3$ is given by $\gamma = \sigma \circ \beta$, where β is a simple closed curve in $\mathbb{R}^2$ and $\mathrm{int}(\beta)$ is entirely contained in U. By $\mathrm{int}(\beta)$ we mean the region of $\mathbb{R}^2$ enclosed by β. Since β is a simple closed curve, the Jordan Curve Theorem (see [1] Section 5.6) tells us the set of points in $\mathbb{R}^2$ which are not in the image of γ is the union of two disjoint connected subsets, which we denote by $\mathrm{int}(\beta)$ and $\mathrm{ext}(\beta)$, where the former is bounded and the latter is unbounded.

Given a curve γ on a surface S, we have $\{\dot{\gamma}, \mathbf{N}, \mathbf{N} \times \dot{\gamma}\}$ forming an orthonormal basis of $\mathbb{R}^3$. We can look at the projections of $\ddot{\gamma}$ onto these basis vectors. Recall that the curvature κ of a unit-speed curve γ is given by $\kappa = \|\ddot{\gamma}\|$.

Definition 3.2. The geodesic curvature κ_g is given by

$$\kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}).$$

Remark 3.3. The sign of the geodesic curvature depends on our choice of orientation of S. It also depends on the orientation of the curve itself. This will become important in later proofs.

Remark 3.4. For a plane curve, $\kappa_g = \kappa_s$ up to a sign. To see this, note that for a plane curve, $\dot{\gamma}$ is always perpendicular to $\mathbf{N}$. Since $\dot{\gamma}$ is perpendicular to $\ddot{\gamma}$ for any unit-speed curve, we see that $\mathbf{N} \times \dot{\gamma}$ is parallel to $\ddot{\gamma}$.

Now we can state the Gauss-Bonnet theorem for simple closed curves. We present the main ideas of the proof, omitting the lengthy computations. For a complete proof, see [3], Theorem 13.1.2.

Theorem 3.5. *Let $\gamma = \sigma \circ \beta$ be a positively-oriented unit-speed simple closed curve on a surface patch σ of length $l(\gamma)$. Then*

$$\int_0^{l(\gamma)} \kappa_g \, ds = 2\pi - \int_{\mathrm{int}(\beta)} K \, dA.$$

Note that for a plane curve, this is consistent with Theorem 3.1 because $\kappa_g = \kappa_s$ and $K = 0$.

Proof. To begin, we consider a smooth orthonormal basis $\{E_1, E_2\}$ of the tangent plane at each point of the surface patch. For example, we can take $E_1 = \sigma_u / \|\sigma_u\|$ and $E_2 = \mathbf{N} \times E_1$, where $\mathbf{N}$ is the standard unit normal. Then $\{E_1, E_2, \mathbf{N}\}$ is a smooth right-handed orthonormal basis of $\mathbb{R}^3$. We want to compute the geodesic curvature κ_g of γ in terms of this basis. Let $\theta(s)$ be the oriented angle $\widehat{\dot{\gamma} E_1}$. This is the angle by which $\dot{\gamma}$ must be rotated counter-clockwise to be parallel to E_1, when looking at the surface from the side to which $\mathbf{N}$ points. It is only defined up to adding a multiple of 2π. We can write

$$\dot{\gamma} = \cos\theta E_1 + \sin\theta E_2. \tag{3.6}$$

Differentiating gives

$$\ddot{\gamma} = \dot{\theta}(-\sin\theta E_1 + \cos\theta E_2) + \cos\theta \dot{E}_1 + \sin\theta \dot{E}_2.$$

Now we can substitute $\mathbf{N} = E_1 \times E_2$ along with the above expressions for $\dot{\gamma}$ and $\ddot{\gamma}$ into the formula $\kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})$. Using the orthonormality of E_1 and E_2, the reader can easily check that we obtain

$$\kappa_g = \dot{\theta} - E_1 \cdot \dot{E}_2.$$

Next, we compute the integral of the above expression. We claim

$$\int_0^{l(\gamma)} \dot{\theta} ds = 2\pi. \tag{3.7}$$

We give only a heuristic argument. First, we claim the integral is a multiple of 2π. Since γ is a simple closed curve, we have $\gamma(s+l) = \gamma(s)$, so $\dot{\gamma}(s+l) = \dot{\gamma}(s)$. In particular, $\dot{\gamma}(l) = \dot{\gamma}(0)$. By (3.6),

$$(\cos(\theta(l)), \sin(\theta(l))) = (\cos(\theta(0)), \sin(\theta(0))),$$

which implies $\theta(l) - \theta(0)$ is a multiple of 2π. By the fundamental theorem of calculus, this is equal to the integral in (3.7).

Now, if $\tilde{\gamma}$ is another simple closed curve contained in the interior of γ, we can find a family of curves γ^τ which are continuous in τ with $\gamma^0 = \gamma$ and $\gamma^1 = \tilde{\gamma}$. The above integral should depend continuously on τ, but it can only assume values that are integer multiples of 2π and nothing in between. By the intermediate value theorem, the integral must be constant in τ, which means it does not depend on the choice of simple closed curve γ. We can therefore take γ to be a very small circle, which is essentially a plane curve, so Theorem 3.1 applies. This proves (3.7).

Now we are left to show

$$\int_0^{l(\gamma)} E_1 \cdot \dot{E}_2 \, ds = \int_{\mathrm{int}(\beta)} K \, dA.$$

Differentiating E_2, we have

$$\begin{aligned}
\int_0^{l(\gamma)} E_1 \cdot \dot{E}_2 \, ds &= \int_0^{l(\gamma)} E_1 \cdot ((E_2)_u \dot{u} + (E_2)_v \dot{v}) \, ds \\
&= \int_\beta (E_1 \cdot (E_2)_u) \, du + (E_1 \cdot (E_2)_v) \, dv \\
&= \int_{\mathrm{int}(\beta)} (E_1 \cdot (E_2)_v)_u - (E_1 \cdot (E_2)_u)_v \, du \, dv \quad \text{by Green's theorem} \\
&= \int_{\mathrm{int}(\beta)} (E_1)_u \cdot (E_2)_v - (E_1)_v \cdot (E_2)_u \, du \, dv.
\end{aligned}$$

We can also show

$$(E_1)_u \cdot (E_2)_v - (E_1)_v \cdot (E_2)_u = \frac{LN - M^2}{(EG - F^2)^{1/2}}$$

by expressing the partial derivatives of E_1 and E_2 in terms of $\{E_1, E_2, \mathbf{N}\}$. For details see [3], Section 13.1.

Using the definition of the area element dA, we can easily see

$$\begin{aligned}
\int_{\mathrm{int}(\beta)} \frac{LN - M^2}{(EG - F^2)^{1/2}} \, du \, dv &= \int_{\mathrm{int}(\beta)} \frac{LN - M^2}{EG - F^2} \, dA \\
&= \int_{\mathrm{int}(\beta)} K \, dA,
\end{aligned}$$

which completes the proof. □

Now we show Theorem 3.5 generalizes from simple closed curves to curvilinear polygons.

Definition 3.8. A curvilinear polygon in $\mathbb{R}^2$ is a continuous map $\beta : \mathbb{R} \to \mathbb{R}^2$ such that for some real number T and some points $0 = t_0 < t_1 < \cdots < t_n = T$ we have
i) $\beta(t) = \beta(t')$ if and only if $t - t'$ is a multiple of T.
ii) β is smooth on each (t_{i-1}, t_i).
iii) The one-sided derivatives $\dot{\beta}^-(t_i)$ and $\dot{\beta}^+(t_i)$ exist for $i = 1, ..., n$ and are non-zero and non-parallel.

The points $\gamma(t_i)$ are called the vertices of the curvilinear polygon. Now let β be a curvilinear polygon in the plane and let σ be a surface patch. Then, $\gamma = \sigma \circ \beta$ is called a curvilinear polygon on the surface patch σ.

As in the proof of Theorem 3.5, we can take $\{E_1, E_2, \mathbf{N}\}$ as a smooth orthonormal basis of $\mathbb{R}^3$. For $i = 1, ..., n$, let $\theta_i^{\pm}$ be the oriented angle between $\dot{\gamma}^{\pm}(t_i)$ and E_1, as in the beginning of the proof of Theorem 3.5. Then $\delta_i = \theta_i^+ - \theta_i^-$ is the exterior angle at the vertex $\gamma(t_i)$ and $\alpha_i = \pi - \delta_i$ is the interior angle.

Theorem 3.9. *Let γ be a positively-oriented unit-speed curvilinear polygon with n edges on a surface patch σ. Then*

$$\int_0^{l(\gamma)} \kappa_g\, ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \int_{\mathrm{int}(\gamma)} K\, dA.$$

Proof. As in the proof of Theorem 3.5, we can find a smooth orthonormal basis $\{E_1, E_2, \mathbf{N}\}$ of $\mathbb{R}^3$, express $\dot\gamma$ and $\ddot\gamma$ in terms of this basis, and compute the geodesic curvature of γ. As before, we get

$$\int_0^{l(\gamma)} \kappa_g\, ds = \int_0^{l(\gamma)} \dot\theta\, ds - \int_{\mathrm{int}(\gamma)} K\, dA.$$

We are left to show

$$\int_0^{l(\gamma)} \dot\theta\, ds = 2\pi - \sum_{i=1}^{n} \delta_i. \tag{3.10}$$

If we approximate a curvilinear polygon with a smooth curve $\tilde\gamma$ by rounding off the corners, then we know the tangent vector $\dot{\tilde\gamma}$ turns an angle of 2π going once around $\tilde\gamma$. Now note that when we integrate around the curvilinear polygon γ we are really summing the integrals along each of the edges of the polygon. So the integral on the left hand side of (3.10) only sees how much the tangent vector turns along the smooth parts of the curve, not along the corners. But $\dot{\tilde\gamma}$ turns more going once around $\tilde\gamma$, since it also turns around the corners of γ. If $\tilde\gamma$ is a very close approximation for γ, then the difference between the integral around $\tilde\gamma$ and γ is only due to these corner contributions, and (3.10) follows. (The rigorous argument goes the same way; see [3], Theorem 13.2.2.) This completes the proof. □

Corollary 3.11. *If the polygon is given by arcs of geodesics (which means $\kappa_g = 0$), then the internal angles satisfy the formula*

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \int_{\mathrm{int}(\gamma)} K\, dA.$$

For example, if we take a geodesic triangle on the unit sphere with interior angles α, β, γ we have $\alpha + \beta + \gamma = \pi + A$, where A is the area of the triangle.

Now that we can say something about the total Gaussian curvature of the part of a surface enclosed by a curvilinear polygon, we seek to cover a surface with adjacent curvilinear polygons. This brings us to the notion of triangulation, which we define in the next section.

4. Gauss-Bonnet Theorem for Compact Surfaces

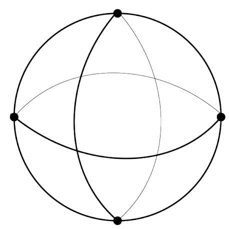
Definition 4.1. Let S be a surface, with atlas consisting of the patches $\sigma_i : U_i \to \mathbb{R}^3$. A *triangulation* of S is a collection of curvilinear polygons each of which is contained, together with its interior, in one of the $\sigma_i(U_i)$ such that
i) Every point of S is in at least one of the curvilinear polygons.
ii) Two curvilinear polygons intersect only at a common vertex or common edge.
iii) Each edge is an edge of exactly two polygons.

Definition 4.2. The Euler characteristic χ of a triangulation is given by

$$\chi = V - E + F,$$

where V, E, F denote the total number of vertices, edges and faces of the triangulation respectively.

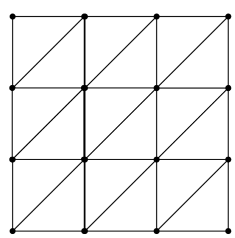
Example 4.3. By intersecting the sphere with the three coordinate planes, we obtain a triangulation with eight triangles.



Therefore,

$$\chi = 6 - 12 + 8 = 2.$$

Example 4.4. To triangulate the torus, we recall that we can think of the torus as the unit square with opposite sides identified. A triangulation of the square is shown in the figure below. When counting vertices and edges, we need to take into account that opposite edges are identified. For example, all four corners of the unit square correspond to the same vertex once we fold the square into a torus.



This triangulation gives $\chi = 9 - 27 + 18 = 0$.

To prove the Gauss-Bonnet theorem for compact surfaces, we use the following topological fact, the proof of which is beyond the scope of this paper.

Theorem 4.5. *Every compact surface has a triangulation with finitely many polygons.*

Now we can state the Gauss-Bonnet theorem for surfaces.

Theorem 4.6. *Let S be a compact surface. Then for any triangulation of S we have*

$$\int_S K \, dA = 2\pi\chi.$$

Proof. Clearly, the left hand side is independent of triangulation, so if we can prove the theorem for a particular triangulation, we have proved it for all possible triangulations. Take a triangulation of S consisting of curvilinear polygons P_i. Assume each of them is contained in a surface patch $\sigma_i : U_i \to \mathbb{R}^3$. Let R_i be such that $\sigma_i(R_i) = P_i$.

The main idea behind the proof is to apply Theorem 3.9 to each of the polygons in the triangulation. To do this, write

$$\int_S K\, dA = \sum_i \int_{R_i} K\, dA.$$

This is valid since the polygons of the triangulation only overlap at vertices and edges, so these overlaps do not affect the integral. Applying Theorem 3.9 to each of the terms on the right hand side, we get

$$\int_S K\, dA = \sum_i \int_0^{l(P_i)} \kappa_g\, ds + \sum_i b_i - \sum_i (n_i - 2\pi),$$

where b_i is the sum of the interior angles of P_i, and n_i is the number of vertices of P_i.

We claim the first term on the right is zero. To see this, note that each edge of the triangulation is traversed twice, once in each direction, since each edge is the edge of precisely two polygons. Additionally, the geodesic curvature changes sign when we traverse the curve in the opposite direction.

Next, $\sum_i b_i = 2\pi V$ because at each vertex of the triangulation, the sum of the interior angles must be 2π.

Finally, $\sum_i (n_i - 2)\pi = \pi \sum_i n_i - 2\pi F$. Since n_i is the number of vertices of P_i, it is also the number of edges. Each edge is counted twice, because of property iii) of the definition of a triangulation. This means we have $\sum_i n_i = 2E$ and this completes the proof. □

Example 4.7. If S is a sphere, we know $\chi(S) = 2$ and the Gauss-Bonnet theorem says

$$\int_S K\, dA = 4\pi,$$

which is in agreement with Example 2.18.

Example 4.8. If S is a torus, then $\chi(S) = 0$, so the Gauss-Bonnet theorem gives

$$\int_S K\, dA = 0.$$

In Example 2.16, we saw the torus has regions of both positive and negative Gaussian curvature; now we know these positive and negative contributions cancel each other out.

As of yet, we cannot apply the theorem to any more surfaces since we do not know any other Euler characteristics. Next, we will compute the Euler characteristic of T_g, a genus g torus.

Theorem 4.9. *$\chi(T_g) = 2 - 2g$.*

Proof. We know the result for $g = 0$ and $g = 1$. Now we proceed by induction. Suppose we know the result for g. We can obtain T_{g+1} by gluing a copy of T_1 onto T_g. We need to fix triangulations of T_g and T_1 so that we can remove an n-gon from each of them and attach them at the n-gon's boundary. From here, it is straightforward to compute how the vertices, faces and edges of the triangulations change. We removed one face, n vertices and n edges from the triangulations of both T_1 and T_g. So we have

$$\begin{aligned} V_{g+1} &= (V_g - n) + (V_1 - n), \\ E_{g+1} &= (E_g - n) + (E_1 - n), \\ F_{g+1} &= (F_g - 1) + (F_1 - 1). \end{aligned}$$

Hence,

$$\begin{aligned} \chi(T_{g+1}) &= V_{g+1} - E_{g+1} + F_{g+1} \\ &= (V_g - E_g + F_g) + (V_1 - E_1 + F_1) - 2 \\ &= 2 - 2g - 2 \\ &= 2 - 2(g+1). \end{aligned}$$ □

In the next section, we say much more about the Euler characteristic, which will add significance to the Gauss-Bonnet theorem.

5. Classification of Surfaces

In this section, we will see the Euler characteristic is invariant under homeomorphism. Moreover, if two oriented surfaces have different Euler characteristics, then they are not homeomorphic. To see this, we will show that for a compact oriented surface S, $\chi(S) = 2 - 2g$ is equivalent to $S \cong T_g$, where T_0 is S^2. Thus we have the remarkable result that any compact oriented surface is homeomorphic to one of the T_g.

For the proof, we will again rely on triangulations of surfaces. Fix a triangulation Δ of S. We will think of Δ as an undirected graph, since it has vertices and edges. Since graphs do not have faces, we have the following notion of Euler characteristic.

Definition 5.1. The Euler characteristic of a graph G is given by $\chi(G) = V - E$, where V is the number of vertices of G and E is the number of edges.

We will work with a spanning tree T of Δ. This means T is a subgraph of Δ which is a tree and contains all the vertices of Δ. The reader can easily verify the existence of such a tree for any connected graph. We prove a fact about trees which will be useful later.

Lemma 5.2. *For a connected graph G, we have $\chi(G) \leq 1$ with equality if and only if G is a tree.*

Proof. Suppose G is a tree. We will show $E = V - 1$. The result is obvious for $V = 1$ and $V = 2$. Suppose the property is true for all trees with fewer than n vertices. Take G to be a tree with n vertices. Choose an edge e connecting vertices v_1 and v_2. Since G is a tree, e is the only path joining v_1 and v_2. Therefore, removing e from G leaves two trees T_1 and T_2 with k and $n-k$ vertices respectively. By the inductive hypothesis, T_1 has $k-1$ edges and T_2 has $n-k-1$ edges. The edges of G consist of the edges of these two trees along with the edge e, totaling $n-1$.

Conversely, if G is an arbitrary graph with n vertices, then the number of edges must be at least $n-1$ for G to be connected.

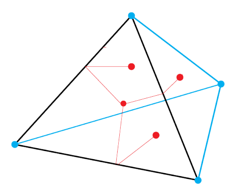
Now suppose we have a connected graph G with n vertices and exactly $n-1$ edges, and suppose G is not a tree, i.e. G contains cycles. Then we can delete edges from G until we are left with a tree. By the previous paragraph, this tree must also have $n-1$ edges, a contradiction. □

In the proof of our next lemma, we work with the dual graph Γ of T, defined as follows. We take the vertices of Γ to be the faces of Δ. We determine whether or not two vertices f_1 and f_2 of Γ are connected by an edge according to the rule

$$f_1 \leftrightarrow f_2 \iff f_1 \cap f_2 \nsubseteq T,$$

where f_1 and f_2 are the faces of Δ corresponding to f_1 and f_2 respectively, and $f_1 \cap f_2$ is the edge between f_1 and f_2. If f_1 and f_2 intersect trivially, then there is no edge between them.

We can represent Γ on the triangulation of S by associating each vertex to an interior point on a face of Δ and connecting the vertices by bent edges. This is illustrated in the figure below, where the tetrahedron can be regarded as a triangulation of a sphere. The tree T is shown in blue and its dual graph Γ is shown in red.



Each of the bent edges of Γ crosses an edge of Δ. By the construction of Γ, it is clear that none of these edges are in T. Therefore, we have a one-to-one correspondence between the edges of Γ and the edges of Δ that are not in T.

Using this fact, we can write

$$\begin{aligned}\chi(\Delta) &= V - E + F \\ &= V(T) - E(T) - E(\Gamma) + V(\Gamma) \\ &= \chi(T) + \chi(\Gamma) \leq 2,\end{aligned}$$

where the last inequality is an application of Lemma 5.2.

Recall that we want to show

$$\chi(S) = 2 - 2g \iff S \cong T_g.$$

The next lemma treats the case $g = 0$.

Lemma 5.3. *Given a compact orientable surface S, the following are equivalent:*

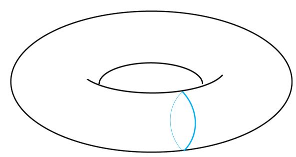
(1) Every simple closed curve separates S.
(2) $\chi(S) = 2$.
(3) $S \cong S^2$.

Proof. We begin by showing (1) $\implies$ (2). From the above discussion, we see showing $\chi(S) = 2$ amounts to showing Γ is a tree. Suppose not, i.e. suppose Γ contains a cycle. Then the representation of Γ on S discussed above corresponds to a simple closed curve γ on S. By assumption, γ separates S into two disconnected components. This means γ splits the vertices of Δ into two collections, but since T is a tree containing all these vertices, it must be that γ crosses an edge of T somewhere. This in turn implies that some edge of Γ crosses an edge of T, which is a contradiction. Therefore Γ is a tree.

To show (2) $\implies$ (3), we show that S is made up of two discs identified along their boundaries. Since $\chi(S) = 2$, Γ is a tree. So if we thicken both T and Γ, we obtain two regions on S that are each homeomorphic to discs. We can keep thickening each tree until their boundaries touch, so we indeed have two discs with identified boundaries.

Finally, (3) $\implies$ (1) follows from the Jordan curve theorem. For details see [1], Section 5.6. □

Before we continue, we consider why condition 1) cannot hold for T_g with $g > 0$. For simplicity, consider $g = 1$. Take a curve on the torus such as the blue curve in the figure below. This curve does not separate the torus, since if we cut along it, our new surface is still one connected component; it looks like a bent cylinder.



We see that the existence of non-separating closed curves on a surface has something to do with the surface having holes in it; so this property is closely related to the genus. It is this relationship that gives us an idea of how to proceed for $g > 0$. We cannot work with the genus of an arbitrary surface, since a priori this concept only makes sense for the T_g. But for any surface, we can ask whether or not a simple closed curve separates S, and this is the approach we use to prove the main theorem.

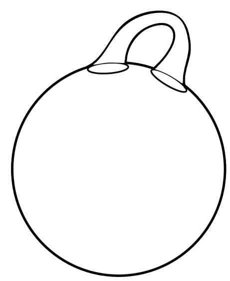
Theorem 5.4 (Classification of Surfaces)**.** *Every compact orientable surface is homeomorphic to the sphere or to one of the T_g.*

Proof. Given S, we ask whether or not every simple closed curve separates S. If yes, then by Lemma 5.3, S is homeomorphic to a sphere and $\chi(S) = 2$. If not, we take a closed curve which does not separate S and thicken it to get a cylinder. Now we cut along the boundary of the cylinder and remove it. Call this removed part A. Let $B = S \setminus A$. The reader can verify

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

The reader can also check we have $\chi(A) = 0$ by finding a triangulation of A. Hence $\chi(S)$ does not change after we remove the cylinder. Next we fill in the two holes obtained from cutting away A to obtain a new surface that we call S_1. Clearly, we can obtain a triangulation of S_1 by adding two faces to any triangulation of S. So we have $\chi(S_1) = \chi(S) + 2$.

Now we ask if every closed curve separates S_1. If yes, then $\chi(S_1) = 2$ and S_1 is homeomorphic to a sphere. This means $\chi(S) = 0$. Moreover, S is homeomorphic to T_1. To see this, we reverse the cutting and gluing process just performed. In other words, if we take out two discs from the surface of a sphere and glue in a cylinder, we have added a handle to our surface, so the new surface is homeomorphic to the genus 1 torus. This is perhaps better explained in the figure below.



If, on the other hand, every closed curve does not separate S_1, then we take a non-separating curve, thicken it to get a cylinder, remove the cylinder, and fill in the two holes. We get a new surface S_2 with $\chi(S_2) = \chi(S_1) + 2$.

We can continue cutting along non-separating curves and filling in the holes, each time increasing the resulting surface's Euler characteristic by 2. We can repeat this process a finite number of times g to get a surface S_g with $\chi(S_g) = 2$. This is clear since Theorem 4.5 implies $\chi(S)$ is finite for any compact surface S.

Then $2 = \chi(S_g) = \chi(S_{g-1}) + 2 = \chi(S_{g-2}) + 2 + 2$. Continuing to substitute in this manner, we get

$$2 = \chi(S_g) = \chi(S) + 2g,$$

which can be rearranged to give

$$\chi(S) = 2 - 2g.$$

We are left to show $S \cong T_g$. Note that to obtain S_g from S we removed g cylinders and filled in $2g$ circles. Now, remove $2g$ circles from S_g and add g handles, as we did for the case $g = 1$. $\square$

6. Singularities of Vector Fields

In this section, we will show how the Gaussian curvature and Euler characteristic of a surface relate to the tangent vector fields that can be defined on the surface.

Definition 6.1. If $\sigma(u, v) : U \to \mathbb{R}^3$ is a surface patch on S and (u, v) are coordinates on U, then

$$\mathbf{V} = \alpha(u, v)\sigma_u + \beta(u, v)\sigma_v$$

is called a *smooth tangent vector field* on S, where α and β are smooth functions.

Definition 6.2. If $\mathbf{V}$ is a smooth tangent vector field on S, a point $p \in S$ for which $\mathbf{V} = 0$ at p is called a *stationary point*.

Definition 6.3. Let p be a stationary point of $\mathbf{V}$ contained in a surface patch $\sigma : U \to \mathbb{R}^3$ of S. Assume p is the only other stationary point of $\mathbf{V}$ in the region $\sigma(U)$. Let ξ be a nowhere vanishing smooth tangent vector field on $\sigma(U)$ (such as σ_u or σ_v). The *multiplicity* of a stationary point p of the tangent vector field $\mathbf{V}$ is given by

$$\mu(p) = \frac{1}{2\pi} \int_0^{l(\gamma)} \frac{d\psi}{ds}\, ds,$$

where $\gamma(s)$ is any positively-oriented unit-speed simple closed curve of length $l(\gamma)$ in $\sigma(U)$ with p in its interior and $\psi(S)$ the oriented angle $\widehat{\xi\mathbf{V}}$ at the point $\gamma(s)$.

We have $\widehat{\xi\mathbf{V}}$ only defined up to multiples of 2π, but we can choose $\psi(s)$ to be a smooth function of s (for a proof see [2], Section 4-4, Lemma 1). Using arguments similar to those in the proof in Theorem 3.5, it can be shown that $\mu(p)$ is an integer and that it does not depend on the choice of curve γ.

Lemma 6.4. *The multiplicity $\mu(p)$ is independent of the choice of reference vector field ξ.*

Proof. Take ξ and $\tilde{\xi}$ to be two nowhere vanishing smooth vector fields on $\sigma(U)$. Let $\psi = \widehat{\xi\mathbf{V}}$ and $\tilde{\psi} = \widehat{\tilde{\xi}\mathbf{V}}$. Let $\theta = \tilde{\psi} - \psi$. We want to show

$$\int_0^{l(\gamma)} \dot{\theta}\, ds = 0, \tag{6.5}$$

but θ is only defined up to multiples of 2π. To avoid this ambiguity, we work with the function

$$\rho = \frac{\xi \cdot \tilde{\xi}}{\|\xi \cdot \tilde{\xi}\|} = \cos\theta.$$

Let β be the curve in U such that $\gamma(s) = \sigma(\beta(s))$. We then have

$$\begin{aligned}
\int_0^{l(\gamma)} \dot{\theta}\, ds &= \int_0^{l(\gamma)} \frac{\dot{\rho}}{\sqrt{1-\rho^2}}\, ds \\
&= \int_\beta \frac{\rho_u\, du + \rho_v\, dv}{\sqrt{1-\rho^2}} \\
&= \int_{\mathrm{int}(\beta)} \left(\frac{\partial}{\partial u}\left(\frac{\rho_v}{\sqrt{1-\rho^2}}\right) - \frac{\partial}{\partial v}\left(\frac{\rho_u}{\sqrt{1-\rho^2}}\right)\right) du\, dv,
\end{aligned}$$

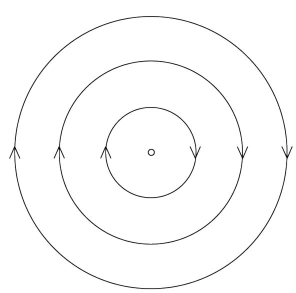
where the last equality is due to Green's theorem. The reader can check that

$$\frac{\partial}{\partial u}\left(\frac{\rho_v}{\sqrt{1-\rho^2}}\right) = \frac{\partial}{\partial v}\left(\frac{\rho_u}{\sqrt{1-\rho^2}}\right) = \frac{\rho_{uv}(1-\rho^2) + \rho\rho_u\rho_v}{(1-\rho^2)^{3/2}},$$

which shows the integral vanishes. □

The function μ is well-defined for any point of $\mathbf{V}$. If we wish to evaluate $\mu(p)$, for p which is not a stationary point, then by the above lemma we can take our non-vanishing reference vector field ξ to be $\mathbf{V}$. Then $\mu(p) = 0$ since $\psi(s) = 0$ for all s.

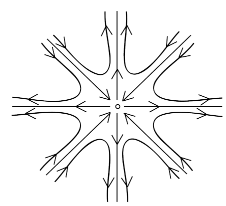
Example 6.6. Consider the vector field in $\mathbb{R}^2$ given by $\mathbf{V}(x, y) = (-y, x)$, which has a stationary point at the origin. In the figure below we show some integral curves of $\mathbf{V}$. This means the curve's tangent vector at (x, y) is given by $\mathbf{V}(x, y)$.



Take $\gamma(t) = (\cos t, \sin t)$ and $\xi = (1, 0)$ everywhere. At $t = 0$, the vector field is pointing straight down, and at $t = \pi/2$, it points to the right. So it is clear that it makes one counter-clockwise rotation as t runs from 0 to 2π. So $\mu(0) = +1$, which means the vector field makes one rotation in the positive direction (counter-clockwise). We can compute this from Definition 6.3. With γ and ξ as specified above, we have $\psi(s) = s - \pi/2$, so

$$\mu(0) = \frac{1}{2\pi} \int_0^{2\pi} \dot{\psi}\, ds = 1.$$

Example 6.7. Let $\mathbf{V}(x, y) = (x^2 - y^2, -2xy)$ which has a stationary point at the origin. Some integral curves are shown below.



Take γ and ξ as in the previous example. From looking at the picture, we can see the vector field rotates 4π in the clockwise direction when going around once counter-clockwise on γ, so $\mu(0) = -2$. We leave it to the reader to compute this from the definition.

We can now use the Gauss-Bonnet theorem to prove the Poincaré-Hopf theorem.

Theorem 6.8 (Poincaré-Hopf)**.** *Let* $\mathbf{V}$ *be a smooth tangent vector field on a compact surface* S *with finitely many stationary points* $p_1, ...p_n$*. Then*

$$\sum_{i=1}^{n} \mu(p_i) = \chi(S).$$

The quantity on the left is called the *index* of $\mathbf{V}$.

Proof. For each i, take γ_i to be a simple closed curve in a surface patch σ_i containing p_i in its interior. Choose the γ_i so that their interiors are disjoint. Let S' denote the part of S outside the interiors of the γ_i. Triangulate S' with curvilinear polygons Γ_j. Note that some of the edges of these polygons will be segments of the curves γ_i. Furthermore, a positive orientation of the Γ_j induces a negative orientation on the γ_i.

Now we can regard the Γ_j along with the γ_i and their interiors as a triangulation of S. By the Gauss-Bonnet theorem,

$$\int_{S'} K\, dA + \sum_{i=1}^{n} \int_{\operatorname{int}(\gamma_i)} K\, dA = 2\pi\chi(S).$$

Next we choose an orthonormal basis $\{F_1, F_2\}$ of the tangent plane of S on each patch σ_i. We can do this by taking $F_1 = (\sigma_i)_u$ and $F_2 = N \times F_1$, for instance. As in the proof of Theorem 3.5, we have

$$\int_{\operatorname{int}(\gamma_i)} K\, dA = \int_0^{l(\gamma_i)} F_1 \cdot \dot{F}_2\, ds. \tag{6.9}$$

On S', choose a smooth orthonormal basis $\{E_1, E_2\}$ of the tangent plane by letting $E_1 = \mathbf{V}/\|\mathbf{V}\|$ and $E_2 = E_1 \times N$. The definition of E_1 makes sense since by construction all the stationary points of $\mathbf{V}$ are not in S'. As in the proof of Theorem 3.5, we have

$$\int_{S'} K\,dA = \sum_j \int_0^{l(\Gamma_j)} E_1 \cdot \dot{E}_2\,ds.$$

Any common edge of two curvilinear polygons is traversed once in each direction, so its contributions to the sum on the right hand side cancel out. Thus, the only contribution is from the integrals along the γ_i, giving

$$\int_{S'} K\,dA = -\sum_{i=1}^{n} \int_0^{l(\gamma_i)} E_1 \cdot \dot{E}_2\,ds, \tag{6.10}$$

where the negative sign is in light of the comment about orientation made in the first paragraph of the proof. Combining (6.9) and (6.10), we obtain

$$2\pi\chi(S) = \int_S K\,dA = \sum_{i=1}^{n} \int_0^{l(\gamma_i)} F_1 \cdot \dot{F}_2 - E_1 \cdot \dot{E}_2\,ds.$$

As in the proof of Theorem 3.5, we have

$$F_1 \cdot \dot{F}_2 = \dot{\phi} - \kappa_g, \quad E_1 \cdot \dot{E}_2 = \dot{\theta} - \kappa_g,$$

where κ_g is the geodesic curvature of γ_i, θ and ϕ are the oriented angles $\widehat{E_1\dot{\gamma}_i}$ and $\widehat{F_1\dot{\gamma}_i}$ respectively. Letting $\psi = \phi - \theta$, we have

$$2\pi\chi = \sum_{i=1}^{n} \int_0^{l(\gamma_i)} \frac{d\psi}{ds}\,ds. \tag{6.11}$$

Note that ψ is the oriented angle $\widehat{F_1 E_1}$. Noting that E_1 is parallel to $\mathbf{V}$ and calling F_1 our reference vector field, we see that ψ is precisely as in Definition 6.3. Dividing both sides of (6.11) by 2π concludes the proof. $\square$

This is a remarkable result, because it seems as though the index should depend on $\mathbf{V}$ but instead it depends only on the topology of S. This theorem allows us to compute the Euler characteristic of a surface by considering the singularities of any smooth tangent vector field on the surface. Perhaps more interestingly, it allows us to say something about the shape of a surface on which we can define certain kinds of vector fields.

Corollary 6.12 (Hairy Ball Theorem)**.** *There does not exist a smooth nowhere vanishing tangent vector field on a sphere.*

Corollary 6.13. *If S admits a smooth nowhere vanishing tangent vector field, then S is homeomorphic to T_1.*

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