דוגמאות: יהי $X = C(I)$

בהינתן פונקציה $g(t)$

אינטגרבילית $\int_I |g(t)|\,dt < \infty$ (א)

נגדיר פונקציונל ליניארי

$$f \in C(I) \mapsto \varphi(f) = \int_I \underbrace{g(t)}\, f(t)\,dt$$

"משקולות"

$$\delta_{t_0}(f) = f(t_0) \quad \xrightarrow{\text{פונקציונל דלתא}} \quad \forall\, t_0 \in I \qquad \text{(ב)}$$

וריאציה $\varphi(t)$ $\overline{\int_I}$

$$\mathrm{var}_I\, f(x) = \sup_{T} \sum_j |\Delta f_j|$$

על כל החלוקות

הקבוצה של כל f שהוא בעלת ...

$\varphi \in C(I)^*$

$\varphi(f) = \int_I f(t)\, d\rho(t)$

$\sum f(x_i)\, \Delta \rho_i$

$\|\varphi\| = \mathrm{var}_I\, \rho$

X^*

$f_n * f$

$\|\rho\|$

[illegible]

$$(H)^* \cong H$$

$$\left(C[a,b]\right)^* \cong V = \{\varphi \mid \operatorname{var}\varphi < \infty\}$$

$$\|\varphi\|_{C^*} = \|\varphi\|_V$$

$$l_p^* \cong l_q \qquad \frac{1}{p}+\frac{1}{q}=1 \qquad 1 \le p < \infty$$

$$l_\infty \supset c_0 \qquad (c_0)^* \cong l_1$$

[illegible]

[illegible]

$(X, \|\ \|)$ [illegible] $\Longrightarrow$ $X^* = \{\varphi\}$ [illegible]

[illegible] X^*

(1) $\varphi_n \xrightarrow{n\to\infty} \varphi$ [illegible]

$\Longleftrightarrow$ $\|\varphi_n - \varphi\| \xrightarrow{n\to\infty} 0$ [illegible]

(2) $\varphi_n \xrightarrow{w} \varphi$ $\Longleftrightarrow$ $\forall x \in X$

$\varphi_n(x) \xrightarrow{n\to\infty} \varphi(x)$

[illegible]

[illegible]

(1) $\ell_2 = \{(x_1, \dots) \mid \sum |x_j|^2 < \infty\}$

$f_k(x) = x_k = \langle x, e_k \rangle$

(i) $f_k \xrightarrow{w} 0$

$f_k(x) = x_k \xrightarrow{k\to\infty} 0 = 0(x)$

(ii) $f_k \not\to 0$

$\|f_k - 0\| = \|e_k\| = 1 \not\xrightarrow{k} 0$

[illegible]

$|f_n(x) - f(x)| \le \|f_n - f\| \, \|x\| \to 0$

$X = C[-1,1]$

$\delta_n(f) = f(\tfrac{1}{n})$ $\qquad (\gamma = \delta_{\frac{1}{n}})$

(i) [illegible]

$\delta_n(f) = f(\tfrac{1}{n}) \xrightarrow[n\to\infty]{} f(0) = \delta_0(f)$

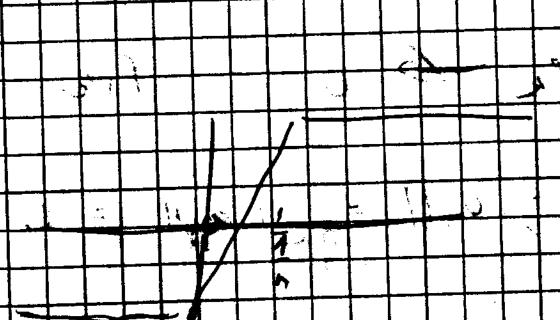
$\Downarrow$

$\delta_{\frac{1}{n}} \xrightarrow{w^*} \delta_0$

(ii) [illegible]

? $\quad \delta_{\frac{1}{n}} \longrightarrow \delta_0$

$\|\delta_{\frac{1}{n}} - \delta_0\| = 2 \not\longrightarrow 0$

f_n  $\Rightarrow \{\delta_{\frac{1}{n}}\}$ [illegible]

$\|f_n\| = 1$

$(\delta_{\frac{1}{n}} - \delta_0)(f_n) = 2$

[illegible] X^* [illegible]

$\dim X < \infty \Rightarrow$

[illegible]

[illegible] $\{v_1, \dots, v_m\}$ [illegible] X

[illegible] $f_n \xrightarrow{w} f$ [illegible]

$$f_n(v_i) \longrightarrow f(v_i)$$

[illegible] $X \to \mathbb{R}^n$

$$\sum \alpha_i v_i \mapsto (\alpha_1 \dots \alpha_m)$$

$$0 < k < \frac{\|\sum_{i=1}^{m} \alpha_i v_i\|}{\|(\alpha_1 \dots \alpha_m)\|} < M$$

$\Leftarrow$ [illegible] $x = \sum_{i=1}^{m} \alpha_i v_i \quad \|x\| = 1$

[illegible] $k \sum_{i=1}^{m} |\alpha_i|^2 \le 1$

[illegible] $\|x\| = 1$

$$|f_n(x) - f(x)| = \left|\sum_{i=1}^{m} \alpha_i [f_n(v_i) - f(v_i)]\right|$$

$$\le \sum_{i=1}^{m} |\alpha_i| \, |f_n(v_i) - f(v_i)| \le m \frac{1}{k} \sum_{i=1}^{m} |f_n(v_i) - f(v_i)|$$

$$\Downarrow$$

$$\|f_n - f\| \xrightarrow{n \to \infty} 0$$

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$X = C[-1,1]$

$$\varphi_n(f) = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)\,dt$$

$$= \int_{-1}^{1} a_n(t)\, f(t)\, dt$$

$$a_n = \frac{n}{2}\, \mathbb{1}_{[-\frac{1}{n}, \frac{1}{n}]}$$

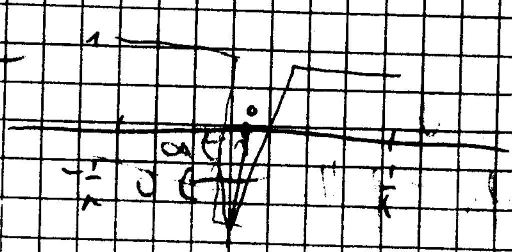
$$\int_{-1}^{1} |a_n(t)|\,dt = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}\,dt = 1$$

(i)

$$\varphi_n(f) = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)\,dt \xrightarrow{n\to\infty} f(0) = \delta_0(f) \quad \Rightarrow \quad \varphi_n \xrightarrow{w^*} \delta_0$$

(ii)

$$\|\varphi_n - \delta_0\| = \|a_n(t)\| + 1 = 2 \not\to 0$$



$f(t) = \begin{cases} 1 \\ \dots \end{cases}$

$\varphi_n \to \delta_0$ [illegible]

$C(I)$ [illegible]

$$\varphi_n(f) = \int_I a_n(t)\, f(t)\, dt$$

$$\|\varphi_n - \delta_0\| = \|a_n(t)\| + 1 \not\to 0$$

[illegible] $(X$ [illegible]$)$

$\{\varphi_n\} \subseteq X^*$ [illegible]

$\exists K : \forall n \quad \|\varphi_n\| \le K$ (1)

$\exists Y \subset X : \forall y \in Y \ \exists \lim \varphi_n(y)$ (2)

$\varphi_n \xrightarrow{w} \varphi : \varphi \in X^*$ [illegible]

[illegible] $\{\varphi_n(x)\}$ [illegible] $x \in X$ [illegible]

[illegible] $\varepsilon > 0$ [illegible] $y \in Y$ [illegible] $\|x - y\| < \varepsilon$

$$|\varphi_n(x) - \varphi_m(x)| \le |\varphi_n(x) - \varphi_n(y)| + |\varphi_n(y) - \varphi_m(y)| + |\varphi_m(y) - \varphi_m(x)|$$

$$\le \|\varphi_n\|\,\|x - y\| + |\varphi_n(y) - \varphi_m(y)| + \|\varphi_m\|\,\|x - y\|$$

$$\le K\varepsilon + \varepsilon + K\varepsilon = \varepsilon(2K + 1)$$

$\forall x \in X \quad \exists \lim_{n\to\infty} \varphi_n(x)$

$\Downarrow$

$f(x) := \lim_{n\to\infty} \varphi_n(x)$ נגדיר

[illegible] $\{\varphi_n\}$ [illegible] f [illegible]

[illegible]

[illegible]

$\forall x \in X \quad |\varphi_n(x)| \le K \|x\| \quad (*)$

$n\to\infty$

$\forall x \in X \quad |f(x)| \le K \|x\|$

$\Downarrow$

$\|f\| \le K$

$\Downarrow$

f [illegible]

[illegible] $(X$ [illegible] [illegible]

$\|\varphi_n\| \le K \Leftarrow \varphi_n \xrightarrow{w} f$ אם

$$L^2(\mathbb{T}) = \left\{ f \in L^2[-\pi,\pi] \;\middle|\; f(-\pi) = f(\pi) \right\}$$

$$f(t) \overset{\text{"="}}{=} \sum_{k\in\mathbb{Z}} \hat{f}(k)\, e^{ikt}$$

$\{e^{ikt}\}$

$$\hat{f}(k) = \langle f, e^{ikt} \rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$$

שאלה: אם $f \in L^2(\mathbb{T})$ רציפה

האם f רציפה בנק' ...

דוגמא: ... f ... רציפה ...

תשובה: ... f ... $S_N(f)$ ...

$$S_N(f)(0) = \sum_{|k|\le N} \hat{f}(k)\, e^{ik\cdot 0}$$

$$= \sum_{|k|\le N} \hat{f}(k) = \sum_{|k|\le N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \left(\sum_{|k|\le N} e^{-ikt} \right) dt$$

$D_N(t)$ = גרעין דיריכלה

$$S_N(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_N(t)\, dt$$

$S_N(f)(0)$ — [illegible]

[illegible] $C(\mathbb{T})$

$(\mathbb{T})$ [illegible]

$$S_N(f)(0) \xrightarrow{N\to\infty} f(0)$$

[illegible] f [illegible] $\left(\sum_{|k|\le N} \hat{f}(k)\, e^{ik0}\right)$

[illegible]

[illegible] $C(\mathbb{T})$

[illegible] $\|S_N\| \le K$ [illegible]

[illegible] $f \in C(\mathbb{T})$ [illegible]

[illegible] $\|S_N\| \le K$ [illegible]

[illegible] $f \in C(\mathbb{T})$

$$\Leftarrow \quad \|S_N\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\, dt$$

$$\varphi(f) = \int_{-\pi}^{\pi} g(t) f(t)\, \frac{dt}{2\pi}$$

$C(\mathbb{T})$

$$\|\varphi\| = \int_{-\pi}^{\pi} |g(t)|\, \frac{dt}{2\pi}$$

8) $\|S_N(\cdot)(0)\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\,dt$

$$D_N(t) = e^{-iNt}\left(1 + e^{it} + e^{2it} + \dots + e^{2Nit}\right)$$

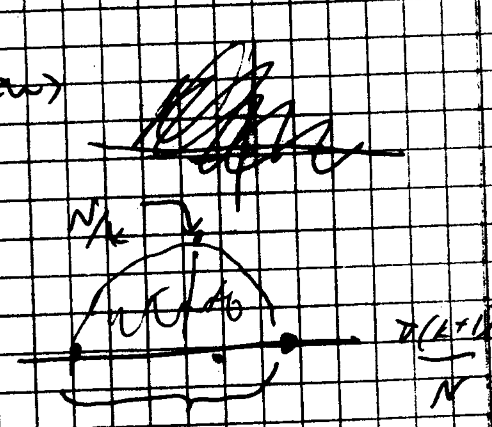
$$= e^{-iNt}\,\frac{1-(e^{it})^{2N+1}}{1-e^{it}}$$

$$= \dots = \frac{\sin\left(N+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$|D_N(t)| =$

[illegible]

$D_N(t) \approx \frac{\sin Nt}{t}$ [illegible]



$\frac{1}{k} \sim \frac{N}{\pi k} \cdot \frac{\pi}{N} \cdot \frac{N}{k}$ $\approx$

[illegible] $\frac{\pi}{N}$

[illegible]

$\Rightarrow \quad |D_n(t)| \approx \sum_{k=1}^{N} \frac{1}{k} \xrightarrow{n\to\infty} \infty$

$\| D_n(t) \| \approx \log N$

$$\| S_N(\cdot)(0) \| = \frac{1}{2\pi} \int_{-\pi}^{\pi} | D_n(t) | \, dt \geq C \int_{-\pi}^{\pi} \log N \, dt$$

$$= c \log N \xrightarrow{N \to \infty} \infty$$

משפט יהי $(X, \|\cdot\|)$ מרחב נורמי ספרבילי

יהי $\{\varphi_n\} \subseteq X^*$ סדרה חסומה

$\|\varphi_n\| \le k$

אז יש תת סדרה $\{\varphi_{n_k}\}$

ו- $\varphi \in X^*$ כך ש-

$$\varphi_{n_k} \xrightarrow{w^*} \varphi$$

הוכחה X ספרבילי $\Rightarrow$ קיימת $\{x_i\}_{i=1}^{\infty}$ צפופה ב-X

$\Downarrow$

$\{\varphi_n(x_1)\}$ סדרה חסומה של מספרים

$|\varphi_n(x_1)| \le \|\varphi_n\| \|x_1\| \le k\|x_1\|$

$\Downarrow$

$\exists\, \varphi_{n^1_k}$ כך ש- $\{\varphi_{n^1_k}(x_1)\}$ מתכנסת

~~נסמן $\{\varphi_{n^1_k}\}$ ...~~

נסתכל על $\{\varphi_{n^1_k}(x_2)\}$ ...

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[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

27.5.13

[illegible]

$f_n \xrightarrow{w} f$

[illegible] $\forall x \in X$ $f_n(x) \to f(x)$

[illegible] H [illegible]

$\exists\, a_n \in H$. $f_n(x) = \langle x, a_n \rangle$

[illegible]

$\forall x \in H$ $\langle x, a_n \rangle \xrightarrow{n\to\infty} \langle x, a \rangle$

$f_n(x)$ $f(x)$

$\forall x \in H$ $\langle x, a_n \rangle \to \langle x, a \rangle$ [illegible] $a_n \xrightarrow{w} a$ [illegible]

[illegible] $\|a_n - a\|_H \xrightarrow{n\to\infty} 0$ [illegible]

[illegible]

[illegible] H [illegible]

$\{a_n\} \subset H$ $\|a_n\| \le k$

[illegible] $\{a_{n_j}\}$ $\exists\, a_{n_j} \xrightarrow{w} a \in H$

יהי H מרחב, $\{e_n\}$ מערכת א"נ

צ"ל: $e_n \xrightarrow{w} 0$

בסל: $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2 \quad \forall x \in H$

$e_n \to 0$ כלומר

בפרט ש"ל ההכרח $\Downarrow$

$\forall x \quad \langle x, e_n\rangle \longrightarrow 0 = \langle x, 0\rangle$

אבל: $\|e_n - e_m\|^2 = 2 \not\longrightarrow 0$

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משפט (Hahn-Banach)

יהי $(X, \|\,\|)$ מרחב נורמי

$Y \subset X$ ת"מ

$f \in Y^*$ (f לינארי רציף על Y)

$\|f\|_{Y^*} < \infty$

אזי קיים $\tilde{f} \in X^*$ כך ש:

$\tilde{f}\big|_Y = f$ וגם

$\|\tilde{f}\|_{X^*} = \|f\|_{Y^*}$ (2)

הוכחה: אם $Y = X$ סיימנו

אחרת נבחר $z \in X \setminus Y$

נגדיר $Y_1 = \operatorname{span}(\{z\} \cup Y)$

נסמן $\|f\|_{Y^*} = \gamma$

$\forall\, y \in Y \quad |f(y)| \le \gamma \|y\|$

[illegible] f, y, ψ_1 [illegible]

$y + \lambda z \in Y_1$ [illegible]

$y \in$

$\psi_1(y + \lambda z) = f_1(y) + \lambda f_1(z)$

$= f(y) + \lambda f_1(z)$ [illegible]

[illegible]

$f_1(z) = ?$ [illegible]

$\forall y \in Y \quad \lambda \in \mathbb{R}$ [illegible]

$|f_1(y + \lambda z)| \le \gamma \, \| y + \lambda z \|$

~~[illegible]~~

$\lambda = 0$ [illegible] $\lambda \ne 0$

$f_1 = ?$

$|f(y + \lambda z)| \le \gamma \, \| y + \lambda z \|$

$|\lambda| \, |f(z + \frac{y}{\lambda})| \qquad |\lambda| \, \gamma \, \| z + \frac{y}{\lambda} \|$

$\Updownarrow$

$|f(z + \frac{y}{\lambda})| \le \gamma \, \| z + \frac{y}{\lambda} \|$

[illegible] $\frac{y}{\lambda} \in Y$ [illegible]

$\forall y \in Y$

$|f(z+y)| \le r\|z+y\|$

$\Updownarrow$

$-r\|z+y\| \le f(z)+f(y) \le r\|z+y\|$

$\Updownarrow$

$-f(y) - r\|z+y\| \le f(z) \le -f(y) + r\|z+y\|$

$B=\{b\}$, $A=\{a\}$

$\forall a \ \forall b$, $a \le c \le b$

$\sup A \le \inf B$

$\forall a \in A,\ b \in B$ $\quad a \le b$

$\forall y_1, y_2 \in Y$

$-f(y_1) - r\|z+y_1\| \le -f(y_2) + r\|z+y_2\|$

$f(y_2-y_1) \le r\left(\|z+y_1\| + \|z+y_2\|\right)$

$f(y_2-y_1) \le r\|y_2-y_1\| = r\|y_2+z-(y_1+z)\|$

$\le r\left(\|y_2+z\| + \|y_1+z\|\right)$

$f(z) = 0$

$|f(z+y)| \le \gamma \|z+y\| \quad \forall y \in Y$

Y_1 f

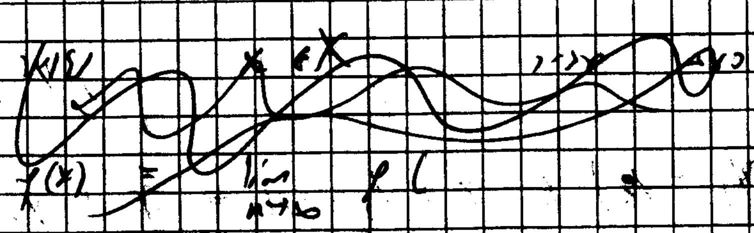
X

$X = \overline{\operatorname{span}}\{z_i\}_{i=1}^{\infty}$

$Y_1 = \operatorname{span}(Y, z_1)$ f

$Y_2 = \operatorname{span}(Y_1, z_2)$ f

$X_1 = \operatorname{span}(Y, z_1, z_2, \dots)$ f



[illegible] x_0 [illegible] 99

[illegible] $x \in X$ [illegible]

[illegible] $\{x_n\}$ $x_n \xrightarrow{n\to\infty} x$

[illegible] $f(x) = \lim_{n\to\infty} f(x_n)$

[illegible]

$$|f(x_n) - f(x_m)| \le \gamma \, \|x_n - x_m\| \longrightarrow 0$$

[illegible]

[illegible] $|f(x_n)| \le \gamma \, \|x_n\|$

$$\Downarrow$$

$$|f(x)| \le \gamma \|x\|$$

[illegible]

[illegible] T f X

[illegible]

[illegible] X [illegible] $\Theta(x)$ [illegible]

[illegible] $\neq$ [illegible]

[illegible]

דוגמא

$Y \subset X$ [illegible] $X - 1$

$Y \subset X$, $\overline{Y} \neq X$ [illegible]

$$f|_Y \equiv 0$$

דוגמא [illegible] $x \in X \setminus \overline{Y}$ [illegible] $f(x) \neq 0$ $\quad f(y) = 0$ $\forall y \in Y$

y [illegible] $\overline{Y} \neq \operatorname{span} X$ $\quad \|f\| = 1$

[illegible] X

דוגמא [illegible] ℓ_p [illegible]

$y_1 = (1, 1, 0, \dots, 0, \dots)$

$y_2 = (0, 1, 1, 0, \dots)$ $\qquad Y = \operatorname{span}\{y_i\}$

$y_3 = (0, 0, 1, 1, 0, \dots)$

$\left(\overline{\operatorname{span}\{y_i\}} = \ell_p \right)$ [illegible] $\quad p = 1,\ 1 < p < \infty,\ p = \infty$

$[(1, 0, 0, \dots)$ [illegible] $]$

דוגמא ℓ_1: $f(x) = \sum \alpha_k x_k$ $\quad \alpha_k = (-1)^k$

$\alpha \in \ell_\infty$

$\forall i \quad f(y_i) = 0$

$$\Downarrow$$

$$f|_{Y = \operatorname{span}\{e_i\}} \equiv 0$$

$f(e_1) = 1 \neq 0$ $\Leftarrow$ Y [illegible] e_1

5 [illegible] $\mathrm{span}_{\mathbb{R}}\{v_i\}$ [illegible] ℓ_∞

[illegible] $\mathrm{span}_{\mathbb{Q}}\{v_i\}$ [illegible]

ℓ_∞ [illegible]

[illegible]

$\alpha \in \ell_q$ $\quad f(\alpha) = \sum \alpha_k x_k$ [illegible] $1<p<\infty$ $\quad \ell_p$ [illegible]

[illegible]

$$\Downarrow$$

$$f(v_i) = 0 \Rightarrow x_k + x_{k+1} = 0$$

$$\Leftrightarrow \exists t \quad x_k = (-1)^k t$$

$$\Downarrow$$

$$\alpha \notin \ell_q \Leftarrow \sum_{k=1}^{\infty} |\alpha_k|^q = \sum_{k=1}^{\infty} t^q = \infty$$

$\Leftarrow$ [illegible] ℓ_p

מסקנות ממשפט האן-בנך

יהי x_0 ... (הוכחה)

$U = \{x \mid \|x\| \le 1\}$

נניח $x_0 \notin U$

נגדיר $\varphi(x_0) = \|x_0\|$

$\Downarrow$

$\varphi(\lambda x_0) = \lambda \|x_0\|$

כאשר $Y = \mathrm{span}\{x_0\}$

$\Downarrow$

$\varphi \in Y^*$, $\|\varphi\| = 1$

$\Downarrow$

לפי משפט האן-בנך קיים $\tilde{\varphi} \in X^*$ $\|\tilde{\varphi}\| = 1$

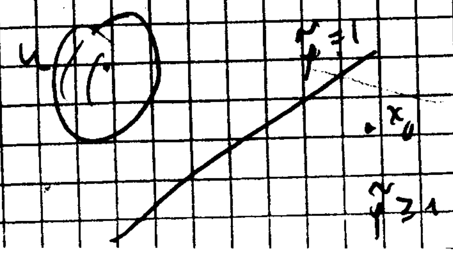
$\Downarrow$

$$\begin{cases} \tilde{\varphi}(x_0) = \varphi(x_0) = \|x_0\| > 1 \\ x \in U \quad |\tilde{\varphi}|_U(x)| \le \|\tilde{\varphi}\| \|x\| = \|x\| \le 1 \end{cases}$$

$\Downarrow$

$|\tilde{\varphi}|_U| \le 1$

$\tilde{\varphi} \le 1$



ציור

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הגדרה: יהיו X Y מרחבים נורמיים

$A: X \to Y$ לינ'

נגדיר נורמה

הגדרה: $\|A\|_{X\to Y} = \sup_{\|x\|\le 1} \|Ax\|$

$\iff \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$

~~$\|Ax\| \le \|A\|\|x\|$~~

טענה: אם A רציף $\to$ אז $\|A\|<\infty$

לכן: $\|A\|<\infty \Leftarrow A$ רציף

הערה:

מסקנה: A רציף $\iff \|A\|<\infty$

$\{e_n\}$ H

$\{\lambda_n\} \subseteq \mathbb{C}$

$A : H \to H$

$$x = \sum_{k=1}^{\infty} c_k e_k \mapsto \sum_{k=1}^{\infty} \lambda_k c_k e_k \qquad \left(\sum_{k=1}^{\infty} |c_k|^2 < \infty \right)$$

$$\sum_{k=1}^{\infty} |\lambda_k c_k|^2 < \infty$$

$\{\lambda_k\}$

$$r = \sup_n |\lambda_n| < \infty$$

$\sup |\lambda_n| < \infty \Rightarrow$

$\|A\| = r$

$x = \sum_{k=1}^{\infty} c_k e_k$

$\|x\| = \left(\sum |c_k|^2 \right)^{1/2}$

$\|Ax\| = \left(\sum |c_k \lambda_k|^2 \right)^{1/2}$

$\Downarrow$

$\|Ax\| \leq r \|x\|$

$\|A\| \leq r$

$n < \lambda_{k_n}$

$c_{k_n} = \frac{1}{n}$

$\sum |c_{k_n}|^2 < \infty$

$\sum \lambda_{k_n} c_{k_n} > \sum \frac{1}{n} = \infty$

$x_i = e_i$

$\|Ax_i\| = |c_i \lambda_i| \leq \lambda_i |c_i| = |\lambda_i| \|x_i\|$

$\lambda(t) \in C(I)$ [illegible]

$A: C(I) \to C(I)$ [illegible]

$$f(t) \mapsto \lambda(t) f(t)$$

γ $\|A\|$ [illegible]

$$\forall f \quad \|A(f)\| = \sup_{t \in I} |\lambda(t) f(t)|$$

$$\leq \max_{t \in I} |\lambda(t)| \cdot \max_{t \in I} |f(t)|$$

$$= \gamma \cdot \|f\|$$

$$\Downarrow$$

$$\|A\| \leq \gamma$$

$f \equiv 1$ [illegible]

$$\|Af\| = \|\lambda\| = \gamma = \|\gamma\| \, \|f\|$$

$$\Downarrow$$

$$\|A\| = \gamma = \|\lambda\|$$

$A: L^2(I) \to L^2(I)$ [illegible]

$\|A\|$ [illegible] A

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$$\|\lambda f\|^2 = \int_I |\lambda f|^2\,dt$$

$$\le \left(\max_{t\in I} |\lambda(t)|\right)^2 \int_I |f(t)|^2\,dt$$

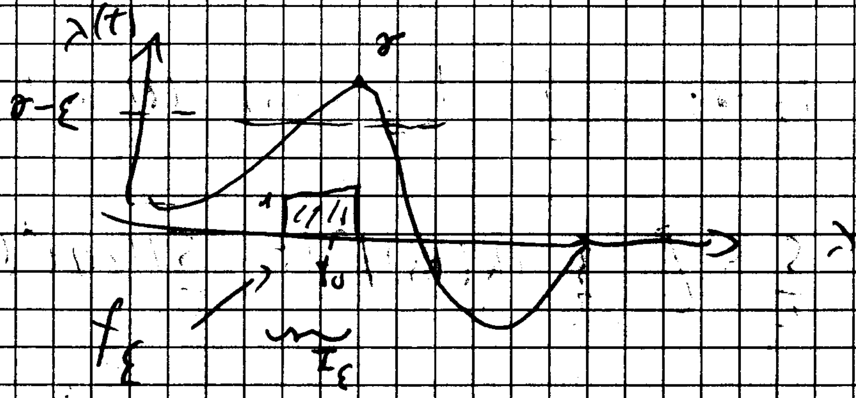
$$= r^2 \|f\|^2$$

$$\Downarrow$$

$$\|\lambda f\| \le r \|f\|$$

$$\Downarrow$$

$$\|A\|_{L^2(I)\to L^2(I)} \le r$$



$$\|A f_\varepsilon\|^2 = \int_{I_\varepsilon} |\lambda(t) f_\varepsilon(t)|^2\,dt$$

$$\ge (r-\varepsilon)^2 \int_{I_\varepsilon} |f_\varepsilon(t)|^2\,dt$$

$$= (r-\varepsilon)^2 \|f_\varepsilon\|^2$$

$\|A\| = r$ [illegible] ε [illegible]

$L^2(I)$

$I=[0,1]$

$K \in C(I\times I) \qquad K: I\times I \to \mathbb{F}$

Lemma

$T: L^2(I) \to L^2(I)$

$f \mapsto T(f)$

$$T(f)(s) = \int_I k(t,s)\, f(t)\, dt$$

$$T(f)(s) = y(s)$$

$$\|y\|^2 = \int_I \left[\left| \int_I k(t,s)\, f(t)\, dt \right|^2 \right] ds$$

$$\le \int_I \left[\int_I |k(t,s)|^2\, dt \int_I |f(t)|^2\, dt \right] ds$$

$$= \int_I \left[\int_I |k(t,s)|^2\, dt \right] ds\, \|f\|^2$$

$$= \underbrace{\iint_{I\times I} |k(t,s)|^2\, dt\, ds}_{\gamma^2}\, \|f\|^2 = \gamma^2 \|f\|^2$$

$$\|Tf\| \le \gamma \|f\|$$

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$$\|J\| \le \sqrt{\iint_{I\times I} |k(t,s)|^2\, dt\, ds}$$

$\|J\| = 1$

$L^2[-\pi, \pi]$

2π

$k(t,s) = \cos(t-s)$

$\|J\| \le \sqrt{}$

$\forall f \in C(I)$

$$\|J(f)\|^2 = \int_I \left| \int_I \cos(t-s) f(t)\, dt \right|^2 ds$$

$$\le \int_I \left(\int_I |f(t)|\, dt \right)^2 ds$$

$$= \left(\int_I |f(t)|\, dt \right)^2$$

$$\le \int_I |f(t)|^2\, dt = \|f\|^2$$

$$\Downarrow$$

$$\|J\| \le 1$$

$$\int_I \int_I k \cos(t-s)\, dt\, ds \to L$$

3.6.13

[illegible] על אופרטורים

X^* [illegible] (מרחב על X)

$$O_p(X) := \{\ A: X \to X \quad \text{אופרטורים חסומים}\ \}$$

$O_p(X)$ הוא מרחב וקטורי.

$$\forall A \in O_p(X): \ \|A\|_{op} = \sup_{0 \neq f \in X} \frac{\|Af\|}{\|f\|} \quad \text{נורמה.}$$

$O_p(X)$ מרחב נורמי.

? [illegible]

(a) 105

[illegible] $Op(X)$ $\to$ [illegible]

[illegible] $\{A_n\}$ [illegible]

$$\|A_n - A_m\|_{op} \xrightarrow{n,m\to\infty} 0$$

[illegible] A [illegible]

$$\|A_n - A\| \xrightarrow{n\to\infty} 0$$

[illegible] A: $Af = ?$

[illegible] $\{A_n f\} \subset X$ [illegible]

[illegible] $\{A_n f\}$ [illegible]

$$\|A_n f - A_m f\| = \|(A_n - A_m) f\|$$

$$\leq \|A_n - A_m\|_{op} \|f\| \xrightarrow{n,m\to\infty} 0$$

$$\Downarrow$$

[illegible] $Af := \lim_n A_n f$

$A \in Op(X)$

(i) A [illegible]

(ii) A [illegible]

[illegible]

$\exists k : \forall n \quad \|A_n\| \le k$

$\Downarrow$

$\|A_n f\| \le k \|f\|$

$\downarrow$

$\|Af\|$

$\Downarrow$

$\forall f \quad \|Af\| \le k \|f\|$

$\Downarrow$

[illegible] $\|A\| \le K$

$\varphi > \quad A_n \longrightarrow A$

[illegible] $\varepsilon > 0$ [illegible]

$\exists N : \forall n, m > N \quad \|A_n - A_m\| < \varepsilon$

$\Downarrow$

$\|A_n f - A_m f\| \le \varepsilon \|f\|$

$\downarrow m \to \infty$

$\|A_n f - A f\| \le \varepsilon \|f\|$

$\Downarrow$

$\forall n > N \quad \|A_n - A\| \le \varepsilon$

$Hom(X \to Y) = \{$ [illegible] $X \to Y \}$

[illegible]

[illegible] Y [illegible] $Hom(X \to Y)$ [illegible]

[illegible]

[illegible] X^* נקרא המרחב הדואלי של X

$$X^* = Hom(X \to \underset{\mathbb{R}}{\mathbb{C}})$$

[illegible]

108 ، A_n : X [illegible] $O_p(X)$ ، [illegible]

(1) $\|A_n - A\|_{op} \to 0$

[illegible] [illegible]

(2) $\forall f$: $A_n f \to A f$

$\left(\|A_n f - Af\|_X \to 0\right)$ [illegible] [illegible]

[illegible] [illegible] $\Leftarrow$ [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible] :

[illegible]

H [illegible] $\{e_n\}$ [illegible] [illegible]

$\Rightarrow \forall f \in H$; $f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$

$S_N(f) = \sum_{k=1}^{N} \langle f, e_k \rangle e_k$

$S_N \in O_p(H)$ ، S_N [illegible] [illegible] [illegible]

$S_N : H \to H$

$f \mapsto S_N(f)$

(a) 109 f [illegible] S_N

$\mathrm{Span}\{e_1, \dots, e_N\} = E_N$

$\|S_N\|$ [illegible]

$\Downarrow$

$\{S_N\}$ [illegible]

(1) [illegible] f

$S_N(f) \longrightarrow \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k = f = I(f)$

[illegible] I

(2) [illegible]

$\|\underbrace{I - S_N}_{fc}\|_{op} \geq 1 \qquad \frac{\|(I - S_N)(e_{N+1})\|}{\|e_{N+1}\|}$

$E_N^\perp$

$\| \qquad \frac{\|e_{N+1}\|}{\|e_{N+1}\|} = 1$

$f = \underset{E_N}{S_N(f)} + \underset{E_N^\perp}{(f - S_N(f))}$

$\Downarrow$

$\|I - S_N\|_{op} \not\to 0$

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$C(\mathbb{T})$, $L^2(\mathbb{T})$ [illegible]

[illegible]

$$A_n : f(t) \longmapsto f\left(t - \tfrac{1}{n}\right)$$

[illegible]

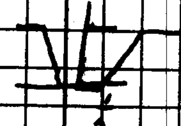
[illegible] $C(\mathbb{T})$ [illegible]

$$\forall f \in C(\mathbb{T}) : A_n f = f\left(t - \tfrac{1}{n}\right) \longrightarrow f(t) = I(f)$$ [illegible]

$$A_n \longrightarrow I$$ [illegible]

[illegible]

$\|f\| = 1$

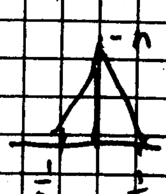
$f(t)$  $f\left(t - \tfrac{1}{n}\right)$ 

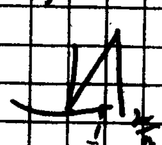
[illegible] $\left\| f\left(t - \tfrac{1}{n}\right) - f(t) \right\| = 1 \Rightarrow \|A_n - I\| \geq 1 \not\to 0$

[illegible] $\|A_n\| \leq 1$ $L^2(\mathbb{T})$

$C_L(\mathbb{T})$

$$A_n f = f\left(t - \tfrac{1}{n}\right) \xrightarrow[L^2]{} f(t) = I(f)$$ [illegible]

$f(t)$  [illegible]

$\|f\| = 1$ $f\left(t - \tfrac{1}{n}\right) =$ 

$$\left\| f\left(t - \tfrac{1}{n}\right) - f(t) \right\| \geq \ldots = \tfrac{n}{2} \longrightarrow \infty$$

$O_p(X)$

$O_p(X)$

$$(A \cdot B)(f) := (A \circ B)f = A(B(f))$$

$$\|A \cdot B\| \le \|A\| \, \|B\|$$

$\forall f$

$$\|ABf\| = \|A(Bf)\| \le \|A\| \, \|Bf\| \le \|A\| \, \|B\| \, \|f\|$$

$$\Downarrow$$

$$\|AB\| \le \|A\| \, \|B\|$$

$A_n \to A$

$B_n \to B$

$A_n B_n \to AB$

$$\|A_n B_n - AB\| = \|A_n B_n - A_n B + A_n B - AB\|$$

$$= \|A_n(B_n - B) + (A_n - A)B\|$$

$$\le \|A_n(B_n - B)\| + \|(A_n - A)B\|$$

$$\le \|A_n\| \, \|B_n - B\| + \|A_n - A\| \, \|B\|$$

$$\longrightarrow 0 + 0$$

[illegible] $O_p(X)$

$\left([illegible] \quad I \right)$

[illegible] $A \in O_p(X)$ [illegible]

[illegible] $B \in O_p(X)$ [illegible]

$$AB = BA = I$$

[illegible] $BA = I$ [illegible] $AB = I$

[illegible]

$$\ell^2 = \{x : \sum |x_i|^2 < \infty\}$$

$$A : x \mapsto (0, x_1, x_2, \dots)$$

$$\|Ax\| = \|x\| \qquad \|A\| = 1 \Rightarrow \begin{array}{l} BA = I \\ AB \neq I \end{array}$$

$$B : x \mapsto (x_2, x_3, x_4, \dots)$$

$$\|B\| = 1$$

$\|B\| \le 1$ [illegible] e_1 [illegible] $\|Bx\| \le 1$

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$AB = I$ $\quad$ $BA = I$

הפיך B $\quad$ הפיך A

לכן A $\quad$ לכן B

A, B הפיכים

משפט: $A \in B(X)$ $\quad$ X ...

A הפיך $\iff$ A ... חסום

$\|A^{-1}\| < \infty$

הערה: לא ! ... נכון ... לא

... $\|A^{-1}\| < \infty$

הגדרה: ... העתקה

נאמר ש- $A: X \to Y$ הפיכה אם

$A^{-1}: Y \to X$

$A \circ A^{-1} = I$

$A^{-1} \circ A = I$

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שאלה

X מ"ב $A \in B(X)$

$A: X \to X$ אופרטור

$\|A\| < 1$

הוכח ש־ $I - A$ הפיך

(תזכורת: בממשיים אם $|q| < 1$ אז $\frac{1}{1-q} = \sum_{i=0}^{\infty} q^i$)

נגדיר: $S_N = I + A + A^2 + \dots + A^N$

נראה ש־ $\{S_n\}$ סדרת קושי.

$n > m$: $\|S_n - S_m\| = \|\sum_{k=m+1}^{n} A^k\|$

$\leq \sum_{k=m+1}^{n} \|A^k\| \leq \sum_{k=m+1}^{n} \|A\|^k = \sum_{k=m+1}^{n} q^k \xrightarrow[n,m\to\infty]{} 0$ $\quad (|q| < 1)$

מכיוון ש־ B מרחב שלם

$\exists \lim_{n\to\infty} S_n = B$

11.5

$$(I-A)^{-1} = B$$

$$(I-A)B = I \quad (i)$$

$$B(I-A) = I \quad (ii)$$

$$(I-A)S_N = (I-A)\left(\sum_{k=0}^{N} A^k\right) \quad (i)$$

$$= \sum_{k=0}^{N} A^k - \sum_{k=1}^{N+1} A^k = I - A^{N+1} \xrightarrow{N} I$$

$$\|A^N\| \le \|A\|^N \xrightarrow{N\to\infty} 0 = \|0\|$$

$$\Downarrow$$

$$(I-A)S_N \longrightarrow I$$

$$\Downarrow$$

$$(I-A)B$$

(ii) באופן דומה

<u>ספקטרום - של אופרטורים</u>

<u>הגדרה:</u> $\lambda\in\mathbb{C}$ נקרא ערך רגולרי

של אופרטור $A: X\to X$

אם $A-\lambda I$ הפיך

$A-\lambda I$ הפיך $\iff$ $\forall g\ \exists! f$ $(A-\lambda I)f=g$ $\iff$ $Af-\lambda f=g$ $\iff$ $Af=\lambda f+g$

<u>משפט</u>

אם $\lambda\in\mathbb{C}$, ו‑ $|\lambda|>\|A\|$

אז λ רגולרי.

<u>הוכחה</u>

$$A-\lambda I = (-\lambda)\left(I-\frac{1}{\lambda}A\right)$$

מכיוון ש $|\lambda|>\|A\| \implies \left\|\frac{1}{\lambda}A\right\|<1$

$\implies I-\frac{1}{\lambda}A$ הפיך לפי משפט קודם

$\implies A-\lambda I$ הפיך

$$(A-\lambda I)^{-1} = -\frac{1}{\lambda}\left(I-\frac{1}{\lambda}A\right)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}A\right)^{k}$$ (נוימן)

סימון $R_\lambda(A)=(A-\lambda I)^{-1}$ רזולבנטה

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<u>הגדרה</u>

$\mathrm{spec}(A) = \{\lambda \in \mathbb{C} : \lambda$ ערך עצמי של $A\}$

<u>טענה:</u> λ ע"ע $\iff$

$\exists f \neq 0 : \; Af = \lambda f$

$\iff$

$(A - \lambda I) f = 0$

$\iff$

$A - \lambda I$ אינו הפיך

$\iff$

$\lambda \in \mathrm{spec}(A)$

<u>דוגמאות</u>

(1) H ספרבילי, $\{e_n\}$ בסיס א"נ

$\{\lambda_n\}$ סדרה חסומה, $\forall n \; |\lambda_n| < k$

נגדיר $A: \; e_n \mapsto \lambda_n e_n$

$\Downarrow$

$\sum_k c_k e_k \mapsto \sum_k \lambda_k c_k e_k$

$\Leftarrow \; \|A\| = \sup |\lambda_n|$ [illegible]

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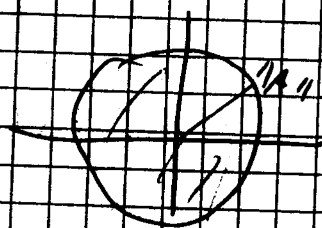
A λ

$\{\lambda_n\} \subseteq spec(A)$

(1)

(2)

$spec(A) \subseteq$



$\|A\| < |\lambda|$ $\Leftarrow$

$\lambda \in \mathbb{C}$

$\lambda \in spec(A)$

$\lambda \notin spec(A) \iff (A-\lambda I)f = g$ $\forall g \, \exists! f$

$f = \sum c_k e_k$ $\qquad g = \sum d_k e_k$

$(A-\lambda I)f = \sum \lambda_k c_k e_k - \lambda \sum c_k e_k = g = \sum d_k e_k$

$\forall k$ $\quad \lambda_k c_k - \lambda c_k = d_k$

$c_k(\lambda_k - \lambda)$

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~~[illegible]~~

$\Updownarrow$

$\forall k \quad c_k = \frac{1}{\lambda_k - \lambda} d_k$ [illegible]

$R_\lambda: \quad g = \sum d_k e_k \longmapsto f = \sum_{k=1}^{\infty} \frac{d_k}{\lambda_k - \lambda} e_k$

$\Updownarrow$

$\|R_\lambda\| = \sup_n \left| \frac{1}{\lambda_n - \lambda} \right| \le k_\lambda$

([illegible] $f \in H$)

$\Updownarrow$

$\forall k \quad |\lambda_k - \lambda| > \varepsilon$ [illegible] $\varepsilon > 0$

$\Updownarrow$

$\lambda \notin \text{clos}\{\lambda_k\}$

$\lambda \notin \text{spec}(A) \iff \lambda \notin \text{clos}\{\lambda_n\}$

$\text{spec}(A) = \text{cls}\{\lambda_n\}$ [illegible]

שאלה 2

$L^2[0,1]$

...

A: $f(t) \mapsto \varphi(t) f(t)$

$\|A\| = \operatorname{ess\,sup} |\varphi(t)|$

$\sup_{t \in A} \{ |\varphi(t)| \}$

$A \neq 0$

$\exists f \neq 0 \quad Af = \lambda f$

φf

$\Updownarrow$

$(\varphi(t) - \lambda) f(t) = 0$

$\mu_L(\{t : \varphi(t) = \lambda\}) > 0$

$\lambda \Leftrightarrow \{\varphi(t) = \lambda\}$

$f(t) =$

[illegible]

$\forall g \ \exists! f \quad Af = \lambda f + g \iff \lambda \notin \mathrm{spec}(A)$

[illegible]

$\Downarrow$

$$p(t) f(t) = \lambda f(t) + g(t)$$

$\Downarrow$

$$f(t) = \frac{1}{p(t) - \lambda} g(t)$$

$$g(t) \mapsto \frac{1}{p(t)-\lambda} g(t) \quad [illegible]$$

[illegible]

$$\exists k \quad \left| \frac{1}{p(t) - \lambda} \right| \le k \quad [illegible]$$

[illegible]

$$\exists \varepsilon \quad |p(t) - \lambda| \ge \varepsilon \quad [illegible]$$

$$\sup \left| \frac{1}{p(t) - \lambda} \right| = \infty \Rightarrow \lambda \in \mathrm{spec}(A)$$

$\lambda \in Spec(A)$

$$\left| \frac{1}{\gamma(t) - \lambda} \right| \leq K$$

$\lambda \notin Spec(A)$

$Spec(A)$

$$A - \lambda I = A - \lambda_0 I - (\lambda - \lambda_0) I$$

$$= (A - \lambda_0 I)\left[I - (\lambda - \lambda_0) R_{\lambda_0} \right]$$

$$R_{\lambda_0} = (A - \lambda_0 I)^{-1}$$

$I-(\lambda-\lambda_0)R_{\lambda_0}$ [illegible] אם

$\|(\lambda-\lambda_0) R_{\lambda_0}\| < 1$ אם

$$= |\lambda-\lambda_0| \, \|R_{\lambda_0}\|$$

$|\lambda-\lambda_0| < \frac{1}{\|R_{\lambda_0}\|} = \varepsilon$ אם

$\Leftarrow$ $B(\lambda_0, \varepsilon)$ [illegible]

[illegible] $\lambda \in B(\lambda_0,\varepsilon)$

$$R_\lambda = R_{\lambda_0} \sum_{k=0}^{\infty} \left[(\lambda-\lambda_0)^k R_{\lambda_0}^k\right]$$

מסקנה R_λ [illegible] λ [illegible]

[illegible]

[illegible]

[illegible]