10 7.6.13

124 [illegible]

$A : X \to X$ [illegible]

$\lambda \in \mathbb{C}$ [illegible]

(1) $A - \lambda I$ [illegible]

$R_\lambda = (A - \lambda I)^{-1}$

(2) λ [illegible]

$\Updownarrow$

$\exists f \neq 0 \quad Af = \lambda f$

$\Updownarrow$

$(A - \lambda I)f = 0$

$\Updownarrow$

$A - \lambda I$ [illegible]

[illegible] $\Rightarrow$

$\forall g \ \exists! f \quad Af + \lambda f = g$

$f = R_\lambda g$

(3) $\mathrm{spec}(A) = \{\lambda \in \mathbb{C} \mid$ [illegible] $\}$

$A - \lambda I$ [illegible] $A - \lambda I$ [illegible]

(3) λ [illegible] $\sigma_c(A)$

[illegible] $A - \lambda I$ [illegible]

[illegible] $A - \lambda I$ [illegible]

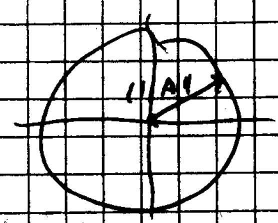
[illegible]

$\mathrm{spec}(A) = \{0\} \cup \{\lambda\} \Leftarrow$

[illegible]

$\|A\| < \lambda \Rightarrow$ [illegible]

$\mathrm{spec}(A) \subseteq$

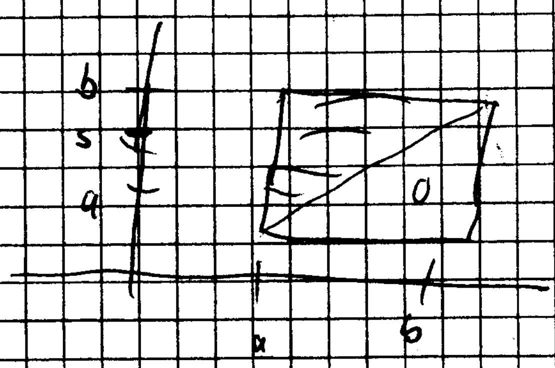


$$R_\lambda = (A - \lambda I)^{-1} = -\frac{1}{\lambda}\left(I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots\right)$$

$$\left(\mathrm{spec}(A) \subsetneq B(0, \|A\|)\right)$$

([illegible]) volterra

$\forall g \ \exists! f$ $$\int_a^s k(s,t)\, f(t)\, dt = \lambda f(s) + g(s)$$



$k(s,t)$

$J: L^2(I)/C(I) \to L^2(I)/C(I)$

$f \mapsto J(f)$

$J(f) = \int_a^s k(s,t)\, f(t)\, dt$

$$\int_a^s f(t)\, dt = g(s) \qquad \left(\begin{array}{c} \lambda = 0 \\ k(s,t) = 1 \end{array} \right)$$

$\Updownarrow$

$f(s) = g'(s)$

$Jf = \lambda f + g$

$\frac{1}{\lambda} \left(\sum_{k=0}^{\infty} \left(\frac{A}{\lambda}\right)^k \right)$

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$J^2 f$ [illegible]

$$(J^2 f)(s) = J(J(f))(s)$$

$$= J\left(\int_a^b k(\sigma,t)\, f(t)\,dt\right)(s)$$

$$= \int_a^b k(s,\sigma)\left[\int_a^b k(\sigma,t)\, f(t)\,dt\right]d\sigma$$

$$= \int_a^b k(s,\sigma)\left[\int_a^b k(\sigma,t)\, f(t)\,dt\right]d\sigma$$

$$= \int_a^b \underbrace{\left[\int_a^b k(s,\sigma)\, k(\sigma,t)\,d\sigma\right]}_{k_2(s,t)} f(t)\,dt$$

[illegible] J^2 [illegible] $k_2(s,t)$

$$k_2(s,t) = \int_a^b k(s,\sigma)\, k(\sigma,t)\,d\sigma$$

$\sigma > s$ [illegible] $\sigma < t$

$(t < s)$ $= \int k(s,\sigma)\, k(\sigma,t)\,d\sigma$

(ב) [illegible] $k_1(s,t)$ [illegible] $\Leftarrow$

$\forall\, s<t :\quad k_1(s,t)=0$ [illegible]

[illegible] $|k(s,t)|\le M$ [illegible]

$$\Downarrow$$

$$|k_1(s,t)| \le \int_t^s |k(s,r)|\,|k(r,t)|\,dr$$

$$\le M^2(s-t) \le M^2(b-a)$$

[illegible] $k_1(s,t)$

$$J^3(f)(s) = \int_a^b k(s,r)\left[\int_a^b k_1(r,t)\,f(t)\,dt\right]dr \Leftarrow$$

$$= \int_a^b \underbrace{\left(\int_a^b k(s,r)\,k_1(r,t)\,dr\right)}_{k_2(s,t)} f(t)\,dt$$

[illegible] k_2 [illegible] k [illegible] $|k|\le M$ [illegible]

$|k_2| \le \int_a^b |k(s,\sigma)|\,|k_1(\sigma,t)|\,d\sigma$

$\le \int_a^b M^3 \int_t^s (\sigma-t)\,d\sigma$

$= M^3 \frac{(\sigma-t)^2}{2!}\Big|_t^s = M^3 \frac{(s-t)^2}{2!}$

[illegible]

$k_n(s,t)$ [illegible]

[illegible]

$|k_{n-1}(s,t)| \le M^n \frac{(s-t)^{n-1}}{(n-1)!} \le M^n \frac{(b-a)^{n-1}}{(n-1)!}$

$\Downarrow$

$\forall \lambda \neq 0$ $\quad \left\| -\frac{1}{\lambda} \sum_{k=0}^{N} \left(\frac{T}{\lambda}\right)^k \right\|$ $\qquad \|T^k\| \le \int_a^b\int_a^b |k_{n-1}(s,\ldots$

$\le \left|\frac{1}{\lambda}\right| \sum_{k=0}^{N} \frac{\|T^k\|}{|\lambda|^k}$ $\qquad \le (b-a)^{n+1}\frac{M^n}{(n-1)!}$

$\le \left|\frac{1}{\lambda}\right| \sum_{k=0}^{N} \frac{M^{k+1}(b-a)^{k+1}}{|\lambda|^k\, k!} \xrightarrow{N\to\infty}$ [illegible]

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$\forall \lambda \neq 0 \quad \exists R_\lambda$ ([illegible] $A-\lambda I$ [illegible])

$\Downarrow$

$\mathrm{spec}(A) = \{0\}$

[illegible]

$A: X \to X$ [illegible]

$\dim X = \infty$ [illegible]

$\dim(\mathrm{Im}\, A) = \dim X_0 = n < \infty$

$X_0 = \mathrm{Im}\, A$

[illegible] $\mathrm{spec}(A)$ [illegible]

$\forall g \ \exists! f \quad Af - \lambda f = g$

$X_0 \ni f_0$

$\Uparrow$

$f = \frac{1}{\lambda} f_0 - \frac{1}{\lambda} g$

$$A f_0 = \lambda f_0 + g_0$$

$$A_0 = A|_{X_0} \quad ; \quad X_0 \to X_0$$

$\mathrm{spec}(A_0) = \{\ldots\}$ [illegible] X_0

$\det(A_0 - \lambda I)$

$\subseteq \mathrm{spec}(A)$

$$\mathrm{spec}(A) = \{\lambda_1, \ldots, \lambda_n, 0\}$$

$\det(A_0 - \lambda I)$ [illegible] $\{\lambda_i\}$ [illegible]

$(A_0 - \lambda I$ [illegible] $)$

[illegible] $\lambda \notin \{\lambda_1, \ldots, \lambda_n, 0\}$

$g \in X$ [illegible] f_0

$$A f_0 = \lambda f_0 + g_0 \Rightarrow f = \frac{1}{\lambda} f_0 - \frac{1}{\lambda} g$$

$$\Rightarrow A f - \lambda f = \frac{1}{\lambda}\left[A f_0 - A g\right] - (f_0 - g)$$

$$= \frac{1}{\lambda}\left[\lambda f_0\right] - f_0 + g = g$$

$A f^1 - \lambda f^1 = A f^2 - \lambda f^2 = g$

$A f_0^1 - \lambda f_0^1 = A f_0^2 - \lambda f_0^2$

$= f_0$

$f_0^1 = f_0^2 = f_0$

$$k(s,t) = \sum_{j=1}^{N} S_j(s)\, T_j(t)$$

$k(s,t)$

$$(Jf)(s) = \int_I \left[\sum_{j=1}^{N} S_j(s)\, T_j(t) \right] f(t)\, dt$$

$$= \sum_{j=1}^{N} S_j(s) \underbrace{\int_I T_j(t) f(t)\, dt}_{a_j}$$

$$= \sum_{j=1}^{N} S_j(s)\, a_j$$

$\Downarrow$

$$\forall f \quad Jf \in \operatorname{span}\{S_1 \dots S_N\}$$

$\Downarrow$

$$\dim(\operatorname{Im} J) \le N < \infty$$

$$Jf(s) = \int_{-\pi}^{\pi} \cos(s-t)\, f(t)\, dt \qquad L^2$$

$\cos(s-t)$

$= \sum S_j(s)\, T_j(t)$

$\operatorname{spec} J = \,?$ (1)

$(Jf)(s) = f(s) + 1$ (2)

[illegible] (7)

$$[A]_{x_0} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$ [illegible] $= x_0 = \mathrm{span}(\sin(s), \cos(s))$

$B = \{\sin(s), \cos(s)\}$ $\Leftarrow$ $\mathrm{spec}(J) = \{0, \pi\}$

$J_0(f) = \ldots \sin(s) \ldots \cos(s)$ $\Leftarrow$ $[J|_{x_0}]^{-1} = \begin{pmatrix} \frac{1}{\pi} & 0 \\ 0 & \frac{1}{\pi} \end{pmatrix}$ (ii)

$f_0 = (J_0 - I)^{-1} g_0 = 0$ $\Leftarrow$ $g_0 = Ag = 0$ $\Leftarrow$ $g = 1$

$f = \frac{1}{\lambda} f_0 - \frac{1}{\lambda} g = -1$ $\Leftarrow$

$Jf = f + g$ $\Leftarrow$ $Jf \equiv 0$ $\quad f(s) = -1$ [illegible]

[illegible]

X [illegible] $A: X \to X$ [illegible]

$\forall \varepsilon > 0 \quad \exists A_\varepsilon : X \to X_\varepsilon \quad ; \quad \|A - A_\varepsilon\| < \varepsilon$

[illegible]

([illegible])

[illegible] J [illegible] $K(s,t) \in L^2(I \times I)$

[illegible] $\varepsilon > 0$ [illegible] J_ε

$\left(\|J - J_\varepsilon\|_{op} \le \|K - K_\varepsilon\| < \varepsilon\right)$ K_ε [illegible]

$I \times I$

[illegible] K_ε

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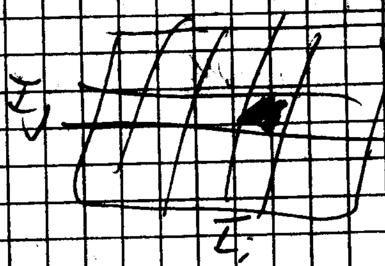

$P_{ij} = I_i \times I_j$

$1_{P_{ij}}(s,t) = 1_{I_i}(s)\, 1_{I_j}(t)$

$k_\varepsilon = \sum_{i,j} c_{ij}\, 1_{I_i}(s)\, 1_{I_j}(t)$

$\| k - k_\varepsilon \| < \varepsilon$

$A: X \to X$

$U = \{ f \mid \|f\| < 1 \}$ $A(U)$

$\{f_n\}$ $\|f_n\| \le 1$

$A f_{n_j} \to$

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[illegible] $A(U)$ [illegible]

[illegible] A [illegible] A_ε [illegible]

$\|A - A_\varepsilon\| < \varepsilon$ [illegible] $A_\varepsilon(U)$

$\|A_\varepsilon\| \geq |A_\varepsilon(u)|$ [illegible] $A_\varepsilon(U)$ [illegible]

[illegible] $A_\varepsilon(U)$ [illegible]

[illegible] ε [illegible] $A_\varepsilon(U)$

[illegible] 2ε [illegible] $A(U)$

$\|f\| < 1$ [illegible] ($\|A f - y\| \leq 2\varepsilon$ [illegible])

$$\|Af - A_\varepsilon f\| = \|(A - A_\varepsilon) f\| \leq \|A - A_\varepsilon\| \, \|f\| < \varepsilon \cdot 1 = \varepsilon$$

[illegible] $y \in Y$ $\|y - A_\varepsilon f\| < \varepsilon$

$$\Rightarrow \|Af - y\| \leq \|Af - A_\varepsilon f\| + \|A_\varepsilon f - y\| < \varepsilon + \varepsilon = 2\varepsilon$$

[illegible] ([illegible] $A e_n \to \lambda_n e_n$)

[illegible] $\Leftrightarrow$ $\|A\| = \sup |\lambda_n| < \infty$

[illegible] $\Leftrightarrow$ $\inf |\lambda_n| > 0$

136 $\lambda_n \to 0 \iff$ [illegible]

$\Leftarrow$ [illegible]

$A_N : H \to \mathrm{span}\{e_1 \ldots e_N\}$

$$x = \sum_{n=1}^{\infty} c_n e_n \mapsto \sum_{n=1}^{N} \lambda_n c_n e_n$$

[illegible] $\varepsilon > 0$ [illegible]

$\exists n_0 \; \forall n > n_0 \quad |\lambda_n| < \varepsilon$

$$\Downarrow$$

$$\|(A - A_N)x\| = \left\|\sum_{n=N}^{\infty} \lambda_n c_n e_n\right\|$$

$$= \sum_{n=N}^{\infty} |\lambda_n c_n|^2 < \varepsilon^2 \sum_{n=N}^{\infty} |c_n|^2 \le \varepsilon^2 \|x\|^2$$

$$\Downarrow$$

$$\|A - A_N\| < \varepsilon^2$$

$\boxed{\Rightarrow}$ A [illegible] $\Leftarrow$ $A(H)$ [illegible]

$x = \sum c_n e_n \quad Ae_{n_k} \to x \quad \{c_n\} \subset$ [illegible]

$$\|Ae_{n_k} - x\|^2 = \|\lambda_{n_k} - x\|^2 = |\lambda_{n_k} - c_{n_k}|^2 + \sum_{n \neq n_k} |c_n|^2$$

$\to 0$ [illegible] $\forall n$ [illegible]

[illegible]

$(\ldots) \quad \{c_n\} \quad)$

$\Leftarrow \quad \lambda_k \to 0$

$\lambda_n \not\to 0$ [illegible] $\exists \varepsilon \; \forall n \; \exists m \; |\lambda_m| > \varepsilon$

$\{\lambda_{m(n)}\}_{n=1}^{\infty} \quad |\lambda_{m(n)}| > \varepsilon$

$\{e_{m(n)}\} \subseteq$ [illegible]

$f \quad 0 - 1$ [illegible]

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10.6.13

~~אופרטורים קומפקטיים~~

X מ"ל (נורמי)

$A: X \to X$ סופי-מימדי אם

$\forall \varepsilon \ \exists A_\varepsilon$: $\|A - A_\varepsilon\| < \varepsilon$

סופי-מימדי, A קומפקטי

הגדרה. A נקרא קומפקטי

$U = \{x \mid \|x\| \le 1\}$

אם $A(U)$ קבוצה קומפקטית

(לכל $\{f_n\} \subset U$ קיימת תת-סדרה f_{n_j} כך ש $A f_{n_j} \xrightarrow{\text{מתכנסת}}$)

טענה במרחב הילברט H כל אופרטור קומפקטי U מקיים

$A(U)$ קבוצה קומפקטית

U

A קומפקטי

138

[illegible]

(1) [illegible] $L^2[a,b]$

[illegible] $k(s,t) \in L^2(I \times I)$

[illegible] $C(I)$

[illegible] $k(s,t)$ $g(t)$

(2) [illegible] A [illegible]

$$A e_n = \lambda_n e_n$$

[illegible] A [illegible] $\Leftrightarrow$ $\lambda_n \to 0$

[illegible] Fredholm

$A : X \to X$ [illegible]

(1) [illegible] $\lambda \neq 0$

[illegible] λ [illegible] A

[illegible] λ [illegible] A

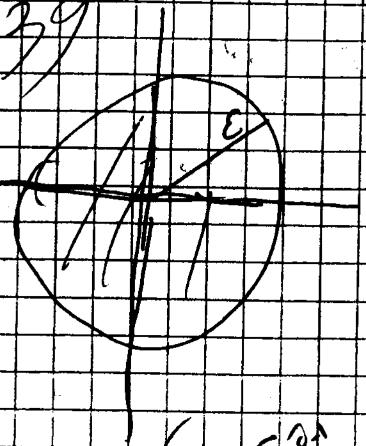
([illegible] λ [illegible])

(2) [illegible]

[illegible] $\{\lambda_n\}$ [illegible] $\lambda_n \to 0$

(3) $\operatorname{spec} A = \{\lambda_n\} \cup \{0\}$

(11) 139



(1) $\epsilon>0$ [illegible]

$|\lambda|>\epsilon$ [illegible]

[illegible] $A-\lambda I$ [illegible] ?

A [illegible]

A_ϵ [illegible] $\|A-A_\epsilon\|<\epsilon$

$B_\epsilon = A-A_\epsilon$ [illegible]

$$A-\lambda I = A_\epsilon + (A-A_\epsilon) - \lambda I = A_\epsilon + (B_\epsilon - \lambda I)$$

[illegible] $|\lambda| > \|B_\epsilon\|$ [illegible]

$$= \left(I + A_\epsilon R_\lambda(B_\epsilon)\right)\underbrace{(B_\epsilon-\lambda I)}_{\text{[illegible]}}$$

$R_\lambda(B_\epsilon) = (B_\epsilon-\lambda I)^{-1}$

[illegible] $I + A_\epsilon R_\lambda(B_\epsilon)$ [illegible]

$D_{\lambda\epsilon} = A_\epsilon R_\lambda(B_\epsilon)$ [illegible]

$D_{\lambda\epsilon}$ [illegible]

$\operatorname{Im} D_{\lambda\epsilon} \subseteq \operatorname{Im} A_\epsilon$

(11) 140

[illegible] ? [illegible] $I + D_{\lambda\varepsilon}$ [illegible]

$\Rightarrow$ [illegible] -1 [illegible] $D_{\lambda\varepsilon}$

$$\left\{ \begin{array}{l} -1 \in spec(D_{\lambda\varepsilon}) \\ D_{\lambda\varepsilon} \text{ [illegible]} \\ spec(D_{\lambda\varepsilon}) = \{\ldots, 0\} \end{array} \right.$$

[illegible] $I + D_{\lambda\varepsilon}$ [illegible]

[illegible] λ [illegible]

[illegible] $I + D_{\lambda\varepsilon}$ [illegible] λ [illegible]

[illegible] $(I + D_\lambda) g = 0$ $\quad g \neq 0$

$$f = (B_\varepsilon - \lambda I)^{-1} g \neq 0$$

$$g = (B_\varepsilon - \lambda I) f$$

$$\Downarrow$$

$$(A - \lambda I) f = \underbrace{\left(I + A_\varepsilon R_\lambda (A_\varepsilon) \right)}_{I + D_\lambda} \underbrace{(B_\varepsilon - \lambda I) f}_{= g} = 0$$

$\Leftarrow$ A [illegible] A [illegible] λ

(11) 147

[illegible]

$$f(\lambda) = \det\left(\left(I + D_{\lambda\epsilon} \right) \Big|_{x_c \to x_\epsilon} \right)$$

[illegible] $f(\lambda) = 0$

[illegible] $f(\lambda)$ [illegible]

$\Leftarrow$ $f(\lambda) - 1$ [illegible]

[illegible] $f(\lambda) = 0$

[illegible]

[illegible]

[illegible] $0 \to$ [illegible]

(3) $0 \in \mathrm{spec}(A)$

$0 \notin \mathrm{spec}(A) \iff A$ [illegible] $\iff I = AA^{-1} \Rightarrow I$ [illegible]

$\Leftarrow$ [illegible] $= I(\ldots)$ [illegible]

[illegible] $C(\mathbb{I})$ [illegible]

[illegible]

A [illegible]

B [illegible]

[illegible] AB, BA [illegible]

[illegible] ε [illegible] A_ε [illegible]

$\|A - A_\varepsilon\| < \varepsilon$

[illegible]

$\|B\| \le M$

$\|AB - A_\varepsilon B\| \le \|A - A_\varepsilon\| \|B\| \le \varepsilon M$

$\|BA - BA_\varepsilon\| \le \|B\| \|A - A_\varepsilon\| \le M \varepsilon$

11 [illegible]

[illegible] $A : H \to H$ [illegible] H

[illegible] $y \in H$

$\varphi_y(x) = \langle Ax, y \rangle$ $\qquad \varphi_y : H \to \mathbb{C}$

[illegible] A [illegible] $\varphi_y(x)$

[illegible] $\langle \cdot , \cdot \rangle$ [illegible]

[illegible]

$$|\varphi_y(x)| = |\langle Ax, y \rangle| \le \|Ax\| \, \|y\|$$

$$\le \|A\| \, \|x\| \, \|y\| = \|A\| \, \|y\| \, \|x\|$$

$$\|\varphi_y\|_{H^*} \le \|A\| \, \|y\|$$

$\varphi_y(x) = \langle x, y^* \rangle$ [illegible] $y^* \in H$

$\langle Ax, y \rangle$

[illegible] $\Leftarrow$ $A^* : H \to H$, $y \mapsto y^*$

A [illegible]

$A^*(\alpha y_1 + y_2)$ [illegible]

$= \alpha A^*(y_1) + A^*(y_2)$ $\qquad y_1 \leftrightarrow y_1^*$

$\langle Ax, \alpha y_1 + y_2 \rangle = \bar{\alpha} \langle Ax, y_1 \rangle + \langle Ax, y_2 \rangle$ $\qquad y_2^* \leftrightarrow y_2$

$= \bar{\alpha} \langle x, y_1^* \rangle + \langle x, y_2^* \rangle = \langle x, \alpha y_1^* + y_2^* \rangle$

$\forall y \quad \|A^* y\|_H = \|\varphi_y\|_{H^*} \leq \|A\|\,\|y\|$

$$\|A^* y\|_H$$

$$\Downarrow$$

$$\|A^*\| \leq \|A\|$$

[illegible] $\Leftarrow$ [illegible] $A : H \to H$ [illegible]

[illegible] $A^* : H \to H$ [illegible]

$\forall x, y \in H$

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

[illegible]

$$(A^*)^* = A$$

[illegible]

$$\langle A^* x, y \rangle = \overline{\langle y, A^* x \rangle}$$

$$= \overline{\langle Ay, x \rangle} = \langle x, Ay \rangle$$

[illegible]

$$\|A\| = \|(A^*)^*\| \leq \|A^*\|$$

$$\Downarrow$$

$$\|A\| = \|A^*\|$$

תרגיל בית

(א) יהי מרחב הילברט H ספרבילי

$\{e_n\}$ בסיס א"נ

$\{\lambda_n\}$ סדרה חסומה

$$A : H \to H$$

$$f = \sum c_n e_n \longmapsto Af = \sum c_n \lambda_n e_n$$

מצא A^*

צ"ל: A^* אופרטור כפל ע"י $\{\bar{\lambda}_n\}$

(הוכחה)

$\forall x, y \in H$

$$\langle Ax, y \rangle$$

$$= \left\langle \sum_n c_n \lambda_n e_n , \sum_n d_n e_n \right\rangle$$

$$= \sum_n \sum_k c_n \lambda_n \bar{d}_k \langle e_n, e_k \rangle$$

$$= \sum_n \sum_k c_k \overline{\bar{\lambda}_k d_k} \langle e_k, e_k \rangle$$

$$= \sum_n \sum_k c_n \overline{\bar{\lambda}_k d_k} \langle e_n, e_k \rangle$$

$$= \left\langle \sum c_n e_n , \sum_k \bar{\lambda}_k d_k e_k \right\rangle$$

$$\overset{\shortparallel}{x} \qquad \overset{\shortparallel}{A^* y}$$

(2) $L^2(I)$ [illegible]

$\lambda(t) : I \to \mathbb{C}$ [illegible]

$A : L^2(I) \to L^2(I)$

$f(t) \mapsto \lambda(t) f(t)$

$\|A\| = \sup |\lambda(t)| < \infty$

[illegible] A^* ?

$\langle Af, g \rangle = \langle f, A^* g \rangle$

$\langle \lambda f, g \rangle$

$\int_I \lambda f \bar{g} \, dt = \int_I f(t) \, \overline{\overline{\lambda(t)} g(t)} \, dt$

$\langle f, A^* g \rangle$ $\left(A^* g = \overline{\lambda(t)} g(t) \right)$

(3) [illegible] $L^2(I)$ [illegible]

$k(s,t) \in L^2(I \times I)$ [illegible]

$(Jf)(s) = \int_I k(s,t) f(t) \, dt$

[illegible] J^* ?

147 $\langle Jf, g \rangle = \int_I \left(\int_I k(s,t) f(t)\, dt \right) \overline{g(s)}\, ds$

$$= \int_I \left[\int_I k(s,t)\, \overline{g}(s)\, ds \right] f(t)\, dt$$

$$= \int_I f(t) \left[\overline{\int_I \overline{k(s,t)}\, g(s)\, ds} \right] dt$$

$$= \langle f, J^* g \rangle$$

$J^*(g)(s) =$, e.o.

$$= \int_I k^*(s,t)\, g(s)\, ds$$

$k^*(s,t) = \overline{k(t,s)}$, e.o.

$\ell^2 = \{ (x_n) \mid \sum |x_n|^2 < \infty \}$ (4)

$S : \ell^2 \to \ell^2$

$(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$

S^*

148 $S^*: (y_1, y_2, \dots) \mapsto (y_2, y_3, \dots)$

$\forall x, y$: $\langle Sx, y \rangle$

$= \langle (0, x_1, \dots), (y_1, \dots) \rangle$

$= \sum_{i=1}^{\infty} x_i \bar{y}_{i+1} = x_1 \bar{y}_2 + x_2 \bar{y}_3 + \dots$

$= \langle (x_1, x_2, \dots), (y_2, y_3, \dots) \rangle$

$= \langle x, S^* y \rangle$

[illegible]

$(A+B)^* = A^* + B^*$ (1)

$(\lambda A)^* = \bar{\lambda} A^*$ (2)

$(AB)^* = B^* A^*$ (3)

$\langle \lambda A x, y \rangle = \lambda \langle Ax, y \rangle = \lambda \langle x, A^* y \rangle$ (2) [illegible]

$= \langle x, \bar{\lambda} A^* y \rangle$

$\langle ABx, y \rangle = \langle Bx, A^* y \rangle =$ (3)

$\langle x, B^* A^* y \rangle$

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$\|AB\| \le \|A\|\|B\|$

$$\|AA^*\| \le \|A\|\|A^*\| = \|A\|^2 \qquad \Leftarrow$$

$$\|A^*A\| = \|A\|^2 \qquad \text{[illegible]}$$

$$\|A\|^2 \le \|A^*A\| \qquad \text{[illegible]}$$

$$\|A^*A\|_{op} = \sup_{\|x\|\le 1} \|A^*Ax\|_H$$

$$\left(\|z\|_H = \sup_{\|y\|\le 1} |\langle z, y\rangle|\right) \to \quad = \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle A^*Ax, y\rangle|$$

$$= \sup_{\substack{\|x\|\le 1\\ \|y\|\le 1}} |\langle Ax, Ay\rangle|$$

$$\ge \sup_{\|x\|\le 1} |\langle Ax, Ax\rangle|$$

$$= \sup_{\|x\|\le 1} \|Ax\|^2 = \|A\|^2$$

$$\|A\| = \sqrt{\|A^*A\|} \qquad \text{[illegible]}$$

150

אופרטורים צמודים לעצמם

אופרטור $A: H \to H$ ו-H הילברט

נקרא צמוד לעצמו אם $A^* = A$

כלומר אם $\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in H$

דוגמאות

(1) A^*A צמוד לעצמו

(2) אופרטור אלכסוני $A: e_n \to \lambda_n e_n$

צמוד לעצמו $\Leftrightarrow$ $\lambda_n \in \mathbb{R} \quad \forall n$

(3) אופרטור כפל $f \to \lambda f$

צמוד לעצמו $\Leftrightarrow$ $\lambda(t) \in \mathbb{R} \quad \forall t$

(4) אופרטור אינטגרלי עם גרעין $k(s,t)$

צמוד לעצמו $\Leftrightarrow$ $k^*(s,t) = k(s,t)$

$\overline{k(t,s)}$

נניח A צמוד לעצמו אזי

$$\langle Af, f\rangle = \langle f, Af\rangle = \overline{\langle Af, f\rangle}$$

$$\Downarrow$$

$$\forall f \quad \langle Af, f\rangle \in \mathbb{R}$$

הערה: גם ההיפך נכון

אם $\langle Af, f\rangle \in \mathbb{R} \quad \forall f \in H$

אז A צמוד לעצמו

<u>תרגיל</u>

יהיו A, B אופרטורים צמודים לעצמם

$\alpha \in \mathbb{R}$

(1) $A \pm B$ צמוד לעצמו

(2) αA צמוד לעצמו

(3) אם AB מתחלפים אזי AB צמוד לעצמו

$$(AB)^* = B^* A^* = BA = AB$$

152 (ב)

הוכח: A צמוד לעצמו אם

$$\langle Af, f\rangle = \langle f, Af\rangle = \overline{\langle f, Af\rangle}$$

$$\Downarrow$$

$$\forall f \quad \langle Af, f\rangle \in \mathbb{R}$$

הוכח כי גם ההפך נכון

אם $\langle Af, f\rangle \in \mathbb{R} \quad \forall f \in H$

אז A צמוד לעצמו

<u>תרגיל</u>

הוכח A, B אופרטורים צמודים לעצמם

$\alpha \in \mathbb{R}$

(1) $A+B$ צמוד לעצמו

(2) αA צמוד לעצמו

(3) אם $AB = BA$ אז AB צמוד לעצמו

$$(AB)^* = B^* A^* = BA = AB$$

153 $E \subset H$

$H = E \oplus E^{\perp}$

$P_E : H \to H$

$x \mapsto x_E$

$x = x_E + x_{E^\perp}$

$\|P_E\| = 1$ (i)

P_E (ii)

$\mathrm{spec}(P_E) = ?$ (iii)

$\|P_E(x)\| = \|x_E\| \leq \|x\|$

$\|P_E(x_E)\| = \|x_E\| \Rightarrow \|P_E\| = 1$ (i)

$x = x_1 + x_2$, $y = y_1 + y_2$ (ii)

E, $E^{\perp}$

$\langle P_E(x), y \rangle = \langle x_1, y_1 + y_2 \rangle = \langle x_1, y_1 \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x, P_E(y) \rangle$

$P_E^* = P_E$

$P_E f = g$ (iii)

$P_E(x) = \lambda$

$\mathrm{Spec}(P_E)$, $\lambda = 1$

spec

$\forall g \; \exists! f \; P_E(f) - \lambda f = g$

$\lambda \neq 0, 1$

$f = f_1 + f_2$, $g = g_1 + g_2$

$f_1 - \lambda f_1 - \lambda f_2 = g_1 + g_2$

$f_2 = \frac{g_2}{-\lambda}$, $f_1 = \frac{g_1}{1-\lambda}$

1.54 טענה: A צמוד לעצמו

אז $Spec(A) \subset \mathbb{R}$

הוכחה (המשך)

יהי λ ע"ע של A.

λ ע"ע $\Leftarrow$ קיים $x \neq 0$ $Ax = \lambda x$

$\Downarrow$

$$\langle Ax, x \rangle = \langle \lambda x, x \rangle = \lambda \langle x, x \rangle$$

$$\|$$

$$\langle x, Ax \rangle = \langle x, \lambda x \rangle = \bar{\lambda} \langle x, x \rangle$$

$\Downarrow$

$\lambda = \bar{\lambda}$

$\Downarrow$

$\lambda \in \mathbb{R}$

טענה: A צמוד לעצמו

אז $\|A^n\| = \|A\|^n$ לכל n

הוכחה: $\|A^n\| \leq \|A\| \dots \|A\|$

$n=2$: $\|A^2\| = \|A^* A\| = \|A\|^2$

$n=4$: $\|A^4\| = \|(A^2)^2\| = \|A^2\|^2 = \|A\|^4$

נכון לכל $n = 2^k$

באופן כללי: $\|A^8\| = \|A^5 A^3\| \leq \|A^5\| \|A^3\|$

$\leq \|A\|^8$

$$\|A^3 A^5\| = \|A^3\| \, \|A^5\| = \|A\|^8$$

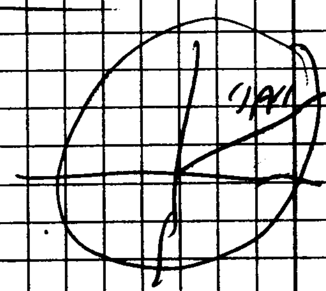
$$\|A^5\| = \|A\|^5$$

$$\|A^3\| \, \|A^5\| < \|A\|^8$$

$n < 2^2$

$spec(A)$

$spec(A) \subset$



$$R_\lambda(A) = -\frac{1}{\lambda}\left(\sum_{k=0}^{\infty} \frac{A^k}{\lambda^k}\right)$$

$$|\lambda| > \|A\|$$

$$r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$$

A

$$|\lambda| > r(A) \qquad R_\lambda(A)$$

$$\left\|\sum \frac{A^k}{\lambda^k}\right\| \le \sum \frac{\|A^k\|}{|\lambda|^k} \le \sum_k \frac{(r(A)+\varepsilon)^k}{|\lambda|^k} \longrightarrow$$

[illegible] 156 [illegible] $\{a_n\}$ [illegible]

[illegible] $a_{n+m} \le a_n \cdot a_m$

[illegible] $a_n^{1/n}$ $\lim_n$ $\exists$

[illegible] $a_n = \|A^n\|$

[illegible] $r(A) = \lim_n \|A^n\|^{\frac{1}{n}}$

[illegible] $r(A) \le \|A\|$ [illegible] $\forall n \quad \|A^n\|^{\frac{1}{n}} \le \|A\|^{n\cdot\frac{1}{n}} = \|A\|$

[illegible] A [illegible] $\|A^n\| = \|A\|^n$

[illegible] $r(A) = \|A\|$

[illegible] $r(A) = \max_{\lambda \in spec(A)} |\lambda|$ [illegible] $r(A)$ [illegible] $spec$ [illegible]

[illegible]

4

157

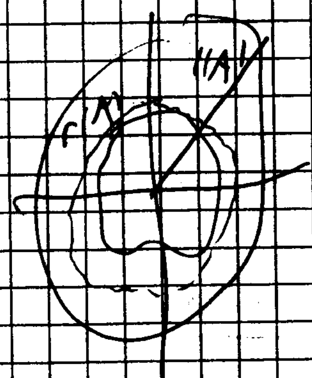
17.6.13

[illegible]

[illegible]

$A: H \to H$ [illegible] $A^* = A$

[illegible] $\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in H$

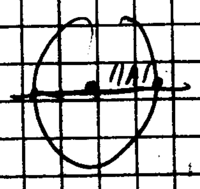


$r(A) = \lim_n \|A^n\|^{1/n}$

$r(A) = \max_{\lambda \in \mathrm{Spec}} |\lambda|$

[illegible] $A = A^*$;

A $r(A) = \|A\|$



$\mathrm{Spec}(A) \subseteq \mathbb{R}$

$\Downarrow$

[illegible] $\|A\|$ [illegible] $-\|A\|$

[illegible] $\mathrm{Spec}(A)$ [illegible]

158

[illegible]

$A : H \to H$ [illegible] H [illegible]

A [illegible]

[illegible] $\{e_n\}$ [illegible] H

$\forall n \quad Ae_n = \lambda_n e_n$ (1)

(2) (1) $\lambda_n \in \mathbb{R}$ [illegible] n

(2) $\lambda_n \to 0$ [illegible] $n \to \infty$

[illegible] $Ae_n = \lambda_n e_n$ $\forall n$

[illegible] $\lambda_n \in \mathbb{R}$ $\forall n$

(2) [illegible] $\lambda_n \in \operatorname{spec}(A) \subset \mathbb{R}$ [illegible]

~~(1)~~ A [illegible] A [illegible]

(1) [illegible] A [illegible]

$\lambda_n \to 0$

הוכחה: נבנה $\{e_n\}$: $A(e_n)=\lambda_n e_n$

ידוע $r(A) = \|A\|$

$\Downarrow$

$\exists \lambda_1 \in spec(A) : |\lambda_1| = \|A\| > 0$

נסמן את הע"ע הזה λ_1 (ידוע A ... $spec(A) = \sigma_p(A) \cup \{0\}$)

$\Downarrow$

$\exists e_1 \quad A e_1 = \lambda_1 e_1$

נורמל $\|e_1\|=1$

ידוע A צמוד לעצמו $spec(A) \subseteq \mathbb{R}$

$\Downarrow$

$\lambda_1 \in \mathbb{R}$

$\Downarrow$

$\lambda_1 = \pm \|A\|$

נגדיר $E_1 = span\{e_1\}$

$H_1 = E_1^{\perp}$

$A|_{H_1}$ צמוד לעצמו קומפקטי.

$A(H_1) \subseteq H_1$

$f \in H_1$ ... Af ... e_1

$\langle f, e_1 \rangle = 0$

$0 \stackrel{?}{=} \langle Af, e_1 \rangle$

$= \langle f, Ae_1 \rangle = \langle f, \lambda_1 e_1 \rangle$

$= \lambda_1 \langle f, e_1 \rangle$

$= 0$

$A|_{H_1} : H_1 \to H_1$

$\left(\|A\|_{H_1 \to H_1} \leq \|A\|_{H \to H} \right)$

$\|A\|_{H_1 \to H_1} > 0$

$A|_{H_1}$

$\forall f, g \in H$

$\langle Af, g \rangle = \langle f, Ag \rangle$

A_ε

$\|A - A_\varepsilon\|_{H \to H} < \varepsilon$

$P_{H_1} \circ A_\varepsilon : H_1 \to H_1$

$\dim \left(P_{H_1} \circ A_\varepsilon \right)$

$\leq \dim \operatorname{Im} A_\varepsilon < \infty$

$\|A - P_{H_1} A_\varepsilon\|_{H_1 \to H_1} = \|P_{H_1}(A - A_\varepsilon)\| \leq \|P_{H_1}\| \|A - A_\varepsilon\|$

$\leq 1 \cdot \varepsilon = \varepsilon$

69

$A: H_1 \to H_1$

$\exists\, e_2 \in H_1$ $\qquad \lambda_2 \in \mathbb{R}$

$\|e_2\| = 1$

$A e_2 = \lambda_2 e_2$

$|\lambda_2| = \|A\|_{H_1 \to H_1} \le \|A\|_{H \to H}$

$\langle e_2, e_1 \rangle = 0$

$E_2 = \mathrm{span}\{e_1, e_2\}$

$H_2 = E_2^{\perp}$

$A(E_2) \subset E_2 \qquad A(H_2) \subset H_2$

$A(E) \subseteq E \qquad A \qquad E$

$A(E^{\perp}) \subset E^{\perp}$

$x \in E^{\perp} \qquad Af \in E^{\perp} \qquad f \in E^{\perp} \Leftarrow$

$\langle f, x \rangle = 0 \Rightarrow \langle Af, x \rangle = \langle f, Ax \rangle = \langle f, x' \rangle = 0$

$\|P_{H_1}\| = 1$

$f = f_1 + f_2$

$H_1 \qquad H_1^{\perp}$

$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$

$\Downarrow$

$\|f_1\| \le \|f\|$

$\|P_{H_1}(f)\| = \|f_1\| \le \|f\|$

$\Downarrow$

$\|P_{H_1}\| \le 1$

169

[illegible] $A|_{H_f}$ [illegible]

[illegible] $\tilde{H} \subset H$ [illegible] A [illegible] ε

[illegible] $A|_{\tilde{H}}$ [illegible]

$$\|A - A_\varepsilon\| < \varepsilon$$

$$\|A - P_{\tilde{H}} A_\varepsilon\|_{\tilde{H} \to \tilde{H}} = \|P_{\tilde{H}}(A - A_\varepsilon)\|_{\tilde{H}} \le \|P\|\,\|A - A_\varepsilon\| < 1 \cdot \varepsilon$$

$\langle e_i, e_j \rangle = \delta_{ij}$

$e_1 \; e_2 \dots e_j \dots$ [illegible]

$|\lambda_1| \ge |\lambda_2| \ge \dots |\lambda_j| \dots$

[illegible]

$H_n = [\mathrm{span}\{e_1 \dots e_n\}]^{\perp}$, $\|A|_{H_n}\|_{H_n \to H_n} \to 0$ [illegible] (1)

[illegible]

$e_1 \dots e_n$

$A e_k = \lambda_k e_k$

$1 \le k \le n$

$A|_{H_n} \equiv 0$

[illegible] $\begin{pmatrix} [illegible] \\ [illegible] \end{pmatrix}$ H_n [illegible]

$\{d_k\}$

$A|_{H_n} = \dots$ [illegible] $A d_k = 0$

$\Downarrow$

[illegible] $\{e_k\}_{k=1}^{n} \cup \{d_k\}$

[illegible]

d_k [illegible]

0 [illegible]

$\Downarrow$

$A = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & 0 \end{pmatrix}$

(2) [illegible]

[illegible]

$e_1, e_2, \dots$

$|\lambda_1| \geq |\lambda_2| \geq \dots$ $\forall k \ |\lambda_k| > 0$

$\forall n \in N \quad A e_n = \lambda_n e_n$

[illegible] $\{e_n\}$ [illegible]

$\overline{E} = \operatorname{span}\{e_n\}_{n=1}^{\infty}$

$H = \overline{E}$

$E^{\perp} \neq 0 \Leftarrow H \neq \overline{E}$

$\left(A|_{E^{\perp}} \equiv 0\right) \qquad A|_{E^{\perp}} : E^{\perp} \to \{0\}$

$|\lambda_n| = \|A|_{H_n}\| \xrightarrow{n} 0$

$\Downarrow$

$A|_{?}$

$\forall n \quad E^{\perp} \subset H_n$

$\Downarrow$

$\|A|_{E^{\perp}}\| \le \|A|_{H_n}\| \longrightarrow 0$

$\Downarrow$

$\|A|_{E^{\perp}}\| = 0$

$\Downarrow$

$A|_{E^{\perp}} \equiv 0$

$\{e_n\} \cup \{\tilde{e}_n\}$ $\qquad E \qquad \{e_n\}$

$E^{\perp} \qquad \{\tilde{e}_n\}$

H

165

[illegible]

[illegible] $L^2(I)$ [illegible]

$k(s,t) \in L^2(I \times I)$ [illegible]

$J : L^2 \to L^2$

$$f \mapsto (Jf)(s) = \int_I k(s,t)\, f(t)\, dt$$

[illegible]

[illegible] J [illegible] $k(s,t) = \overline{k(t,s)}$

[illegible] $\{f_n\}_n$

[illegible] λ_n [illegible]

$\lambda_n \to 0$

$$J(f_n) = \lambda_n f_n$$

[illegible]

$$\int_I k(s,t)\, f(t)\, dt = \lambda f(s)$$

166

$I=[-\pi,\pi]$ [illegible]

$K(s,t) = k(s-t)$

[illegible] $k(x) \in L^2(I)$ [illegible] 2π -

[illegible]

$K(s,t)$ [illegible]

$k(s-t)$

$= k(t-s)$

$= \overline{k(t-s)}$

$= \overline{K(t,s)}$

[illegible] k [illegible]

[illegible]

[illegible]

$e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$ $n \in \mathbb{Z}$

[illegible]

$(Je_n)(s) = \int_{-\pi}^{\pi} k(s-t)\,\frac{e^{int}}{\sqrt{2\pi}}\,dt$

$= \left[u=s-t\right] = \int_{-\pi}^{\pi} k(u)\,\frac{e^{in(s-u)}}{\sqrt{2\pi}}\,du$

[illegible] 2π

$= e^{ins} \underbrace{\int_{-\pi}^{\pi} k(u)\,\frac{e^{-inu}}{\sqrt{2\pi}}\,du}_{\hat{k}(n)}$

$= \hat{k}(n)\, e^{ins}$ $\Rightarrow \operatorname{spec}(J) = \{\hat{k}(n)\} \cup \{0\}$

$(\lambda_n = \hat{k}(n))$

167 $\quad I = [-1, 1]$

$$(Jf)(s) = \int_0^s f(t)\,dt$$

$F(s)$

$F(0)=0$

$\|J\|$

$$\|J\| = \sup_{f \neq 0} \frac{\left\| \int_0^s f(t)\,dt \right\|_{L^2}}{\|f\|_{L^2}}$$

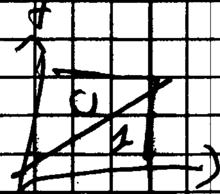
$Jf(s) = s \qquad f = 1$

$$\|Jf\| = \left(\int_0^1 s^2\,ds \right)^{1/2} = \frac{1}{\sqrt{3}} \qquad \|f\|_{L^2} = 1$$

$$\frac{1}{\sqrt{3}} \le \|J\|$$

J

$$K(s,t) = \begin{cases} 1 & t < s \\ 0 & t > s \end{cases}$$



$$k(s,t) = \overline{k(t,s)}$$

(12) 168

$\|J^* J\| = \|J\|^2$ [illegible]

[illegible]

[illegible] $k^*(s,t) = \overline{k(t,s)}$ [illegible]

$$= \begin{cases} 0 & t < s \\ 1 & t > s \end{cases}$$

[illegible] $J^* J$ [illegible]

$$(k^* * k)(s,t) = \int_0^1 k^*(s,\tau)\, k(\tau,t)\, d\tau$$

$$= \int_{\max\{s,t\}}^{1} d\tau = \begin{cases} 1-s & t < s \\ 1-t & t > s \end{cases}$$

$A = J^* J$ [illegible]

$$(Af)(s) = \int_0^1 (k^* * k)\, f(t)\, dt$$

$$= (1-s)\int_0^s f(t)\, dt + \int_s^1 (1-t)\, f(t)\, dt$$

$\|A\| = \|J\|^2$ [illegible] A

[illegible] A

$\Downarrow$

$$\|A\| = \max_{\lambda_n} |\lambda_n|$$

(*) $$(1-s)\int_0^s f(t)\,dt + \int_s^1 (1-t) f(t) = \lambda f(s)$$

[illegible] $f(s)$ $\Rightarrow$ [illegible]

[illegible] $f(s)$ [illegible]

$\Leftarrow$ $f(s)$ [illegible]

([illegible] s [illegible])

(**) $$-\int_0^s f(t)\,dt + (1-s)\cancel{f(s)} - \cancel{(1-s)f(s)} = \lambda f'$$

([illegible])

$$-f(s) = \lambda f''(s)$$

$\Downarrow$

$$f''(s) = -\frac{1}{\lambda} f(s)$$

170

עבור $\lambda > 0$

$$f(s) = c_1 \cos\left(\frac{s}{\sqrt{\lambda}}\right) + c_2 \sin\left(\frac{s}{\sqrt{\lambda}}\right)$$

תנאי ב (*) $s=1$ עבור $f(1)=0$

תנאי ב (**) $s=0$ עבור $f'(0)=0$

⇓

$$f'(s) = -\frac{c_1}{\sqrt{\lambda}} \sin\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{c_2}{\sqrt{\lambda}} \cos\left(\frac{s}{\sqrt{\lambda}}\right)$$

$$0 = f'(0) = \frac{c_2}{\sqrt{\lambda}} \Rightarrow c_2 = 0$$

$$0 = f(1) = c_1 \cos\left(\frac{1}{\sqrt{\lambda}}\right) \Rightarrow \frac{1}{\sqrt{\lambda}} = \frac{\pi}{2} + \pi n$$

$n \in \mathbb{N}$

⇓

$$\lambda_n = \frac{1}{\left(\frac{\pi}{2} + \pi n\right)^2} \qquad n \in \mathbb{N}$$

פונקציות עצמיות מנורמלות

$$e_n(s) = \sqrt{2} \cos\left(\left(\frac{\pi}{2} + \pi n\right) s\right)$$

נורמה

לנרמל

[illegible]

[illegible] $\{e_n\}_{n=0}^{\infty}$ [illegible]

$\Downarrow$

$$\|A\| = \max_n |\lambda_n| = \lambda_0 = \frac{4}{\pi^2}$$

$\Downarrow$

$$\|J\| = \sqrt{\|A\|} = \frac{2}{\pi}$$

$f(s) = \cos \frac{\pi}{2} s$ [illegible]

$$\|Jf\|^2 = \langle Jf, Jf \rangle = \langle J^*Jf, f \rangle$$

$$= \langle Af, f \rangle = \lambda_0 \langle f, f \rangle = \lambda_0 \|f\|^2$$

$\Downarrow$

$$\frac{\|Jf\|}{\|f\|} = \sqrt{\lambda_0}$$

בחינה

זמן: 3 שעות

5 שאלות מהן 2

שאלות קצרות על נושאים ספציפיים

נושאים

1 מרחבים נורמיים

$L^*(Z)$

$C(Z)$

e_1, e_2, e_∞

e_p

הגדר - ...

בסיס ספרבילי

משפט ...

... - ...

... בין ...

- ... ...

... ...

(... ... ...
...)

2 מרחבי הילברט

- ... ...

- ... ...

- ... ...

173

[illegible]

[illegible]

[illegible]

4) [illegible]

[illegible]

[illegible] (spec [illegible]

[illegible] R_s [illegible]

[illegible]

[illegible]

spec [illegible]

(spec [illegible])

174

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

11^{00} 27/6

12^{00} 17/7