63 $\|h\| = 1$ [illegible] $h \in G^\perp$ [illegible]

$$H = G \oplus G^\perp$$ [illegible]

$\forall x \in H$: $x = g + \lambda h$

G $\quad$ $G^\perp$ ← [illegible]

$$= g + \langle x, h\rangle h$$

$$\Downarrow$$

$$f(x) = f(g) + \langle x, h\rangle f(h)$$

$$= \langle x, \underbrace{\overline{f(h)}\, h}_{a} \rangle$$

[illegible]

$$\forall x \quad \langle x, b\rangle = f(x) = \langle x, a\rangle$$

$$\Downarrow$$

$$\forall x \quad \langle x, b-a\rangle = 0$$

$$\Downarrow$$

$$b-a \in H^\perp$$

$$\Downarrow$$

$$b-a = 0$$

$$\Downarrow$$

$$b = a$$

64

$H = \ell_2$

$f(x) = \sum_{k=1}^{\infty} \alpha_k x_k$

$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$

$\|f\| = \left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2}$

$1 < p, q < \infty$

$\frac{1}{p} + \frac{1}{q} = 1$

$\alpha \in \ell_q$

$\forall x \in \ell_p \qquad f(x) = \sum_{k=1}^{\infty} \alpha_k x_k$

$\Downarrow$

$|f(x)| \le \left(\sum_{k=1}^{\infty} |\alpha_k|^q\right)^{1/q} \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$

$\|\alpha\|_q \qquad \|x\|_p$

$\|f\| \le \|\alpha\|_q$

$\Downarrow$

$\alpha \in \ell_q$

ℓ_p

$f : l_p \to \mathbb{C}$ [illegible]

$a_k = f(0,\dots,0,\underset{k}{1},0,\dots,0)$ [illegible]

$\Downarrow$

$$\left(\sum_{k=1}^{n} |a_k|^q\right)^{1/q} = \sup_{\substack{(x_1,\dots,x_n) \\ \sum_{k=1}^{n}|x_k|^p \le 1}} \left|\sum_{k=1}^{n} a_k x_k\right|$$

$$= \sup_{\|x\|_p \le 1} \left| f(x_1, x_2, \dots, x_n, 0, 0, \dots) \right| \le \|f\|$$

$\Downarrow$

$$\|a\|_q \le \|f\| < \infty$$

[illegible] $\|f\| \le \|a\|_q$

$\Downarrow$

$$\|f\| = \|a\|_q$$

$f(x) = \sum_{k=1}^{\infty} a_k x_k$ [illegible] l_1 [illegible]

$a \in l_\infty$ [illegible]

[illegible]

$$f(x) = \sum_{k=1}^{\infty} a_k x_k \quad \forall x \in l_p \quad \binom{1<p}{p\neq 1}$$

[illegible] $x = (x_k)_{k=1}^{\infty}$ l_p [illegible] x^n [illegible]

$(x^n)_k = \begin{cases} x_k & k \le n \\ 0 & \end{cases}$ [illegible] $\|x - x^n\|_p = \left(\sum_{k=n+1}^{\infty} |x_k|^p\right)^{1/p}$ [illegible]

$\sum_{k=1}^{n} a_k x_k = f(x^n) \xrightarrow{n\to\infty} f(x)$ [illegible]

ℓ_∞

$a \in \ell_1 \longmapsto f(x) = \sum_{k=1}^{\infty} a_k x_k$

$c_0 = \{ x = (x_n)_n \mid x_n \to 0 \} \subseteq \ell_\infty$

$\|x\| = \sup_n |x_n|$

c_0, ℓ_∞

$f(x) = \sum_{k=1}^{\infty} a_k x_k$, $a \in \ell_1$

$x_k \xrightarrow{k} 0$, $x^n \to x$

$|x_k| \le |x_k - (x^n)_k| + |(x^n)_k| \le \|x - x^n\| + |(x^n)_k|$

$\lim_k |x_k| \le \|x - x^n\| \xrightarrow{n\to\infty} 0$

$a_k = f(0, \dots, \overset{k}{1}, 0, \dots)$

$\sum_{k=1}^{n} |a_k| = \sup_{\|x\|_\infty \le 1} \left| \sum_{k=1}^{n} a_k x_k \right| = \sup |f(x_1, \dots, x_n, 0, \dots)| \le \|f\|$

$\sum_{k=1}^{n} a_k x_k = f(x_1, \dots, x_n, 0, \dots) \to f(x)$

$f(x) = \sum_{k=1}^{\infty} a_k x_k$

$|f(x)| \le \|a\|_1 \|x\|_\infty$

$\|a\|_{\ell_1} \le \|f\|$

$x_k \to 0$

$k \ge k_0$, $|x_k| < \varepsilon$

$\|x - x^{k_0}\| < \varepsilon$

[illegible]: [illegible] $C(I)$, $\|f\| = \sup_{x \in I} |f(x)|$

[illegible] $\varphi(f) = \int_I g(t) f(t)\,dt$

$$\Downarrow$$

$$|\varphi(f)| \le \int_I |g(t)|\,|f(t)|\,dt$$

$$\le \|f\| \int_I |g(t)|\,dt$$

$$\Downarrow$$

$$\|\varphi\| \le \int_I |g(t)|\,dt$$

[illegible] $|\varphi(f)| \le \|f\| \int_I |g(t)|\,dt$

[illegible] $f = \operatorname{sign} g(t)$

$$= \begin{cases} 1 & g(t) > 0 \\ -1 & g(t) < 0 \end{cases}$$ [illegible]

[illegible] $\operatorname{sign} g \notin C(I)$ [illegible]

$$\Downarrow$$

[illegible] $\{f_n(t)\}$ [illegible] $\operatorname{sign} g(t)$

$$\Downarrow$$

$$\lim_n |\varphi(f_n)| = \int_I |g(t)|\,dt$$

$$\Downarrow$$

$$\|\varphi\| = \int_I |g(t)|\,dt$$

$C(I)$

$\Gamma_{t_0} : f \longrightarrow f(t_0)$

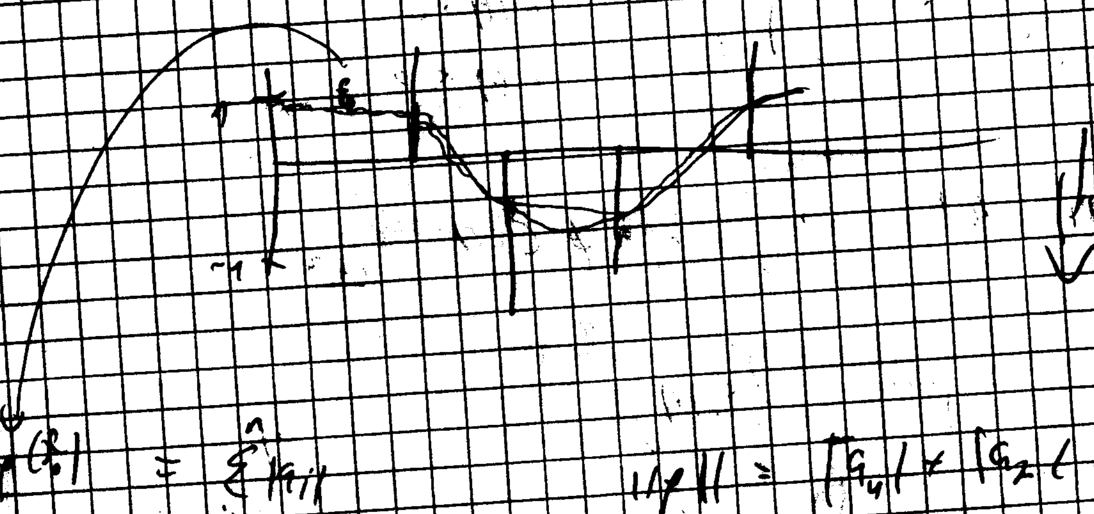
$\|\delta_{t_0}\| = 1$

$\|f\| \le 1$

$f = 1$

$f : C(I) \to \mathbb{R}$

$f \mapsto \left(a_1 \delta_{t_1} + a_2 \delta_{t_2} + \dots a_n \delta_{t_n}\right)$



$f(\xi) = \sum_{i=1}^{n} |a_i|$ $\qquad \|f\| = |a_1| + |a_2| + \dots$

(6) [illegible]

$(C(I), \|\cdot\|_\infty)$ [illegible] $L^2(I)$ [illegible] $C(I)$

$I = [-1,1]$ $\qquad \varphi(f) = \int_{-1}^{0} f(t)\,dt - \int_{0}^{1} f(t)\,dt \qquad (1)$

$I=[0,\pi]$ $\qquad \varphi(f) = \int_0^\pi \frac{\sin(nt)}{t} f(t)\,dt \qquad (2)$

$I=[0,\pi]$ $\qquad \varphi(f) = \int_0^\pi \cos t\, f(t)\,dt - f\left(\frac{\pi}{2}\right) \qquad (3)$

$L^2(I)$ (1)

$\varphi(f) = \langle f, g\rangle \qquad g(x) = \begin{cases} 1 & -1 \le x \le 0 \\ -1 & 0 < x \le 1 \end{cases}$

$\|\varphi\|_{L^2} = \|g\| = \int_{-1}^{1} 1\,dx = 2$ [illegible] $L^2(I)$ [illegible]

$\|g_n\| \le 1 \quad g_n$

[illegible] $C(I)$

$\|\varphi\|_{C(I)} = \sup_{\|f\|=1} \|\varphi(f)\| \ge 2$

g_n [illegible]

$\forall \|f_n\| = 1$

$|\varphi(f)| \le \int_{-1}^{0} |f(t)|\,dt + \int_0^1 |f(t)|\,dt \le \int_{-1}^{0} 1 + \int_0^1 1 = 2$

$\Downarrow$

$\|\varphi\|_{C(I)} = 2$

$\varphi(f) = \langle \frac{\sin nt}{t}, f \rangle \qquad L^2(I) \qquad (2)$

$\Downarrow$

$\|\varphi\|_{L^2} = \left\| \frac{\sin nt}{t} \right\| = \left(\int_0^\pi \left(\frac{\sin nt}{t}\right)^2 dt \right)^{1/2}$

$\|x\| = 1$ $C[I] \to$

$|f(x)| \le \int_0^\pi \frac{\sin t}{t} |x(t)| \, dt \le \int_0^\pi \left|\frac{\sin t}{t}\right| dt$

$C[I]$

$f(x) = \int_I g(t)\, x(t) \, dt$

$\|f\| = \int_I |g(t)|$

$\|f\|_{C[I]} = \int_0^\pi \left|\frac{\sin t}{t}\right| dt$

$C[I] \to$

$g(x) = \begin{cases} 1 & 0 \le x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < x \le \pi \end{cases}$

g_n

$\|g_n\| = 1$

$\|f\|_{C[I]} = \int_0^\pi |\cos t| \, dt + 1$

$y = \cos t$

$f(t) = \langle f, g \rangle + f(\frac{\pi}{2})$ $L^2(I) \to$

$f = \frac{g}{\|g\|}$ $\|f\| = 1$ $|\langle f, g \rangle|$

$\sup |f(x)| = f\left(\frac{g}{\|g\|}\right) = \|g\|$

2.5.13

הגדרה: פונקציונל ליניארי –
$(X, \| \ \|)$ מרחב נורמי
$f: X \to \mathbb{C}$ (או $\mathbb{R}$) ליניארי

$$\|f\| = \sup_{\|x\| \le 1} |f(x)| = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \ne 0} \frac{|f(x)|}{\|x\|}$$

משפט $\|f\| < \infty \iff f$ רציף.

נסמן $X^* = \{ f \text{ ליניארי רציף} \}$ (ב)

דוגמה. אם $X = H$ מרחב הילברט

$$H^* = X^* = \{ f_a = \langle \cdot, a \rangle \}_{a \in H}$$

$$H \underset{\text{אנטי-לינארי}}{\cong} H^*$$

$$a \mapsto f_a$$