פתרון תרגיל 7

.1

<u>**פתרון**</u>

נציב $x=\pi-t$ ז"א $g(t)=\pi-t$. $g(\pi)=0, g(0)=\pi$.

$dx=-dt \Leftarrow x=\pi-t$.

$$\int_0^\pi \frac{x\sin x}{1+\sin^2 x}dx=-\int_\pi^0\frac{(\pi-t)\sin(\pi-t)}{1+\sin^2(\pi-t)}dt=\int_0^\pi\frac{\pi\sin t-t\sin t}{1+\sin^2 t}dt=\int_0^\pi\frac{\pi\sin t}{1+\sin^2 t}dt-\int_0^\pi\frac{t\sin t}{1+\sin^2 t}dt$$

סה"כ קיבלנו

$$2\int_0^\pi\frac{x\sin x}{1+\sin^2 x}dx=\int_0^\pi\frac{\pi\sin t}{1+\sin^2 t}dt \Leftarrow \int_0^\pi\frac{x\sin x}{1+\sin^2 x}dx=\int_0^\pi\frac{\pi\sin t}{1+\sin^2 t}dt-\int_0^\pi\frac{t\sin t}{1+\sin^2 t}dt$$

נשאר לחשב את האינטגרל $\int_0^\pi\frac{\pi\sin x}{1+\sin^2 x}dx$.

נחשב תחילה את האינטגרל הלא מסוים $\int\frac{\pi\sin x}{1+\sin^2 x}dx=\int\frac{\pi\sin x}{2-\cos^2 x}dx$. נציב $t=\cos x$ ואז $dt=-\sin x dx$

$$\int\frac{\pi\sin x}{1+\sin^2 x}dx=-\pi\int\frac{1}{2-t^2}dt=-\frac{\pi}{2\sqrt2}\int\left(\frac{1}{\sqrt2-t}+\frac{1}{\sqrt2+t}\right)dt=$$

$$=-\frac{\pi}{2\sqrt2}\left(-\ln(\sqrt2-t)+\ln(\sqrt2+t)\right)=-\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+t}{\sqrt2-t}\right)=-\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+\cos x}{\sqrt2-\cos x}\right)$$

$$\frac{\pi\sin x}{1+\sin^2 x}dx=\left[-\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+\cos x}{\sqrt2-\cos x}\right)\right]_0^\pi=-\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+\cos\pi}{\sqrt2-\cos\pi}\right)-\left[-\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+\cos0}{\sqrt2-\cos0}\right)\right]$$

$$\frac{\pi}{2\sqrt2}\ln\left(\frac{\sqrt2+1}{\sqrt2-1}\right)^2=\frac{\pi}{\sqrt2}\ln\frac{\sqrt2+1}{\sqrt2-1}$$

נקבל ש

$$\int_0^\pi\frac{x\sin x}{1+\sin^2 x}dx=\frac{\pi^2}{4}$$

.3

נחשב $\int_{\frac1e}^{e}|\ln x|dx$

מתקיים: $|\ln x|=\begin{cases}-\ln x & ; \quad x\in\left[\frac1e,1\right];\\ \ln x & ; \quad x\in[1,e]\end{cases}$, ולכן:

$$(dx=dv;\ \ln x=u) \quad \Rightarrow \quad \int\ln x dx=x\ln x-\int dx=x\ln x-x+C$$

$$\int_{\frac1e}^{e}|\ln x|dx=\int_{\frac1e}^{1}(-\ln x)dx+\int_1^e\ln x dx=(-x\ln x+x)\Big|_{\frac1e}^{1}+(x\ln x-x)\Big|_1^e=2(1-e^{-1})$$

.4

שאלה 2

א $\displaystyle\int_1^3 x^3\sqrt{x^2-1}\,dx = \int_1^3 x^2\cdot x\sqrt{x^2-1}\,dx = \int_0^8 (1+t)\sqrt{t}\cdot\frac{1}{2}\,dt =$

הצבה: $x^2-1=t$ שינוי גבולות:

$\Rightarrow x^2=1+t$ $2x\,dx=dt$ $x=1 \Rightarrow 1=1+t \Rightarrow t=0$

$x=\sqrt{1+t}$ $x\,dx=\frac{1}{2}dt$ $x=3 \Rightarrow 9-1=t \Rightarrow t=8$

$$= \frac{1}{2}\int_0^8\left(\sqrt{t}+t\sqrt{t}\right)dt = \frac{1}{\cancel{2}}\,\frac{t^{3/2}}{3/\cancel{2}}\Big|_0^8 + \frac{1}{\cancel{2}}\,\frac{t^{5/2}}{5/\cancel{2}}\Big|_0^8 = \frac{8^{3/2}}{3}+\frac{8^{5/2}}{5} =$$

$$= \frac{5\cdot 2^{9/2}+3\cdot 2^{15/2}}{15} = \frac{5\cdot 2^4\sqrt{2}+3\cdot 2^7\cdot\sqrt{2}}{15} = \frac{464\sqrt{2}}{15}$$