EXERCISES FOR CHAPTER 6: Taylor and Maclaurin Series

1. Find the first 4 terms of the Taylor series for the following functions:

(a) $\ln x$ centered at a=1, (b) $\dfrac{1}{x}$ centered at a=1, (c) $\sin x$ centered at $a = \dfrac{\pi}{4}$.

Solution

(a) $f(x) = \ln x$. So $f^{(1)}(x) = \dfrac{1}{x}$, $f^{(2)}(x) = -\dfrac{1}{x^2}$, $f^{(3)}(x) = \dfrac{2}{x^3}$, $f^{(4)}(x) = -\dfrac{6}{x^4}$ and so

$$\ln x = \ln 1 + (x-1)\times 1 + \frac{(x-1)^2}{2!}\times(-1) + \frac{(x-1)^3}{3!}\times(2) + \frac{(x-1)^4}{4!}\times(-6) + \ldots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

(b) $f(x) = \dfrac{1}{x}$. So $f^{(1)}(x) = -\dfrac{1}{x^2}$, $f^{(2)}(x) = \dfrac{2}{x^3}$, $f^{(3)}(x) = -\dfrac{6}{x^4}$ and so

$$\frac{1}{x} = 1 + (x-1)\times(-1) + \frac{(x-1)^2}{2!}\times(2) + \frac{(x-1)^3}{3!}\times(-6) + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 - \cdots$$

(c) $f(x) = \sin x$. So $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$ and so

$$\sin x = \frac{\sqrt{2}}{2} + (x - \frac{\pi}{4})\times(\frac{\sqrt{2}}{2}) + \frac{(x-\frac{\pi}{4})^2}{2!}\times(-\frac{\sqrt{2}}{2}) + \frac{(x-\frac{\pi}{4})^3}{3!}\times(-\frac{\sqrt{2}}{2}) + \cdots$$

$$= \frac{\sqrt{2}}{2}\left(1 + (x-\frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{2} - \frac{(x-\frac{\pi}{4})^3}{6} + \cdots\right)$$

2. Find the first 3 terms of the Taylor series for the function $\sin \pi x$ centered at a=0.5. Use your answer to find an approximate value to $\sin(\dfrac{\pi}{2} + \dfrac{\pi}{10})$

Solution

$f(x) = \sin \pi x$. So $f^{(1)}(x) = \pi \cos \pi x$, $f^{(2)}(x) = -\pi^2 \sin \pi x$, $f^{(3)}(x) = -\pi^3 \cos \pi x$, $f^{(3)}(x) = \pi^4 \sin \pi x$ and so

$$\sin \pi x = \sin\frac{\pi}{2} + \frac{(x-\frac{1}{2})^2}{2!}\times(-\pi^2) + \frac{(x-\frac{1}{2})^4}{4!}\times(\pi^4) + \ldots$$

$$= 1 - \pi^2\frac{(x-\frac{1}{2})^2}{2!} + \pi^4\frac{(x-\frac{1}{2})^4}{4!} + \ldots$$

$$\sin\pi\left(\frac{1}{2}+\frac{1}{10}\right)=1-\pi^2\frac{\left(\frac{1}{10}\right)^2}{2!}+\pi^4\frac{\left(\frac{1}{10}\right)^4}{4!}+\dots$$
$$=1-0.0493+0.0004$$
$$=0.9511$$

(To 4 decimal places the GDC gives the same answer.)

3. Find the Taylor series for the function x^4+x-2 centered at a=1.
Solution
$f(x)=x^4+x-2$. $f^{(1)}(x)=4x^3+1$, $f^{(2)}(x)=12x^2$, $f^{(3)}(x)=24x$, $f^{(4)}(x)=24$ and all other derivatives are zero. Thus

$$x^4+x-2=0+(x-1)\times 5+\frac{(x-1)^2}{2!}\times 12+\frac{(x-1)^3}{3!}\times 24+\frac{(x-1)^4}{4!}\times 24$$
$$=5(x-1)+6(x-1)^2+4(x-1)^3+(x-1)^4$$

4. Find the first 4 terms in the Taylor series for $(x-1)e^x$ near x=1.
Solution
Either find the Taylor series for e^x and then multiply by $(x-1)$:
$f(x)=e^x \Rightarrow f(1)=e$
$f^{(1)}(x)=e^x \Rightarrow f^{(1)}(1)=e$
$f^{(2)}(x)=e^x \Rightarrow f^{(2)}(1)=e$
$f^{(3)}(x)=e^x \Rightarrow f^{(3)}(1)=e$
so that
$$(x-1)e^x\approx(x-1)\left(e+(x-1)e+\frac{(x-1)^2}{2!}e+\frac{(x-1)^3}{3!}e\right)$$
$$\approx(x-1)e+(x-1)^2e+\frac{(x-1)^3}{2}e+\frac{(x-1)^4}{6}e$$
or with a bit more work,
$f(x)=(x-1)e^x \Rightarrow f(1)=0$
$f^{(1)}(x)=(x-1)e^x+e^x \Rightarrow f^{(1)}(1)=e$
$f^{(2)}(x)=(x-1)e^x+2e^x \Rightarrow f^{(2)}(1)=2e$
$f^{(3)}(x)=(x-1)e^x+3e^x \Rightarrow f^{(3)}(1)=3e$
$f^{(4)}(x)=(x-1)e^x+4e^x \Rightarrow f^{(4)}(1)=4e$
so that
$$(x-1)e^x\approx 0+(x-1)e+\frac{(x-1)^2}{2!}2e+\frac{(x-1)^3}{3!}3e+\frac{(x-1)^4}{4!}4e$$
$$\approx(x-1)e+(x-1)^2e+\frac{(x-1)^3}{2}e+\frac{(x-1)^4}{6}e$$

5. Find the first 3 terms in the Maclaurin series for (a) $\sin^2 x$, (b) $\dfrac{x}{\sqrt{1-x^2}}$, (c) xe^{-x}, (d) $\dfrac{x}{1+x^2}$.

Solution

(a)

$f(x)=\sin^2 x \Rightarrow f(0)=0$

$f^{(1)}(x)=2\sin x\cos x=\sin 2x \Rightarrow f^{(1)}(0)=0$

$f^{(2)}(x)=2\cos 2x \Rightarrow f^{(2)}(0)=2$

$f^{(3)}(x)=-4\sin 2x \Rightarrow f^{(3)}(0)=0$

$f^{(4)}(x)=-8\cos 2x \Rightarrow f^{(4)}(0)=-8$

$f^{(5)}(x)=16\sin 2x \Rightarrow f^{(5)}(0)=0$

$f^{(6)}(x)=32\cos 2x \Rightarrow f^{(6)}(0)=32$

and so

$$\sin^2 x \approx 0+0+\frac{x^2}{2!}\times 2+0-\frac{x^4}{4!}\times 8+0+\frac{x^6}{6!}\times 32$$

$$\approx x^2-\frac{x^4}{3}+\frac{2x^6}{45}$$

In later exercises you will see more efficient ways to do this expansion.

(b)

$$f(x)=\frac{x}{\sqrt{1-x^2}} \Rightarrow f(0)=0$$

$$f^{(1)}(x)=\frac{\sqrt{1-x^2}-x\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}=\frac{1}{(1-x^2)^{3/2}} \Rightarrow f(0)=1$$

$$f^{(2)}(x)=-\frac{3}{2}(1-x^2)^{-5/2}(-2x) \Rightarrow f(0)=0$$

$$f^{(4)}(x)=-\frac{3}{2}(1-x^2)^{-5/2}(-2)+\frac{3}{2}\frac{5}{2}(1-x^2)^{-5/2}(-2x)(-2x) \Rightarrow f(0)=3$$

(c)

$f(x)=xe^{-x} \Rightarrow f(0)=0$

$f^{(1)}(x)=e^{-x}-xe^{-x} \Rightarrow f^{(1)}(0)=1$

$f^{(2)}(x)=-e^{-x}-e^{-x}+xe^{-x} \Rightarrow f^{(2)}(0)=-2$

$f^{(3)}(x)=2e^{-x}+e^{-x}-xe^{-x} \Rightarrow f^{(3)}(0)=3$

so that

$$xe^{-x} \approx x-2\frac{x^2}{2!}+3\frac{x^3}{3!}$$

$$\approx x-x^2+\frac{x^3}{2}$$

(d) $f(x)=\dfrac{x}{1+x^2}$. Notice that since $\dfrac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$, it follows that $\dfrac{x}{1+x^2}=x-x^3+x^5-x^7+\cdots$.

Of course by evaluating derivatives we also have:

$$f^{(0)}(0)=0$$

$$f^{(1)}(x)=\frac{(1+x^2)-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}\Rightarrow f^{(1)}(0)=1$$

$$f^{(2)}(x)=\frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)2x}{(1+x^2)^4}=\frac{2x(x^2-3)}{(1+x^2)^3}\Rightarrow f^{(2)}(0)=0$$

$$f^{(3)}(x)=-\frac{6(1-6x^2+x^4)}{(1+x^2)^4}\Rightarrow f^{(3)}(0)=-6$$

$$f^{(4)}(x)=\frac{24x(5-10x^2+x^4)}{(1+x^2)^5}\Rightarrow f^{(4)}(0)=0$$

$$f^{(5)}(x)=-\frac{120(-1+15x^2-15x^4+x^6)}{(1+x^2)^6}\Rightarrow f^{(5)}(0)=120$$

so that the first three terms are:

$$\begin{aligned}\frac{x}{1+x^2}&=0+x\times 1+\frac{x^2}{2}\times 0+\frac{x^3}{6}\times(-6)+\frac{x^4}{24}\times 0+\frac{x^5}{120}\times 120\\&=x-x^3+x^5\end{aligned}$$

as before.

6. Find the Maclaurin series for $\ln(1+x)$ and hence that for $\ln\left(\dfrac{1+x}{1-x}\right)$.

Solution

The Maclaurin series for $\ln(1+x)$ is standard:

$\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\cdots$ and so $\ln(1-x)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\cdots$.

Hence $\ln\left(\dfrac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=2(x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\cdots)=2\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$.

7. The function $\ln(1+x)$ is to be approximated by the first three terms of its Maclaurin series, i.e. $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}$. Estimate the maximum value of x for which the approximation agrees with the exact value to 3 decimal places.

Solution

The absolute value of the remainder term in Lagrange form is $\dfrac{6}{(1+c)^4}\dfrac{x^4}{4!}=\dfrac{x^4}{4(1+c)^4}$ where c is between 0 and x. The maximum value of the remainder term is obtained when $c=0$ and so equals $\dfrac{x^4}{4}$. We must then have $\dfrac{x^4}{4}<5\times 10^{-4}$ and so $x<0.211$.

8. By using a suitable Maclaurin series given in the text find the sum to infinity of the following infinite series: (a) $\pi-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\frac{\pi^7}{7!}+\ldots$, (b) $1-\frac{e^2}{2!}+\frac{e^4}{4!}-\frac{e^6}{6!}+\ldots$

Solution

(a) $\sin\pi=0$ (b) $\cos e$

9. Using $\frac{1}{1+x}\equiv 1-x+x^2-x^3+\ldots$ find the Maclaurin series for the function $\frac{1}{2+x}$.

Solution

Since $\frac{1}{2+x}=\frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$ we have that

$$\begin{aligned}\frac{1}{2+x}&=\frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)\\&=\frac{1}{2}(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots)\\&=\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\end{aligned}$$

10. (a) Evaluate the integral $\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$ by first finding the Maclaurin approximation to the integrand with 3 terms. (b) Evaluate the integral exactly and compare. (c) The fourth term in the Maclaurin expansion of $\sin^2 x$ is $-\frac{x^8}{315}$. Confirm that your estimate of the integral is consistent with the alternating series estimation theorem.

Solution

(a) The Maclaurin series for was shown in Exercise 5 to be $\sin^2 x\approx x^2-\frac{x^4}{3}+\frac{2x^6}{45}$. Hence

$$\int_0^{\pi/6}\sin^2 x\,dx\approx\int_0^{\pi/6}(x^2-\frac{x^4}{3}+\frac{2x^6}{45})dx=(\frac{x^3}{3}-\frac{x^5}{15}+\frac{2x^7}{315})\Big|_0^{\pi/6}\approx 0.0452941.$$

(b) The exact integral is

$$\int_0^{\pi/6}\sin^2 x\,dx=\int_0^{\pi/6}\frac{1-\cos 2x}{2}dx=(\frac{x}{2}-\frac{\sin 2x}{4})\Big|_0^{\pi/6}=\frac{\pi}{12}-\frac{\sin(\pi/3)}{4}$$

$$=\frac{\pi}{12}-\frac{\sin(\pi/3)}{4}=\frac{2\pi-3\sqrt{3}}{24}\approx 0.045293$$

(c) The error is expected to have the same sign and less (in absolute value) as the first term neglected i.e. $\int_0^{\pi/6} -\frac{x^8}{315}dx = -\left.\frac{x^9}{9\times 315}\right|_0^{\pi/6} \approx -1.04336\times 10^{-6}$. The actual error is $0.045293 - 0.0452941 = -1.033\times 10^{-6}$ consistently with the alternating series estimation theorem.

11. Find the Maclaurin series for $x\sin x$.
Solution
Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, it follows that $x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$.

12. (a) Find the Maclaurin series for $(1+x)^m$ where *m* is not necessarily an integer and hence show that the formula for the binomial series works for non-integral exponents as well. (b) Use your answer to find the expansion of $\frac{1}{\sqrt{1-x^2}}$ up to the term in x^6.
Solution
$f(x) = (1+x)^m \Rightarrow f(0) = 1$
$f^{(1)}(x) = m(1+x)^{m-1} \Rightarrow f^{(1)}(0) = m$

$f^{(2)}(x) = m(m-1)(1+x)^{m-2} \Rightarrow f^{(2)}(0) = m(m-1)$
$f^{(3)}(x) = m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f^{(3)}(0) = m(m-1)(m-2)$
$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k} \Rightarrow f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1)$
and so

$$\begin{aligned}(1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \\ &\quad + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots\end{aligned}$$

This is an infinite series. If *m* is a positive integer the series will stop when $k = m$ and will agree with the standard binomial expansion.
(b)

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= (1-x^2)^{-1/2} \\ &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3x^4}{8} + \frac{5x^6}{16} + \cdots\end{aligned}$$

13. *For Physics students.* The kinetic energy of a relativistic particle is given by $K = (\gamma - 1)mc^2$ where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Here *m* is the constant mass of the particle, *v* its speed and c is the constant speed of light. Use the result of the previous problem to show that for $v \ll c$, $K \approx \frac{1}{2}mv^2$.

Solution

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$\approx 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\cdots,$$

$$\approx 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2+\cdots$$

and so $K \approx (1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots-1)mc^2 = \frac{1}{2}mv^2+\frac{3}{8}mv^2\left(\frac{v}{c}\right)^2+\cdots$. This is approximately $K \approx \frac{1}{2}mv^2$ if $v \ll c$ since the neglected terms are "small".

14. Find the Maclaurin series for $\sin^2 x$ using the series for $\cos 2x$. Hence find $\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^4}$.

Solution

Since $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos 2x = 1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots$ it follows that

$$\sin^2 x = \frac{1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\cdots\right)}{2}$$

$$= \frac{2x^2-\frac{2x^4}{3}+\frac{64x^6}{720}+\cdots}{2}$$

$$= x^2-\frac{x^4}{3}+\frac{2x^6}{45}+\cdots$$

$$\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^4} = \lim_{x\to 0}\frac{x^2-\frac{x^4}{3}+\frac{2x^6}{45}+\cdots-x^2}{x^4}$$

$$= \lim_{x\to 0}(-\frac{1}{3}+\frac{2x^2}{45}+\cdots)$$

$$= -\frac{1}{3}$$

15. Find the first 3 terms in the Maclaurin series for $\cos(\sin x)$. Hence or otherwise find $\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2}$.

Solution

From $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ and $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$ we have that

$$
\begin{aligned}
\cos(\sin x) &= 1-\frac{(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)^2}{2!}+\frac{(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)^4}{4!}-\cdots \\
&= 1-\frac{x^2}{2!}+\frac{2x^4}{3!2!}+\frac{x^4}{4!}+\cdots \\
&= 1-\frac{x^2}{2}+\frac{5x^4}{24}+\cdots
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2} &= \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+\frac{5x^4}{24}+\cdots)}{x^2} \\
&= \lim_{x\to 0}\frac{(\frac{x^2}{2}-\frac{5x^4}{24}+\cdots)}{x^2} \\
&= \lim_{x\to 0}\left(\frac{1}{2}-\frac{5x^2}{24}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

16. Find the first 3 terms in the Maclaurin series for $\sin(\sin x)$. Hence or otherwise find $\lim_{x\to 0}\frac{x-\sin(\sin x)}{x^3}$

Solution

We know that $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$ so

$$
\begin{aligned}
\sin(\sin x) &= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)-\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^3}{3!}+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^5}{5!}+\cdots \\
&= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^3}{3!}+\frac{3x^5}{3!3!}+\frac{x^5}{5!}+\cdots \\
&= x-\frac{x^3}{3}+\frac{x^5}{10}+\cdots
\end{aligned}
$$

Thus

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin(\sin x)}{x^3} &= \lim_{x\to 0}\frac{x-(x-\frac{x^3}{3}+\frac{x^5}{10}+\cdots)}{x^3}\\ &= \lim_{x\to 0}\frac{\frac{x^3}{3}-\frac{x^5}{10}-\cdots}{x^3}\\ &= \lim_{x\to 0}\left(\frac{1}{3}-\frac{x^2}{10}\right)\\ &= \frac{1}{3}\end{aligned}$$

17. The equation $e^{-2x}=3x^2$ has a root near $x=0$. By finding a suitable polynomial approximation to e^{-2x} find an approximation to this root.

Solution
We use the Maclaurin series for the exponential.

$e^{-2x}\approx 1+(-2x)+\dfrac{(-2x)^2}{2!}=1-2x+2x^2$. Hence

$$\begin{aligned}1-2x+2x^2 &= 3x^2\\ x^2+2x-1 &= 0\\ x=\frac{-2\pm\sqrt{8}}{2} &= \begin{cases}0.414\\ -1.41\end{cases}\end{aligned}$$

We want the positive root. The actual root is 0.390.

18. (a) Find the Maclaurin series for the function $f(x)=\ln(1+x)$ and hence that for $\dfrac{\ln(1+x)}{x}$. (b) By keeping the first four terms in the Maclaurin series for $\dfrac{\ln(1+x)}{x}$ integrate the function $\dfrac{\ln(1+x)}{x}$ from $x=0$ to $x=1$. (c) Estimate the magnitude and sign of the error made in evaluating the integral in this way. (d) Confirm that your results are consistent with the alternating series estimation theorem by evaluating the integral on the GDC.

Solution
(a) With $f(x)=\ln(1+x)$ we have $f(0)=\ln(1)=0$.

$$f^{(1)}(x)=\frac{1}{1+x}\Rightarrow f^{(1)}(0)=1$$
$$f^{(2)}(x)=-(1+x)^{-2}\Rightarrow f^{(2)}(0)=-1$$
$$f^{(3)}(x)=(-1)(-2)(1+x)^{-3}\Rightarrow f^{(3)}(0)=(-1)(-2)$$
$$\vdots$$
$$f^{(k)}(x)=(-1)^{k-1}(k-1)!(1+x)^{-k}\Rightarrow f^{(k)}(0)=(-1)^{k-1}(k-1)!$$

Hence

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ and so $\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$.

(b)

$$\int_0^1 \frac{\ln(1+x)}{x} dx = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots \Big|_0^1$$
$$= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$
$$= 0.7986$$

(c) The error is less in magnitude than the first term neglected i.e. $\frac{1}{5^2} = 0.0400$ and has the same sign (i.e. positive) so the estimate is an underestimate.

(d)The value of the integral is 0.8225. The error is thus 0.8225-0.7986=+0.0239.

19. Find the first 3 terms in the Maclaurin series for $\sqrt{1 - x + x^2}$.

Solution

Work first with the function $f(x) = \sqrt{1-x}$ and then substitute $x - x^2$ for x in the answer.

$f(0) = 1$

$$f'(x) = \frac{1}{2}\frac{-1}{\sqrt{1-x}} \Rightarrow f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{2}(-\frac{1}{2})(1-x)^{-3/2}(-1) \Rightarrow f''(0) = -\frac{1}{4}$$

So we have that:

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

and so

$$\sqrt{1-x+x^2} = 1 - \frac{1}{2}(x - x^2) - \frac{1}{8}(x-x^2)^2 + \cdots$$
$$= 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^2}{8} + \cdots$$
$$= 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots$$

20. (a) Find the Maclaurin series for the function $f(x) = \frac{1}{1+x^2}$. (b) Hence by integrating each term, show that $\arctan x = \sum_1^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$. (c) Show that this series converges for $-1 \le x \le 1$ and hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Solution

(a) The Maclaurin series for $f(x)=\dfrac{1}{1+x}$ is $1-x+x^2-x^3+\cdots$ so that the series for $f(x)=\dfrac{1}{1+x^2}$ is $1-x^2+x^4-x^6+\cdots$. Including the remainder term we have:

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots+(-1)^n x^{2n}+(-1)^{n+1}\frac{x^{2n+2}}{1+c^2}$$

(b) Integrating

$$\int_0^x \frac{1}{1+t^2}dt=\int_0^1(1-t^2+t^4-t^6+\cdots+(-1)^n t^{2n})dt$$

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^{n+1}\frac{x^{2n+1}}{2n+1}+R_n$$

where $|R_n|=\left|\int_0^x \frac{t^{2n+2}}{1+c^2}dt\right|$.

Hence $\arctan x=\sum_1^\infty (-1)^{n+1}\dfrac{x^{2n-1}}{2n-1}$.

(c) For the remainder term it is true that:

$|R_n|=\left|\int_0^x \frac{t^{2n+2}}{1+c^2}dt\right|\le\left|\int_0^x t^{2n+2}\,dt\right|=\dfrac{|x|^{2n+3}}{2n+3}$ since $\dfrac{1}{1+c^2}\le 1$, and so $\lim_{n\to\infty}|R_n|=\lim_{n\to\infty}\dfrac{|x|^{2n+3}}{2n+3}=0$ provided $-1\le x\le 1$.

Thus the series converges for $-1\le x\le 1$. Setting $x=1$ gives

$$\arctan 1=\frac{\pi}{4}=\sum_1^\infty(-1)^{n+1}\frac{1}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots.$$

21. Find the Maclaurin series for the function $\dfrac{1}{1-x^2}$ by using partial fractions or otherwise.

Solution

The answer is obtained at once using the known series for $\dfrac{1}{1-x}$ and letting $x\to x^2$.

With partial fractions,

$\dfrac{1}{1-x^2}=\dfrac{1}{(1-x)(1+x)}=\dfrac{A}{1-x}+\dfrac{B}{1+x}$. By the cover up method, $A=\dfrac{1}{2}, B=\dfrac{1}{2}$ so that

$\dfrac{1}{1-x^2}=\dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right)$. Since

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$$
$$\frac{1}{1+x}=1-x+x^2-x^3+x^4+\cdots$$

it follows that

$$\frac{1}{1-x^2} = \frac{1}{2}\left(2+2x^2+2x^4+\cdots\right)$$
$$= 1+x^2+x^4+\cdots$$

22. Find the first 4 terms in the Maclaurin series for the function $\frac{x+1}{x^2-5x+6}$ by first finding the partial fraction decomposition of the function.

Solution

$$\frac{x+1}{x^2-5x+6} \equiv \frac{x+1}{(x-2)(x-3)}$$
$$\equiv \frac{A}{x-2}+\frac{B}{x-3}$$

By the cover up method, $A=-3,\quad B=4$, so that

$$\frac{x+1}{x^2-5x+6} \equiv \frac{-3}{x-2}+\frac{4}{x-3}$$

We know that

$\frac{1}{1+x} \equiv 1-x+x^2-x^3+\cdots$ and so $\frac{1}{1-x} \equiv 1+x+x^2+x^3+\cdots$ so that

$$\frac{-3}{x-2} = \frac{3}{2-x} = \frac{3}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{3}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right) = \frac{3}{2}+\frac{3x}{4}+\frac{3x^2}{8}+\frac{3x^3}{16}+\cdots$$

and

$$\frac{4}{x-3} = -\frac{4}{3-x} = -\frac{4}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = -\frac{4}{3}\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots\right) = -\frac{4}{3}-\frac{4x}{9}-\frac{4x^2}{27}-\frac{4x^3}{81}-\cdots$$

so that

$$\frac{x+1}{x^2-5x+6} = \frac{3}{2}+\frac{3x}{4}+\frac{3x^2}{8}+\frac{3x^3}{16}+\cdots-\frac{4}{3}-\frac{4x}{9}-\frac{4x^2}{27}-\frac{4x^3}{81}-\cdots$$
$$= \frac{1}{6}+\frac{11x}{36}+\frac{49x^2}{216}+\frac{179x^3}{1296}+\cdots$$

23. (a)Find the first three non-zero terms of the Maclaurin series for $f(x)=e^{-x^2}\sin x$.

(b) Hence or otherwise show that $\lim_{x\to 0}\frac{f(x)-x}{x^3} = -\frac{7}{6}$.

Solution

We use

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

so that

$$\begin{aligned}
e^{-x^2}\sin x &= (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - x^2(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) + \frac{x^4}{2!}(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots) \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - x^3 + \frac{x^5}{3!} + \frac{x^5}{2!} + \cdots \\
&= x - \frac{7x^3}{6} + \frac{27x^5}{40} + \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x) - x}{x^3} &= \lim_{x\to 0}\frac{x - \frac{7x^3}{6} + \frac{27x^5}{40} + \cdots - x}{x^3} \\
&= \lim_{x\to 0}\frac{-\frac{7x^3}{6} + \frac{27x^5}{40} + \cdots}{x^3} \\
&= \lim_{x\to 0}(-\frac{7}{6} + \frac{27x^2}{40} + \cdots) \\
&= -\frac{7}{6}
\end{aligned}$$

24. Find the Maclaurin series for the functions e^x and $\sin x$, and hence expand $e^{\sin x}$ up to the term in x^4. Hence integrate $\int_0^1 e^{\sin x}\, dx$.

Solution

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

so that

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \frac{\sin^4 x}{4!} + \cdots$$

$$= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^3}{3!} + \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^4}{4!} + \cdots$$

$$= 1 + x - \frac{x^3}{3!} + \cdots + \frac{x^2 - \frac{2x^4}{3!} + \cdots}{2!} + \frac{x^3 + \cdots}{3!} + \frac{x^4 + \cdots}{4!} + \cdots$$

$$= 1 + x - \frac{x^3}{3!} + \frac{x^2}{2!} - \frac{2x^4}{2!3!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

Hence

$\int_0^1 e^{\sin x}\, dx = \int_0^1 (1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots)dx = x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40}\Big|_0^1 = \frac{394}{240} \simeq 1.6417$. The GDC gives $\int_0^1 e^{\sin x}\, dx = 1.63187$.

25. The Maclaurin series for e^z converges for all z including the case when z is a complex number. Using this fact, write the Maclaurin series for $e^{i\theta}$ and hence prove Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Hence deduce the extraordinary relation $e^{i\pi} + 1 = 0$.

Solution

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)$$

$$= \cos\theta + i\sin\theta$$

Substituting $\theta = \pi$ gives $e^{i\pi} = \cos\pi + i\sin\pi = -1 \Rightarrow e^{i\pi} + 1 = 0$.

26. (a) Find the Maclaurin series for the function $f(x) = \frac{1}{1+x}$. (b) By substituting $x \to e^{-x}$, find an expansion of $\frac{1}{1+e^{-x}}$ in powers of e^{-x}. (c) Use the series you got in (b) to show that $\int_0^\infty \frac{e^{-x}}{e^{-x}+1}dx = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$. (d) Evaluate the integral $\int_0^\infty \frac{e^{-x}}{1+e^{-x}}dx$ directly and hence deduce the value of $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$.

Solution

(a) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_0^\infty (-1)^n x^n$

(b)

$$\frac{1}{1+e^{-x}} = 1 - e^{-x} + e^{-2x} - e^{-3x} + \cdots = \sum_{0}^{\infty} (-1)^n e^{-nx}$$

$$\Rightarrow \frac{e^{-x}}{1+e^{-x}} = e^{-x} - e^{-2x} + e^{-3x} - \cdots = \sum_{1}^{\infty} (-1)^{n+1} e^{-nx}$$

(c) $\displaystyle\int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx = \sum_{1}^{\infty} (-1)^{n+1} \int_0^{\infty} e^{-nx} dx = -\sum_{1}^{\infty} (-1)^{n+1} \left.\frac{e^{-nx}}{n}\right|_0^{\infty} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

(d) $\displaystyle\int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx == -\ln(1+e^{-x})\Big|_0^{\infty} = \ln 2$. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

27. (a) Starting from $\cos\theta \le 1$ and integration over the range $[0, x]$ establish that $\sin x \le x$. By integrating this inequality again show that $\cos x \ge 1 - \dfrac{x^2}{2}$. By integrating once more establish that $\sin x \ge x - \dfrac{x^3}{6}$. (b) Use these results and the squeeze theorem to prove that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Solution

(a) $\cos\theta \le 1 \Rightarrow \displaystyle\int_0^x \cos\theta \, d\theta \le \int_0^x 1 \, d\theta$

i.e.

$\sin\theta\big|_0^x \le \theta\big|_0^x \Rightarrow \sin x \le x$

(b) From $\sin\theta \le \theta$ we have that

$$\int_0^x \sin\theta \, d\theta \le \int_0^x \theta \, d\theta$$

$$-\cos\theta\big|_0^x \le \left.\frac{\theta^2}{2}\right|_0^x$$

$$-\cos x + 1 \le \frac{x^2}{2}$$

i.e. that $1 - \dfrac{x^2}{2} \le \cos x$.

(c) Similarly,

$$\int_0^x (1 - \frac{\theta^2}{2}) \, d\theta \le \int_0^x \cos\theta \, d\theta$$

$$\left.\theta - \frac{\theta^3}{6}\right|_0^x \le \sin\theta\big|_0^x$$

$$x - \frac{x^3}{6} \le \sin x$$

i.e. that $\sin x \ge x - \dfrac{x^3}{6}$.

(d) So we have that

$$x \ge \sin x \ge x - \frac{x^3}{6}$$
$$1 \ge \frac{\sin x}{x} \ge 1 - \frac{x^2}{6}$$
$$\lim_{x\to 0} 1 \ge \lim_{x\to 0} \frac{\sin x}{x} \ge \lim_{x\to 0}(1 - \frac{x^2}{6})$$
$$1 \ge \lim_{x\to 0} \frac{\sin x}{x} \ge 1$$
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

28. Use the method of the previous problem to prove that $1 - \frac{x^2}{2} + \frac{x^4}{24} \ge \cos x \ge 1 - \frac{x^2}{2}$.

Solution

From the last problem, $\cos x \ge 1 - \frac{x^2}{2}$. Integrating once, gives $x - \frac{x^3}{6} \le \sin x$ and once more,

$$\int_0^x (\theta - \frac{\theta^3}{6})\, d\theta \le \int_0^x \sin\theta\, d\theta$$
$$\left.\frac{\theta^2}{2} - \frac{\theta^4}{24}\right|_0^x \le -\cos\theta\Big|_0^x$$
$$\frac{x^2}{2} - \frac{x^4}{24} \le -\cos x + 1$$
$$-1 + \frac{x^2}{2} - \frac{x^4}{24} \le -\cos x$$
$$1 - \frac{x^2}{2} + \frac{x^4}{24} \ge \cos x$$

so that $1 - \frac{x^2}{2} + \frac{x^4}{24} \ge \cos x \ge 1 - \frac{x^2}{2}$.

29. (a) Find the Maclaurin series for $\ln(1+x)$. (b) Use the alternating series estimation theorem to prove that $x - \frac{x^2}{2} \le \ln(1+x) \le x$. (c) Hence or otherwise prove that $\ln(n+1) - \ln n < \frac{1}{n}$. (d) Using your result in (c) show that $1 + \frac{1}{2} + \cdots + \frac{1}{n} > \ln(n+1)$. (e) Hence deduce that the harmonic series $\sum_1^\infty \frac{1}{k}$ diverges.

Solution

(a) $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$

(b) For any value of x we have an alternating series and so the sum of the series lies between 2 consecutive partial sums and so $x-\frac{x^2}{2}\le \ln(1+x)\le x$.

(c) $\ln(1+x)\le x$. Set $x=\frac{1}{n}$, then $\ln(1+\frac{1}{n})<\frac{1}{n}\Rightarrow \ln(\frac{n+1}{n})<\frac{1}{n}\Rightarrow \ln(n+1)-\ln n<\frac{1}{n}$

(Note $x=\frac{1}{n}\ne 0$.)

(d)

$$\left.\begin{array}{l}\frac{1}{1}>\ln 2-\ln 1\\ \frac{1}{2}>\ln 3-\ln 2\\ \cdots\\ \frac{1}{n}>\ln(n+1)-\ln n\end{array}\right\}_{\text{add side by side}}\Rightarrow 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>\ln(n+1)$$

(e) We then have that $\lim_{n\to\infty}\ln(n+1)=\infty$ and so $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

30. (a) Find the Maclaurin series for the function $f(x)=\frac{1}{1+x}$ and hence that for $f(x)=\frac{1}{1+x^2}$. (b) By integrating both sides of the Maclaurin series for $f(x)=\frac{1}{1+x^2}$, show that the Maclaurin series for the function $f(x)=\arctan x$ is $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots=\sum_{0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$. (c) Using the Maclaurin series for $f(x)=\arctan x$ up to and including the term with x^{13}, show that $\int_0^{1/\sqrt{3}}\arctan x\,dx\approx 0.158459$. (c) Show that the exact value of the integral is $\int_0^{1/\sqrt{3}}\arctan x\,dx=\frac{1}{\sqrt{3}}\frac{\pi}{6}-\frac{1}{2}\ln\left(\frac{4}{3}\right)$. (d) Hence deduce that an approximate value of π is $\pi\approx 3.14159$. (e) To how many decimal places is this approximation expected to be accurate?

Solution

(a) The series for $f(x)=\frac{1}{1+x}$ is standard, $\frac{1}{1+x}=1-x+x^2-x^3+\cdots$ and so also $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$.

(b)

$$\begin{aligned}\arctan x &= \int_0^x \frac{dt}{1+t^2}\\ &= \int_0^{1/\sqrt{3}} \left(1-t^2+t^4-\cdots\right)dt\\ &= \left(t-\frac{t^3}{3}+\frac{t^5}{5}-\cdots\right)_0^x\\ &= x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^{n+1}\frac{x^{2n+1}}{2n+1}+\cdots\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{1/\sqrt{3}} \arctan x\,dx &\approx \int_0^{1/\sqrt{3}}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\frac{x^{13}}{13}\right)dx\\ &= \left(\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\frac{x^{10}}{90}-\frac{x^{12}}{132}+\frac{x^{14}}{182}\right)_0^{1/\sqrt{3}}\\ &\approx 0.158459\end{aligned}$$

The error is the next term neglected i.e. $-\int_0^{1/\sqrt{3}}\frac{x^{15}}{15}dx = -\frac{x^{16}}{240}\bigg|_0^{1/\sqrt{3}} = -6.3\times 10^{-7}$.

(d)

$$\begin{aligned}\int_0^{1/\sqrt{3}} \arctan x\;dx &= x\arctan x\Big|_0^{1/\sqrt{3}} - \int_0^{1/\sqrt{3}}\frac{x}{1+x^2}dx\\ &= \frac{\arctan\frac{1}{\sqrt{3}}}{\sqrt{3}}-\frac{1}{2}\ln(1+x^2)\Big|_0^{1/\sqrt{3}}\\ &= \frac{\frac{\pi}{6}}{\sqrt{3}}-\frac{1}{2}\ln(1+\frac{1}{3})\Big|+\frac{1}{2}\ln(1)\Big|\\ &= \frac{1}{\sqrt{3}}\frac{\pi}{6}-\frac{1}{2}\ln\left(\frac{4}{3}\right)\end{aligned}$$

(e) $\frac{1}{\sqrt{3}}\frac{\pi}{6}-\frac{1}{2}\ln\left(\frac{4}{3}\right)=0.158459 \Rightarrow \pi = 6\sqrt{3}\left(0.158459+\frac{1}{2}\ln\left(\frac{4}{3}\right)\right)\approx 3.14159$

(f) The error in (c) was estimated to be at most 6.3×10^{-7} in absolute value so we can expect agreement with π to 5 decimal places (due to rounding).

31. *This problem is hard.* Consider the function $f(x)=\sin(n\arcsin x)$ for $|x|<1$. (a) Find the first and second derivatives of the function $f(x)$. (b) Hence show that $(1-x^2)f''(x)-xf'(x)+n^2f(x)=0$. (c) Use mathematical induction to prove that for $k=0,1,2,\cdots$ $(1-x^2)f^{(k+2)}(x)-(2k+1)xf^{(k+1)}(x)+(n^2-k^2)f^{(k)}(x)=0$ where $f^{(k)}(x)$ denotes the *k*th derivative of $f(x)$. (d) Hence derive the Maclaurin series for $f(x)$. (e) Deduce that if n is odd, the function $f(x)$ is a polynomial of degree n.

(f) Using your answer in (d) or otherwise, find $\lim_{x\to 0}\frac{\sin(n\arcsin x)}{x}$.

Solution

(a) $f'(x)=\cos(n\arcsin x)\frac{n}{\sqrt{1-x^2}}$ and $f''(x)=\frac{nx\cos(n\arcsin x)}{(1-x^2)^{3/2}}-\frac{n^2\sin(n\arcsin x)}{1-x^2}$.

(b)

$$(1-x^2)f''(x)-xf'(x)+n^2f(x)=$$

$$=(1-x^2)\left(\frac{nx\cos(n\arcsin x)}{(1-x^2)^{3/2}}-\frac{n^2\sin(n\arcsin x)}{1-x^2}\right)-\frac{xn\cos(n\arcsin x)}{\sqrt{1-x^2}}+n^2\sin(n\arcsin x)$$

$$=\frac{nx\cos(n\arcsin x)}{(1-x^2)^{1/2}}-n^2\sin(n\arcsin x)-\frac{xn\cos(n\arcsin x)}{\sqrt{1-x^2}}+n^2\sin(n\arcsin x)$$

$$=0$$

(c) The proposition was proven true for $k=0$. Assume that it is also true for some $k\geq 0$ i.e. that $(1-x^2)f^{(k+2)}(x)-(2k+1)xf^{(k+1)}(x)+(n^2-k^2)f^{(k)}(x)=0$.

Then differentiating once we find

$$(1-x^2)f^{(k+3)}(x)-2xf^{(k+2)}(x)-(2k+1)f^{(k+1)}(x)+(2k+1)xf^{(k+2)}(x)+(n^2-k^2)f^{(k+1)}(x)=0$$

$$(1-x^2)f^{(k+3)}(x)+(2k+3)xf^{(k+2)}(x)+(n^2-k^2-2k-1)f^{(k+1)}(x)=0$$

$$(1-x^2)f^{(k+3)}(x)+(2k+3)xf^{(k+2)}(x)+(n^2-(k+1)^2)f^{(k+1)}(x)=0$$

which is precisely what needs to be proved. The statement is true for k=0, and when assumed true for k it was proved true for k+1. So it is true for all $k\in\mathbb{N}$.

(d) $f(0)=\sin(n\arcsin 0)=0$, $f'(0)=\cos(n\arcsin 0)\frac{n}{\sqrt{1-0^2}}=n$ and $f''(0)=0$. Using $(1-x^2)f^{(k+2)}(x)-(2k+1)xf^{(k+1)}(x)+(n^2-k^2)f^{(k)}(x)=0$ evaluated at x=0 we get $f^{(k+2)}(0)=(k^2-n^2)f^{(k)}(0)$ and since $f''(0)=0$ all even derivatives vanish. The odd derivatives are:

$f^{(3)}(0)=(1^2-n^2)f^{(1)}(0)=(1^2-n^2)n$

$f^{(5)}(0)=(3^2-n^2)f^{(3)}(0)=(3^2-n^2)(1^2-n^2)n$ and so on.

Thus,

$$\sin(n\arcsin x)=nx+\frac{(1^2-n^2)n}{3!}x^3+\frac{(3^2-n^2)(1^2-n^2)n}{5!}x^5+\cdots$$

(e) If n is odd, the series will stop and the function is thus a polynomial of degree n.

(f) $\lim_{x\to 0}\frac{\sin(n\arcsin x)}{x}=\lim_{x\to 0}\frac{nx+\frac{(1^2-n^2)n}{3!}x^3+\cdots}{x}=n$.

32. (a) Write down the Maclaurin series for the function $f(x)=\sin x$ and hence that for $f(x)=\dfrac{\sin x}{x}$. (b) Use this series to find an approximate value of the integral $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ by keeping the first three terms in the Maclaurin series. (c) Estimate the error made in evaluating this integral in this way. (d) Confirm your answer by numerically evaluating the integral on your GDC.

Solution

(a) $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots+(-1)^{n+1}\dfrac{x^{2n+1}}{(2n+1)!}+\cdots$ and so

$\dfrac{\sin x}{x}=1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\cdots+(-1)^{n+1}\dfrac{x^{2n}}{(2n+1)!}+\cdots$. Hence

(b)

$$\begin{aligned}\int_0^1 \frac{\sin x}{x}dx &= \int_0^1\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+(-1)^n\frac{x^{2n}}{(2n+1)!}+\cdots\right)dx\\ &= x-\frac{x^3}{3\times 3!}+\frac{x^5}{5\times 5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)(2n+1)!}+\cdots\Bigg|_0^1\\ &\approx 1-\frac{1}{3\times 3!}+\frac{1}{5\times 5!}\\ &\approx 0.946111\end{aligned}$$

(c) The series is alternating so by the alternating series estimation theorem the error is less than the absolute value of the fourth term in the series i.e. $\left|-\dfrac{1}{7\times 7!}\right|=\left|-0.0000283447\right|$.

(d) The value of the integral is 0.946083. We have that $S-S_3=-0.0000280447$. The error has the same sign as the first term neglected (the fourth) and is smaller (in absolute value) in accordance with the alternating series estimation theorem.

33. Consider the improper integrals $I=\displaystyle\int_1^\infty \sin(\frac{1}{x})\,dx$ and $J=\displaystyle\int_1^\infty \sin^2(\frac{1}{x})\,dx$. (a) By making the substitution $u=\dfrac{1}{x}$, show that $I=\displaystyle\int_0^1\frac{\sin u}{u^2}\,du$ and $J=\displaystyle\int_0^1\frac{\sin^2 u}{u^2}\,du$. (b) By using appropriate MacLaurin series, determine whether the integrals *I* and *J* exist.

Solution

(a) $u=\dfrac{1}{x}$ and so $du=-\dfrac{1}{x^2}dx$. Thus $I=-\displaystyle\int_1^0\frac{\sin u}{u^2}\,du=\int_0^1\frac{\sin u}{u^2}\,du$ and similarly for *J*. (Note the change in the limits of integration.)

(b) We have that

$I=\int_0^1 \frac{\sin u}{u^2}du=\int_0^1\left(\frac{1}{u}-\frac{u}{3!}+\frac{u^3}{5!}-\cdots+(-1)^n\frac{u^{2n-1}}{(2n+1)!}+\cdots\right)du$. All terms are finite except for the first, $\int_0^1 \frac{1}{u}du=\ln u\big|_0^1=-\lim_{\varepsilon\to 0^+}\ln\varepsilon$ that diverges. Hence the integral does not exist.

For J use $\frac{\sin^2 u}{u^2}=\frac{1-\cos 2u}{2u^2}$ so that

$$
\begin{aligned}
J&=\int_0^1 \frac{1}{2u^2}\left(1-(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots+(-1)^n\frac{u^{2n}}{(2n)!}+\cdots)\right)du\\
&=\int_0^1 \frac{1}{2u^2}\left(\frac{u^2}{2!}-\frac{u^4}{4!}+\cdots-(-1)^n\frac{u^{2n}}{(2n)!}+\cdots)\right)du\\
&=\frac{1}{2}\int_0^1\left(\frac{1}{2!}-\frac{u^2}{4!}+\cdots-(-1)^n\frac{u^{2n-2}}{(2n)!}+\cdots)\right)du
\end{aligned}
$$

The integral is therefore finite.

34. *For Physics students.* In Probability, the exponential probability distribution function is defined by $\frac{dP}{dt}=\lambda e^{-\lambda t}$ where λ is a positive constant. A particular application of this distribution is to the decay of radioactive nuclei. In that case, the probability that a particular nucleus will decay at some time T in the interval from $t=0$ to $t=\tau$ is given by $P(0\le T\le\tau)=\int_0^\tau \lambda e^{-\lambda t}\,dt$. (a) Show that the p.d.f. $\frac{dP}{dt}=\lambda e^{-\lambda t}$ is well defined i.e. show that $\int_0^\infty \lambda e^{-\lambda t}\,dt=1$. (b) Evaluate $P(0\le T\le\tau)=\int_0^\tau \lambda e^{-\lambda t}\,dt$. (c) The probability that a particular nucleus will decay within a time interval called the half-life is ½. Deduce that the half-life is given by $\frac{\ln 2}{\lambda}$. (d) Using your answer to (b) in the case where $\lambda\tau\ll 1$, explain why the constant λ is called the probability of decay per unit time. (Hint: Do a MacLaurin expansion of the answer in (b).)

Solution

(a) $\int_0^\infty \lambda e^{-\lambda t}\,dt=\lim_{\Lambda\to\infty}\int_0^\Lambda \lambda e^{-\lambda t}\,dt=\lambda\lim_{\Lambda\to\infty}\frac{e^{-\lambda t}}{-\lambda}\bigg|_0^\Lambda=-\lim_{\Lambda\to\infty}(e^{-\lambda\Lambda}-1)=1$.

(b) $P(0\le T\le\tau)=\int_0^\tau \lambda e^{-\lambda t}\,dt=\lambda\frac{e^{-\lambda t}}{-\lambda}\bigg|_0^\tau=1-e^{-\lambda\tau}$.

(c) $\frac{1}{2}=1-e^{-\lambda T}\Rightarrow e^{-\lambda T}=\frac{1}{2}\Rightarrow T=\frac{\ln 2}{\lambda}$

(d) $P(0 \le T \le \tau) = 1 - e^{-\lambda\tau} \approx 1 - (1 - \lambda\tau + \frac{\lambda^2\tau^2}{2!} + \ldots) \approx \lambda\tau$. Thus, approximately, $\lambda = \frac{P(0 \le T \le \tau)}{\tau}$, is the probability of decay per unit time if $\lambda\tau \ll 1$.

35. Consider the infinite series $\sum_1^\infty (\sqrt[3]{n^3+1} - n)$. (a) Find the first three terms of the Maclaurin expansion of the function $f(x) = \sqrt[3]{1+x}$. (b) Use your result in (a) to show that for large n the general term of the infinite series behaves as $\frac{1}{3n^2}$. (c) Hence show that the infinite series $\sum_1^\infty (\sqrt[3]{n^3+1} - n)$ converges.

Solution

(a) $f(x) = (1+x)^{1/3}$. Hence $f(0) = 1$ and

$$f'(0) = \frac{1}{3}(1+x)^{-2/3}\Big|_{x=0} = \frac{1}{3},$$

$$f''(0) = \frac{1}{3}(-\frac{2}{3})(1+x)^{-5/3}\Big|_{x=0} = -\frac{2}{9}$$

Thus, $f(x) = (1+x)^{1/3} \approx 1 + \frac{x}{3} - \frac{2x^2}{9} + \cdots$

(b) $\sqrt[3]{n^3+1} = n\sqrt[3]{1+\frac{1}{n^3}}$. When n is large, $\frac{1}{n^3}$ is small and we may approximate $\sqrt[3]{1+\frac{1}{n^3}}$ by the first few terms of the MacLaurin series for $f(x) = (1+x)^{1/3}$ with $x = \frac{1}{n^3}$. In other words,

$$\sqrt[3]{1+\frac{1}{n^3}} \approx 1 + \frac{1}{3n^3} - \frac{2}{9n^6} + \cdots.$$

Hence the general term $\sqrt[3]{n^3+1} - n$ behaves as

$$n\sqrt[3]{1+\frac{1}{n^3}} - n \approx n(1 + \frac{1}{3n^3} - \frac{2}{9n^6} + \cdots) - n = \frac{1}{3n^2} - \frac{2}{9n^5} \approx \frac{1}{3n^2}$$ when n is large.

(c) Applying the limit comparison test with the series $\sum_1^\infty \frac{1}{n^2}$ we find

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^3+1} - n}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{3n^2} - \frac{2}{9n^5}}{\frac{1}{n^2}} = \lim_{n\to\infty}(\frac{1}{3} - \frac{2}{9n^3}) = \frac{1}{3}.$$

Hence the series converges since $\sum_1^\infty \frac{1}{n^2}$ does, being a p-series with $p = 2$.