מבוא לחדו"א 2 – פתרון תרגיל 2

1. חלוקת פולינומים:

א. $\int \frac{x^3+9x^2+19x-5}{x+5}\,dx$

חלוקת פולינומים:

$$\begin{array}{r|l} x^3+9x^2+19x-5 & x+5 \\ -(x^3+5x^2) & \hline x^2+4x-1 \\ \hline 4x^2+19x-5 & \\ -(4x^2+20x) & \\ \hline -x-5 & \\ -(-x-5) & \\ \hline 0 & \end{array}$$

$$= \int (x^2+4x-1)\,dx = \boxed{\frac{x^3}{3}+2x^2-x+C}$$

ב. $\int \frac{3x^3}{x+5}\,dx$

$$\begin{array}{r|l} 3x^3 & x+5 \\ -(3x^3+15x^2) & \hline 3x^2-15x+75 \\ \hline -15x^2 & \\ -(-15x^2-75x) & \\ \hline 75x & \\ -(75x+375) & \\ \hline -375 & \end{array} \qquad -\frac{375}{x+5}$$

$$\int \frac{3x^3}{x+5}\,dx = \int \left(3x^2-15x+75-\frac{375}{x+5}\right)dx =$$

$$= \boxed{x^3-\frac{15}{2}x^2+75x-375\ln|x+5|+C}$$

ד.

$$\int \frac{(x-1)^3}{x^3-x}\,dx = \int \frac{(x-1)^{2\cancel{3}}}{x(\cancel{x-1})(x+1)}dx \overset{=}{} \int \frac{(x-1)^2}{x(x+1)}\,dx$$

$$= \int \frac{x^2-2x+1}{x^2+x}\,dx = \int \frac{x^2+x}{x^2+x}\,dx + \int \frac{-3x+1}{x(x+1)}\,dx$$

$$= x + \int \frac{-3x+1}{x(x+1)}\,dx$$

פירוק שברים חלקיים:

$$\frac{A}{x} + \frac{B}{x+1} = \frac{-3x+1}{x(x+1)}$$

$$A(x+1) + Bx = -3x+1$$

$$\left.\begin{aligned} A + B &= -3 \\ A &= 1 \end{aligned}\right\} \quad B = -4$$

$$= x + \int \frac{1}{x}\,dx + \int \frac{-4}{x+1}\,dx =$$

$$\boxed{x + \ln|x| - 4\ln|x+1| + C}$$

כפל בצמוד שלב 2

(ב) $\int \frac{3x}{3+\sqrt{3x+9}}\,dx = \int \frac{3x}{3+\sqrt{3x+9}} \cdot \frac{3-\sqrt{3x+9}}{3-\sqrt{3x+9}}\,dx =$

$= \int \frac{3x\cdot(3-\sqrt{3x+9})}{9-3x-9}\,dx = \int (-3+\sqrt{3x+9})\,dx =$

$= -3x + \frac{(3x+9)^{\frac{3}{2}}}{\frac{3}{2}\cdot 3} + C = \boxed{-3x + \frac{2}{9}\cdot\sqrt{(3x+9)^3} + C}$

(ג) $\int \frac{5x}{\sqrt{x+4}-2}\,dx = \int \frac{5x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\,dx =$

$= \int \frac{5x\cdot(\sqrt{x+4}+2)}{x+4-4}\,dx = \int 5\sqrt{x+4}\,dx + \int 10\,dx =$

$= 5\cdot\frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} + 10x + C = \boxed{\frac{10}{3}\sqrt{(x+4)^3} + 10x + C}$

(ד) $\int \frac{1}{1-\sin x}\,dx = \int \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx =$

$= \int \frac{1+\sin x}{1-\sin^2 x}\,dx = \int \frac{1+\sin x}{\cos^2 x}\,dx =$

$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos^2 x}\,dx = \boxed{\tan x - \frac{1}{\cos x} + C}$

↓

ראו סרטון

שאלה 3 אינטגרלים של פונקציות רציונליות

$\frac{ax+b}{x^2+px+q}$

(ב) $\int \frac{x^2-2x}{x^2-4x+3}\,dx$

נתחיל בחלוקת פולינומים

$$\begin{array}{r|l} x^2-4x+3 & 1 \\ \hline x^2-2x & \\ \hline 2x-3 & \end{array}$$

$$\Rightarrow 1 + \frac{2x-3}{x^2-4x+3}$$

$$= \int 1\,dx + \int \frac{2x-3}{x^2-4x+3}\,dx =$$

$$= x + \int \frac{2x-4}{x^2-4x+3}\,dx + \int \frac{1}{x^2-4x+3}\,dx =$$

לפי הנגזרת

$$= x + \ln|x^2-4x+3| + \int \frac{1}{(x-1)(x-3)}\,dx$$

$$\frac{A}{x-1} + \frac{B}{x-3} = \frac{1}{(x-1)(x-3)}$$

$$A(x-3) + B(x-1) = 1$$

$$\begin{cases} A+B=0 \\ -3A-B=1 \end{cases} \Rightarrow \quad -3A+A=1$$

$$-2A=1$$

$$A=-\tfrac{1}{2} \Leftarrow \quad B=\tfrac{1}{2}$$

$$= \boxed{x + \ln|x^2-4x+3| - \tfrac{1}{2}\ln|x-1| + \tfrac{1}{2}\ln|x-3| + C}$$

ב. $\int \frac{x^3}{x^2+6x+10}\,dx = (\#)$

חלוקת פולינומים

$$\begin{array}{r|l} x^3 & x^2+6x+10 \\ \hline x^3+6x^2+10x & x-6 \\ \hline =-6x^2-10x & \\ -6x^2-36x-60 & \\ \hline =\ 26x+60 & \end{array}$$

$$(\#) = \int \left(x-6+\frac{26x+60}{x^2+6x+10}\right)dx = \int (x-6)\,dx +$$

$$+13\int \frac{2x+6}{x^2+6x+10}\,dx - 18\int \frac{1}{x^2+6x+10}\,dx =$$

$$= \frac{x^2}{2} - 6x + 13\ln|x^2+6x+10| - 18\int \frac{1}{(x+3)^2+1}\,dx =$$

$$= \boxed{\frac{x^2}{2} - 6x + 13\ln|x^2+6x+10| - 18\arctan(x+3) + C}$$

חלוקת פולינומים

ג. $\int \frac{x^2+1}{x^2+6x+9}\,dx =$

$$\begin{array}{r|l} x^2+1 & x^2+6x+9 \\ \hline x^2+6x+9 & 1 \\ \hline -6x-8 & \end{array}$$

$$\int 1\,dx - \int \frac{6x+8}{(x+3)^2}\,dx = x - 6\int \frac{x+3}{(x+3)^2}\,dx + 10\int \frac{1}{(x+3)^2}\,dx =$$

$$= \boxed{x - 6\ln|x+3| - \frac{10}{x+3} + C}$$

4. אינטגרל מהצורה

$$\frac{ax+b}{\sqrt{x^2+px+q}}$$

א. $$\int \frac{2x+3}{\sqrt{x^2+4x+13}}\,dx = \underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+13}}\,dx}_{I} - \underbrace{\int \frac{1}{\sqrt{x^2+4x+13}}\,dx}_{II}$$

$$I = 2\sqrt{x^2+4x+13}$$

לצורך האינטגרל השני נשתמש בנוסחה:

$$\int \frac{1}{\sqrt{(x+b)^2+a^2}}\,dx = \ln\left((x+b)+\sqrt{(x+b)^2+a^2}\right)$$

$$II = \int \frac{1}{\sqrt{(x+2)^2+3^2}}\,dx = \ln\left((x+2)+\sqrt{(x+2)^2+3^2}\right)$$

סה"כ $$= \boxed{2\sqrt{x^2+4x+13} + \ln\left(x+2+\sqrt{x^2+4x+13}\right) + C}$$

ב. $$\int \frac{5}{\sqrt{5-9x^2-12x}}\,dx = 5\int \frac{1}{\sqrt{9-(3x+2)^2}}\,dx$$

נשתמש באינטגרל

$$\int \frac{1}{\sqrt{m^2-(x+n)^2}}\,dx = \arcsin\left(\frac{x+n}{m}\right)$$

חלוקה בגלל פנימית

$$= 5\arcsin\left(\frac{3x+2}{\sqrt{9}}\right)\cdot\frac{1}{3} + C = \boxed{\frac{5}{3}\arcsin\left(x+\frac{2}{3}\right) + C}$$

2. $\int \frac{17}{\sqrt{x^2-6x+9}}\,dx = 17\int \frac{1}{\sqrt{(x-3)^2}}\,dx =$

$= 17\int \frac{1}{|x-3|}\,dx = \boxed{17\ln|x-3| + C}$

9. $\int \frac{1}{\sqrt{x^2+7x+5}}\,dx = \int \frac{1}{\sqrt{(x+3.5)^2-7.25}}\,dx$

$= \boxed{\ln\left((x+3.5)+\ln\sqrt{(x+3.5)^2-7.25}\right) + C}$