Contents

Part I
Introduction

Tropical algebraic geometry is a field of mathematics that has been growing during the last 10 years (ref. [15],[4],[14],[18],[20],[21]). Its main purpose is to generalize the process of translating complicated geometric problems into solvable combinatorial problems. This idea is achieved by matching regular geometric objects with piece-wise linear ones.

Let us define the main algebraic structure of tropical geometry, the max-plus algebra over a totally ordered group (such as $(\mathbb{R},+),(\mathbb{Q},+)$) with the following operations:

$$a \oplus b = max(a,b)$$

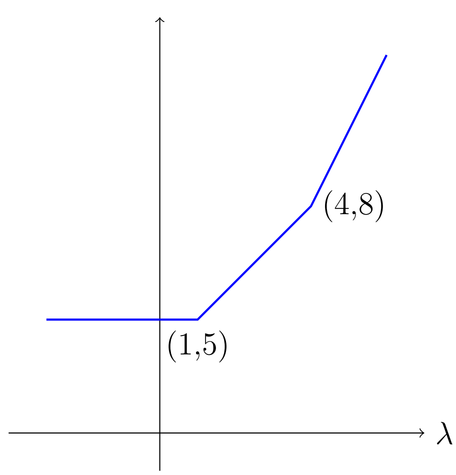
$$a \odot b = a + b$$

In the following sections we will present a short survey of the field (mainly from an algebraic point of view), and present an algebraic extension to the max-plus algebra called Exploded Layered Tropical algebra (ELT algebra for short). ELT algebra is the main algebraic structure we will use throughout this thesis.

1 Tropical Polynomials And Their Roots

We wish to study algebraic geometry over the max-plus algebra. In order to do so, we should look at roots of polynomials. However, the usual definition of roots is useless since zero $(-\infty)$ does not play its classical role in this algebra. For instance, let us look at the max-plus polynomial $p(\lambda) = \lambda^2 \oplus 4\lambda \oplus 5$. The equation $p(a) = -\infty$ is false for every a.

Figure 1: The graph of $p(\lambda) = \lambda^2 \oplus 4\lambda \oplus 5$



Looking at the graph of the polynomial $p(\lambda)$ (figure 1), we can see that it is comprised of three line segments that meets at the points 1 and 4. One can also notice that p can be factored as

$$p = (\lambda \oplus 1) \odot (\lambda \oplus 4).$$

We would expect these points to be the roots of p. In order to define these roots algebraically, Zur Izhakian has introduced supertropical geometry ([7]).

We investigate these possible roots and see that they come about when two monomials are equal. Indeed, these are the points where the graph of the polynomial changes its slope. Therefore Izhakian has built a structure that defines the sum of two equal elements as a "ghost" element. Izhakian treats these ghosts as zeros since we want to view them as roots.

Further research led Izhakian and Rowen ([9]) to the idea of a graded algebra. Not only do we "remember" the sum as a "ghost", we also keep a layer element that gives us more information. For example, assuming a natural or a tangible element is of layer one, then the sum of three tangible elements is of layer three. In the broad perspective we will see the graded algebra as a lesser degeneration of the classical geometry than the supertropical algebra, which is lesser than tropical geometry.

In this thesis we introduce an extension of a non-archimedean valuation, from the field of Puiseux series into an exploded layered tropical algebra (or ELT algebra for short), where the coefficient of the leading monomial determine the layers. Given this new structure we further refine the definition of a root to be a point where the layer of the evaluation of the polynomial is zero. This structure is similar to Parker's "exploded" semiring and holomorphic curves ([16]). Parker uses exploded manifolds to define and compute Gromov-Witten invariants.

While this definition does not coincide with the "ghost" definition above, it does coincide with the classical tropical definition (i.e., a root is a point of equality between two or more evaluations of monomials). Furthermore, we will see that this definition produces clear formalizations and interesting results for polynomials, algebraic varieties and linear algebra.

Example 1.1. *The roots of the polynomial* $f(\lambda) = {}^{[a]}0\lambda^2 + {}^{[1]}2$ *in ELT algebra over the complex field are* ${}^{[\frac{\pm i}{\sqrt{a}}]}1$.

Indeed,

$$f({}^{[\frac{\pm i}{\sqrt{a}}]}1) = {}^{[a]}0\left({}^{[\frac{\pm i}{\sqrt{a}}]}1\right)^2 + {}^{[1]}2 = {}^{[a]}0\left({}^{[\frac{-1}{a}]}2\right) + {}^{[1]}2 = {}^{[-1]}2 + {}^{[1]}2 = {}^{[0]}2.$$

2 Linear Algebra

Tropical linear algebra, also known as Max-Plus linear algebra, has been studied for more than 50 years (ref. [2]). While tropical geometry deals with geometric combinatorial problems, topical linear algebra deals with algebraic non-linear combinatorial problems (for instance, the assignment problem [13]). Tropical linear algebra may also be used as a mean to study the tropical algebraic geometry (for instance, the tropical resultant). Notable work in this field can be found at [2],[3],[10],[11] and [19].

In tropical linear algebra, a set of vectors is linearly dependent if for some non-trivial linear combination the maximal entry at each column is obtained at least twice. For instance, the vectors

$$w_1 = (1, 2, 0), w_2 = (0, 3, 2), w_3 = (0, 0, 0)$$

are linearly dependent. Indeed,

$$(1 \odot w_1) \oplus (w_2) \oplus (2 \odot w_3) = (2, 3, 1) \oplus (0, 3, 2) \oplus (2, 2, 2) = (2, 3, 2).$$

In ELT linear algebra, a set of vectors is linearly dependent if for some non-trivial linear combination, all of the layers equal zero. For instance, the vectors

$$v_1 = ({}^{[1]}1, {}^{[1]}2, {}^{[1]}0), v_2 = ({}^{[1]}0, {}^{[1]}3, {}^{[1]}2), v_1 = ({}^{[-1]}0, {}^{[1]}0, {}^{[1]}0)$$

are linearly dependent. Indeed,

$${}^{[1]}1v_1 + {}^{[-1]}0v_2 + {}^{[1]}2v_3 = ({}^{[0]}2, {}^{[0]}3, {}^{[0]}2).$$

We notice that while $u_1 = (1, 1), u_2 = (1, 1)$ are clearly linearly dependent in tropical algebra, the two vectors

$$u_1' = ({}^{[1]}1, {}^{[-1]}1), u_2' = ({}^{[-1]}1, {}^{[1]}1)$$

are independent. Geometrically the span of these two vectors is equal a span of one vector. However, the layers of these two spans differs. Naturally we would like to know what is the maximal size of an independent set.

One of the challenging results presented in this thesis is that the size of a maximal independent set is exactly n. Furthermore, the maximal number of linearly independent rows of a matrix (called row rank) is always equal to the maximal number of independent columns.

The addition of layers also enables us to extend the usual tropical permanent with the ELT determinant, having the layer multiplied by the sign. For example consider the matrix with rows v_1, v_2, v_3

$$A = \begin{pmatrix} {}^{[1]}1 & {}^{[1]}2 & {}^{[1]}0 \\ {}^{[1]}0 & {}^{[1]}3 & {}^{[1]}2 \\ {}^{[-1]}0 & {}^{[1]}0 & {}^{[1]}0 \end{pmatrix}.$$

Using a rather natural definition, we obtain the determinant

$$|A| = {}^{[1]}0\left({}^{[1]}1 \cdot {}^{[1]}3 \cdot {}^{[1]}0\right) + {}^{[1]}0\left({}^{[1]}2 \cdot {}^{[1]}2 \cdot {}^{[-1]}0\right) = {}^{[0]}4.$$

Since the layer of the determinant is zero, the rows of the matrix are linearly dependent, as we have seen before.

3 Main Results

In this thesis we present a few main results:

1. Complete characterization of univariate polynomials factorization
2. A geometric counterexample to unique factorization of polynomials in two variables.
3. Unique factorization for primary polynomials in any number of variables.
4. Formalize the natural definitions of linear dependence and determinant, and prove that a matrix is singular if and only if its rows are linearly dependent.
5. Prove that the row rank and the column rank of a matrix are equal.
6. Formalize and prove an exploded-layered version of Payne's generalization [17] of Kapranov's theorem (for the hypersurface case).
7. Prove that the Sylvester matrix of two polynomials is singular if and only if they have a common tropical root. In this proof we use exploded layered tropical linear algebra and Kapranov's theorem.

4 Tropical Geometry

Tropical geometry was described in the introduction as a general theory that enables one to translate complicated geometric problems into a solvable combinatorial one. Since this is a broad definition, there is more than one way to approach tropical geometry.

4.1 Amoebas

Amoebas ([5]) are the geometrical objects defined as images of varieties by the function $F : (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by

$$F(z_1, z_2, ..., z_n) = (\log_t |z_1|, \log_t |z_2|, ..., \log_t |z_n|).$$

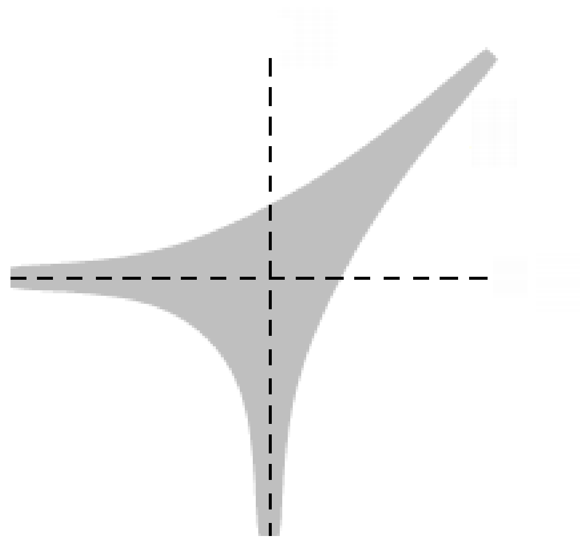
These amoebas have different asymptotic limits in different directions, and the idea is to take the lines to which the amoebas converge. We achieve this goal by narrowing the amoeba to have zero width.

Tropical curves are defined as the limit obtained as t tends to ∞. These tropical curves are piecewise linear objects as we wished.

Example 4.1. *[5] Consider the polynomial*

$$f(x, y) = x + y + 1 \in \mathbb{C}[x, y].$$

Figure 2: The amoeba $F(V_f \cap (\mathbb{C}^*)^2)$

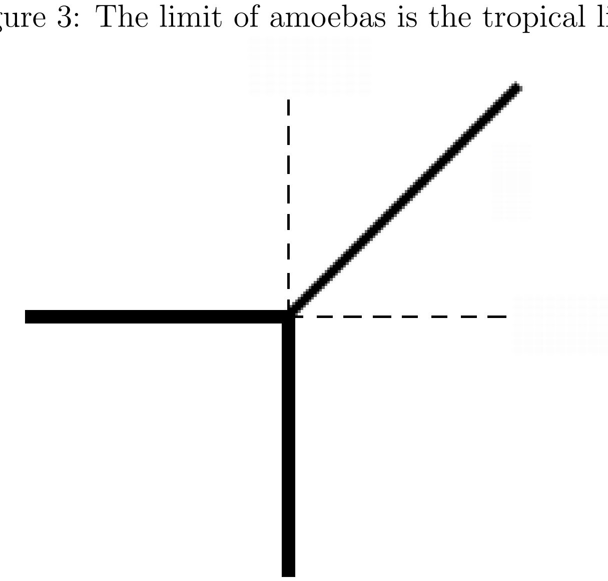


The variety of f is $V_f := \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$. Figure 2 is a graph of an amoeba of V_f.

The tentacles of the amoeba are created by points $(log_t(|x|), log_t(|y|))$ such that $|x|$ or $|y|$ tends to 0 or ∞.
For example, the set of points $(x, x-1) \in V_f$ such that $|x| \to \infty$ creates the north-east tentacle. If $|x| \to 0$ the west tentacle is created (since $|y| \to 1$), and $x \to -1$ creates the south tentacles.

When taking the limit $t \to \infty$ we obtain the tropical curve (Figure 3).

Figure 3: The limit of amoebas is the tropical line



We can see that the max-plus behavior arises from the definition of the tropical curves as the limit of the log action on regular curves. For any t, and two points $t^a, t^b \in V_f$ we obtain two points in the amoeba a, b. The point $t^a t^b = t^{a+b}$ is sent to the point $a+b$ in the amoeba, thus $a \odot b := a + b$.

Now we consider $log_t(t^a + t^b)$ when t tends to ∞. If $a \geq b$ then

$$log_t(t^a + t^b) = log_t(t^a(1 + t^{b-a})) = a + log_t(1 + t^{b-a}),$$

since $b - a \leq 0$ then $log_t(t^a + t^b) \to a$ as t tends to ∞. Thus $a \oplus b := \max\{a, b\}$.

4.2 Non-Archimedean amoebas

4.2.1 Non-Archimedean valuations

First we recall the definition of a (non-Archimedean) valuation. Let K be a field, then the function $v : K \to \mathbb{R} \cup \{\infty\}$ is called a *valuation* if the following properties hold:

1. $v(x) = \infty \iff x = 0$.
2. $\forall x, y \in K : v(xy) = v(x) + v(y)$.
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Let v be a valuation over a field K with $char(K) \neq 2$, we recall the following basic properties: Since $v(1) = v(1 \cdot 1) = v(1) + v(1)$ then

$$v(1) = 0.$$

Next, $0 = v(1) = v((-1) \cdot (-1)) = v(-1) + v(-1)$ so $v(-1) = 0$ and for all $x \in K$

$$v(-x) = v((-1) \cdot x) = v(x).$$

For all $x \in K$, since $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$ then

$$v(x^{-1}) = -v(x).$$

We saw that $v(\pm 1) = 0$. Assume $v(x) = 0$ for some x.
If $v(x) < 0$ then

$$v(1 + x) \geq \min\{v(1), v(x)\} = \min\{0, v(x)\} = v(x).$$

Also,

$$0 > v(x) = v(1 + x - 1) \geq \min\{v(1 + x), v(-1)\} = \min\{v(1 + x), 0\}$$

therefore $v(x) \geq v(1 + x)$. Together we obtain that

$$v(x) < 0 \Rightarrow v(1 + x) = v(x).$$

Now, for all $x, y \in K$ if $v(x) \neq v(y)$ then $v(x) - v(y) \neq 0$ and so $v(xy^{-1}) \neq 0$. Thus for all $x, y \in K$ such that $v(x) \neq v(y)$, either $v(x^{-1}y) < 0$ or $v(y^{-1}x) < 0$.

Assume that $v(x^{-1}y) < 0$. Then

$$v(x+y) = v(x(1+x^{-1}y)) = v(x)+v(1+x^{-1}y) = v(x)+v(x^{-1}y) = v(x)-v(x)+v(y) = v(y).$$

Thus if $v(x) \neq v(y)$ then $v(x+y) = v(x)$ or $v(x+y) = v(y)$. In other words,

$$v(x+y) > \min\{v(x), v(y)\} \Rightarrow v(x) = v(y).$$

That is the reason that the equality between the valuation of two elements is central in our theory.

4.2.2 Puiseux series

Consider the field of *Puiseux series* $K(t) = \{\sum_{j\in I} a_j t^j\}$, $I \subset \mathbb{R}$ a totally ordered set. There is a valuation:

$$v : K(t) \to \mathbb{R} \cup \{\infty\}$$

given by

$$v\Big(\sum_{j\in I} a_j t^j\Big) = \min(I).$$

Now using this valuation, we can define the *non-Archimedean amoeba* (cf. [6]) of a hypersurface of $K(t)^n$ as the image of the function $F : K(t)^n \to \mathbb{R}^n$, given by $F(x_1, ...x_n) = (-v(x_1), ..., -v(x_2))$.

Let X be the hypersurface in $K(t)^n$ obtained by the polynomial $\sum a_J f^J$ where $a_J \in K(t)$,$J = (i_1, ..., i_n)$ and $f^J = x_1^{i_1} \cdots x_n^{i_n}$. Kapranov's theorem states that the non-Archimedean amoeba of X coincides with the corner-roots of the tropical polynomial $\sum -v(a_J)\lambda^I$.

This approach to tropical geometry (ref. [4],[1]) is closely related to our work as it is algebraic rather than analytic.

5 Exploded Layered Tropical Algebra

In this section we define the algebraic structure we use throughout this thesis, which is inherited from the work of Parker ([16]) and Izhakian and Rowen ([8],[9],[10],[11]). The purpose of this tropical algebraic structure is to formalize some of the results of tropical geometry, prove classical theorems that could not even be formulated in tropical geometry, and hopefully solve open classical problems. This field of mathematics is a new and exciting area.

Definition 5.1. Let L be a set closed under addition and multiplication and F a totally ordered group (such as $(\mathbb{R},+)$ or $(\mathbb{Q},+)$). An *ELT algebra* is the set $\{{}^{[\ell]}\lambda | \lambda \in F, \ell \in L\}$ together with the semi-ring structure:

1. ${}^{[\ell_1]}\lambda + {}^{[\ell_2]}\lambda = {}^{[\ell_1 +_L \ell_2]}\lambda$,
2. If $\lambda_1 > \lambda_2$ then ${}^{[\ell_1]}\lambda_1 + {}^{[\ell_2]}\lambda_2 = {}^{[\ell_1]}\lambda_1$,
3. ${}^{[\ell_1]}\lambda_1 \cdot {}^{[\ell_2]}\lambda_2 = {}^{[\ell_1 \cdot_L \ell_2]}(\lambda_1 +_F \lambda_2)$.

Let R be an ELT algebra. We write $s : R \to L$ for the function which extracts the coefficient:

$$s({}^{[\ell]}\lambda) = \ell,$$

and $t : R \to F$ for the function which extracts the tangible value:

$$t({}^{[\ell]}\lambda) = \lambda.$$

We extend the total order on F to a partial order on R in the natural way:

$${}^{[\ell_1]}\lambda_1 \geq {}^{[\ell_2]}\lambda_2 \iff \lambda_1 \geq_F \lambda_2.$$

Example 5.2. *Zur Izhakian's supertropical geometry (ref. [7]) is equivalent to an ELT algebra with $L = \{1, 2\}$ such that*

$$1 + 1 = 2, 1 + 2 = 2, 2 + 2 = 2$$

and

$$1 \cdot 1 = 1, 1 \cdot 2 = 2, 2 \cdot 2 = 2.$$

The supertropical "ghost" element 1^ν is equivalent to ${}^{[2]}1$ in the ELT notation, and the tangible element 1 to ${}^{[1]}1$.

Therefore, in this thesis we will refer to this specific ELT algebra as a supertropical algebra.

Example 5.3. *The classical max-plus algebra is equivalent to the trivial ELT algebra with $L = \{1\}$. We call this case tropical algebra.*

5.1 Non-Archimedean amoebas

Let R be an ELT algebra over the reals with layers from an algebraically closed field $L = \mathbb{F}$ of characteristic 0, and let K be the field of Puiseux series with coefficients in $\mathbb{F}$ and powers in $\mathbb{R}$ or $\mathbb{Q}$. We define an ELT tropicalization function from K^* to R in the following way. Assume $x \in K^*$ is a series with leading monomial ct^{α}, then we define

$$ELTrop(x) := {}^{[c]}(-\alpha).$$

Let $f \in K[x_1, ..., x_n]$ be a polynomial, and write $f = \sum a_I x^I$. Then

$$ELTrop[f] := \sum ELTrop(a_I)\lambda^I \in R[\lambda_1, ..., \lambda_n].$$

Definition 5.4. If $f \in K[x_1, ..., x_n]$ is a polynomial, then its *ELT variety* is the following set of ELT roots

$$V(f) := \{a \in R^n : s\Big(ELTrop[f](a)\Big) = 0_{\mathbb{F}}\}.$$

In these terms, Kapranov's theorem states that the non-Archimedean amoeba $V(f)$ coincides with the pointwise ELT tropicalization of the classical variety of f.

5.2 The element $-\infty$

The element $-\infty$ is essential for some of the results we obtain in this thesis, therefore we may add it in the following way.

Define $\overline{R} := R \cup \{-\infty\}$ such that for all $a \in \overline{R}$:

$$a + (-\infty) = (-\infty) + a = a,$$
$$a \cdot (-\infty) = (-\infty) \cdot a = (-\infty).$$

We also define

$$s(-\infty) := 0_{\mathbb{F}},$$

and

$$ELTrop(0_K) = -\infty.$$

Part II
Factorization and basic properties of polynomials

6 Introduction

In this part, we focus on the problem of factorization of polynomials and the necessary requirements from the layer structure L. The factorization of tropical polynomials is important for a number of reasons: first, the factors of a certain polynomial help us split the variety of the polynomial into smaller varieties; second, the way polynomials factor affects the algebraic structure of the polynomials ring and its ideals. This is important since we aim to create an extensive algebraic base. Also, Gathmann (ref. [5]) explained the importance of factorization of tropical polynomials and its connection to ordinary polynomial factorization.

We will show that in this structure most polynomials in one indeterminate factor uniquely. We will also show that polynomials in several variables, in which all monomials have the same tangible value at some point (called primary polynomials), factor uniquely.

Example 6.1. For example, consider the variety of three geometric lines which intersect at (0,0).

$$f = (x + y)(x + 0)(y + 0) = (x + y + 0)(xy + x + y).$$

This variety may factor into the three geometric lines, or a tropical line and a tropical quadratic factor. The distinction between these two cases is encapsulated in the layer of the intersection point. We will see that since this is a primary polynomial it factors uniquely in our expanded structure.

7 Basic Definitions

We wish to define polynomials over the ELT algebra with $F = \mathbb{R}$. One must note that unlike polynomials over classical algebra, two ELT polynomials may be equal everywhere yet contain different monomials.

Example 7.1. *Consider the two ELT polynomials*

$$f(x) = x^2 + {}^{[1]}2,$$

and,

$$g(x) = x^2 + x + {}^{[1]}2.$$

For each $x \in R$ such that $x > 1$, the monomial x^2 dominates the other monomial since $x^2 > x, 2$. Thus, in this case, $f(x) = g(x) = x^2$.

If $t(x) = 1$ then $f(x) = x^2 + {}^{[1]}2$. In the polynomial g, the monomials $x^2, {}^{[1]}2$ dominates the monomial x and so $g(x) = f(x)$ as well.

The last case is $x < 1$ in which similarly $f(x) = g(x) = {}^{[1]}2$.

Therefore f and g are equal at every point of R even though they contain different monomials.

For this reason we define ELT polynomials as **functions**.

Definition 7.2. An ELT *polynomial* p is a function $p : R^n \to R$ of the form

$$p(\lambda_1, ..., \lambda_n) = \sum_{I \in G} a_I \lambda^I,$$

where $G \subseteq \mathbb{N}^n$ is a finite set and for all $I \in G$ the coefficient a_I is in R.

We denote the set of all such polynomials as $R[\lambda_1, ..., \lambda_n]$.

Definition 7.3. Let f be a polynomial of the form $f = \sum_{i=1}^{n} h_i$, where h_i are monomials. Write $f_h = \sum_{h_i \neq h} h_i$. Then:

1. A monomial h is called **inessential at a point** a if $f_h(a) = f(a)$. If h is inessential at every point, then h is called **inessential**.

2. A monomial h is called **essential at a point** a if it is not inessential at a (needed for layer zero) and $f(a) = h(a)$. If such a point exists, then h is called **essential**.

3. A monomial h is called **quasi-essential at a point** a if it is neither essential nor inessential at a. If h is neither essential nor inessential, then it is called **quasi-essential**.

Example 7.4. Consider $f = \lambda^2 + {}^{[1]}1\lambda + {}^{[1]}2$. Then λ^2 is essential at each tangible point with value greater than 1. At 1 all of the monomials are quasi-essential, and for tangibles lesser than 1 the fixed monomial 2 is essential.

Definition 7.5. The monomial of a univariate polynomial f with the highest power of λ is called the *leading* monomial. The monomial with the lowest power is called the *tail* monomial. Any other monomial is called a *middle* monomial.

Definition 7.6. A *corner root* of an ELT polynomial is a point at which at least two monomials dominate. i.e., $(c_1, ..., c_n) \in R^n$ is a corner root of $p(\lambda_1, ..., \lambda_n) = \sum_{I \in G} a_I \lambda^I$ if the set $\{I \in G | t(a_I c^I) = t(p(c_1, ..., c_n))\}$ is of order 2 at least.

In other words, a point a is a corner root of a polynomial $f = \sum_{i=1}^{n} h_i$ if h_k is quasi-essential at a for some $1 \leq k \leq n$.

Lemma 7.7. *A polynomial f in one variable has only finitely many corner roots.*

Proof. By definition, a corner root is a point where two monomials h_1, h_2 have the same tangible value. There are only finitely many different possible pairs of monomials, each contributing at most one root.

□

Definition 7.8. Let f and g be two polynomials in several variables. We say that f and g are *root equivalent* if

$$s\Big(f(c)\Big) = s\Big(g(c)\Big),$$

for every corner root c.

Definition 7.9. Let f be a multivariate polynomial. If all of the monomials of f have the same tangible value at the point a, then f is called **primary in** a.

Lemma 7.10. *Let f be primary in a. Then f is of the form $f = c\sum {}^{[b_I]}0 a_I \lambda^I$ where $I = (i_1, ..., i_n) \in (\mathbb{N}\cup\{0\})^n$, $a = (a_1, ..., a_n)$, $a_I = a_1^{-i_1} \cdots a_n^{-i_n}$, $b_I \in L$, $b_I \neq 0$, $\lambda^I = \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, and c is tangible.*

Proof. First we let c be the tangible value of $f(a)$. Let $d\lambda_I$ be any monomial of f. Then it is quasi-essential or essential at a. Thus the tangible value of $da_1^{i_1} \cdots a_n^{i_n}$ must be c. Therefore, $d = {}^{[b_I]}0a_1^{-i_1} \cdots a_n^{-i_n}c$ where b_I is the layer of d. $b_I \neq 0$ since otherwise the monomial could not be quasi-essential. $\square$

Definition 7.11. The *essential part* of a general polynomial $f = \sum_{i\in I} h_i$ at a point a is the polynomial

$$f_a = \sum_{k\in K} h_k$$

such that $k \in K \subseteq I$ if and only if h_k is not inessential at a.

It is fairly clear that the essential part at a is always primary at a.

8 The Expanded Structure

We wish to obtain a basic algebraic result - unique factorization - for polynomials over the ELT structure. In their paper (ref. [8]), Izhakian and Rowen showed that unique factorization fails in their original supertropical structure even when considering polynomials as functions.

As we explained in the introduction, we try to expand the structure of L. First we consider $L = \mathbb{N}$ with the usual operations. Considering $R[\lambda]$, the polynomials in one variable over this structure, we wish to know if there is unique factorization. A rather simple counterexample arises which we will explore.

We would hope generally that a polynomial would factor according to the variety of its roots; for each of the connected parts there would be one factor (not necessarily irreducible). In the one variable case, each such part is a single point.

Now we are ready to study the following polynomial over the supertropical algebra:

$$f = \lambda^2 + {}^{[2]}1\lambda + {}^{[1]}0.$$

In this example, the roots are $\lambda = 1$ and $\lambda = -1$. We expect that the above polynomial will factor into $(\lambda + {}^{[2]}1)({}^{[2]}1\lambda + 0)$, since these are the roots with the correct layers. We are rather close but not exactly there, as we get

$$(\lambda+{}^{[2]}1)({}^{[2]}1\lambda+0) = {}^{[2]}1\lambda^2+0\lambda+{}^{[2]}1{}^{[2]}1\lambda+{}^{[2]}1 = {}^{[2]}1\lambda^2+{}^{[2]}1\lambda+{}^{[2]}1 = {}^{[2]}1(\lambda^2+{}^{[2]}1\lambda+0) = {}^{[2]}1f.$$

The polynomial f is irreducible and also $(\lambda + {}^{[2]}1)$ and $({}^{[2]}1\lambda + 0)$ are irreducible, which contradicts unique factorization of $g = {}^{[2]}1f = (\lambda + {}^{[2]}1)({}^{[2]}1\lambda + 0)$.

However, these factorizations are not inherently different. The main problem is the lack of an inverse for the layer. If we add fractional positive layers, i.e. take $L = \mathbb{Q}^+$, the unique factorization of the polynomial will be

$${}^{[1/2]}(-1)(\lambda + {}^{[2]}1)({}^{[2]}1\lambda + 0) = (\lambda + {}^{[2]}1)(\lambda + {}^{[1/2]}(-1)).$$

9 Positive Rational Layers

For this section we fix $L = \mathbb{Q}^+$.

9.1 Monomial equality

We will show that polynomials which are equal as functions must consist of the same monomials (other than the inessential ones). Moreover, we will see that given enough points of equality, two polynomials are equal everywhere.

Lemma 9.1. *Let $k > l \in \mathbb{N}$ and $a, b \in R$. The polynomial $f = a\lambda^k + b\lambda^l$ has a corner root $x \in R$. For any substitution $\lambda < x$, the second monomial is essential and for any substitution $\lambda > x$, the first monomial is essential.*

Proof. A tangible point x is a corner root if $t(ax^k) = t(bx^l)$, and therefore $t(x^{k-l}) = t(ba^{-1})$. Given our assumptions, such a corner root exists. If $y > x$ then $t(y^{k-l}) > t(x^{k-l}) = t(ba^{-1})$ and therefore $ay^k > by^l$; similarly if $y < x$ then $ay^k < by^l$. We see that the corner root is like a scale; on one side one monomial is essential and on the other side the second monomial is essential. This is the piecewise linear behavior of supertropical algebras. □

Corollary 9.2. *Given a polynomial in one variable*

$$f = a_{r_1}\lambda^{r_1} + ... + a_{r_n}\lambda^{r_n}$$

such that $r_1 > r_2 > ... > r_n$, $n > 2$ and $a_{r_i} \neq -\infty$, then f has a finite set of corner roots

$$x_k > x_{k-1} > ... > x_1, k > 0.$$

The monomials $a_{r_1}\lambda^{r_1}+a_{r_2}\lambda^{r_2}+...+a_{r_i}\lambda^{r_i}$ are quasi-essential at x_k for some $i>0$. The monomial $a_{r_1}\lambda^{r_1}$ is essential at $\lambda > x_k$, the monomial $a_{r_i}\lambda^{r_i}$ is essential at $x_k>\lambda>x_{k-1}$, and the monomials between them are inessential at $\lambda \neq x_k$.

The monomials $a_{r_i}\lambda^{r_i}+...+a_{r_j}\lambda^{r_j}$ are quasi-essential at x_{k-1} for some $j>i$. The rightmost monomial is essential at $x_{k-1}>\lambda>x_{k-2}$, and the monomial between it and the leftmost monomial are inessential at $\lambda\neq x_{k-1}$.

This continues until the rightmost monomial is $a_{r_n}\lambda^{r_n}$ which is essential at $\lambda<x_1$.

Lemma 9.3. *If $f=\sum h_k$ is a multivariate polynomial with a non-empty set of corner roots, then for all i, h_i is quasi-essential on at least one corner root of f.*

Proof. Recall that h_i is not inessential. Therefore, let c be a point so that h_i is essential or quasi-essential at c. If c is a corner root we are done, so we assume that it is not.

The monomial h_i is in several variables and is not inessential at some point $c=(c_1,...,c_n)$. Consider f as a polynomial of one variable by fixing all of the variables other than λ_j at c (we specialize $\lambda_r=c_r$). There is only a finite number of monomials, and so there is only a finite number of corner roots between h_i and any other monomial h_n of f; we denote them as x_k. First, assume that $\{x_k\}$ is not an empty set. Tangibles are from an ordered monoid. Thus we can sort these corner roots along with the point c by size. Considering the fact that h_i is quasi-essential at c, we take the closest corner root in the array after which h_i is inessential to be x_s. Since this corner root is the closest to c, h_i and h_s are quasi-essential at x_s, for otherwise there is another monomial h_w bigger than h_i but then x_w is between c and x_s which is false. We obtained x_s as a corner root on which h_i is quasi-essential.

However, if the set of corner roots $\{x_k\}$ is empty we choose a variable other then j. If the corner roots sets are empty for all variables, it follows that the polynomial has no corner roots altogether, which is absurd. Indeed, if f has any corner root, it must have at least one monomial which differs in at least one variable power from h_i. Leaving this variable free we obtain a non empty set of roots. □

We will prove that given $s(f({}^{[x]}a))$ for all x, one can know all of the monomials that are quasi-essential at a. As a consequence, one can know all of the monomials of a polynomial f, due to the lemma above that assures us that each monomial is at least quasi-essential at some corner root.

Lemma 9.4. *Let f be a univariate polynomial, and let g be the polynomial which contains all of the monomials of f which are not inessential at some given point a. Then g is of the form $g=c_a\sum_{k=0}^{n}{}^{[b_k]}0\lambda^k a^{n-k}$, where c_a is tangible.*

Proof. Clearly, all of the monomials of g have the same tangible value at a. Let n be the degree of g, and define $c_a=t(g(a))a^{-n}$. Let h be a monomial of g of degree k. Then $c_a a^n=t(g(a))=t(h(a))=ba^k$. Therefore $t(b)=c_a a^{n-k}$, as desired. □

Lemma 9.5. *If f and g are two polynomials in one variable (with non-empty corner root sets), then f and g are root-equivalent and are equal at one point $\iff$ f and g have the same monomials $\iff$ $f = g$ everywhere.*

Proof. Fix a tangible corner root a and look at the sum of all non-inessential monomials at a;

$$c_a \sum_{k=0}^{n} {}^{[b_k]}0\lambda^k a^{n-k}.$$

Next we look at the root at layer i: $f({}^{[i]}a) = c_a \sum_{k=0}^{n} {}^{[b_k \cdot i^k]}a^n$. Now its layer

$$s(f({}^{[i]}a)) = s(c_a \sum_{k=0}^{n} {}^{[b_k \cdot i^k]}a^n) = \sum_{k=0}^{n} b_k i^k$$

(we include the layer of c_a into the b_k's, and therefore $s(c_a) = 1$). This equation holds for all $i \in \mathbb{N}$. A similar argument for g yields $s(g({}^{[i]}a)) = \sum d_k i^k$, with e_a as the constant. Our goal is to show that $b_k = d_k$ and that $c_a{=}e_a$. This will prove that f and g have the same monomials, and the rest of the proof is trivial.

Next let us prove that $b_k = d_k$. We take a number $m \in \mathbb{N}$ so that mb_k and mc_k are in $\mathbb{N}$ for all k (this is the *lcm* of all of the denominators). Now take i to be $\max_k(mb_k, md_k) + 1$. $\sum mb_k i^k$ is written to the base i, but this form is unique. Recalling $f({}^{[i]}a) = g({}^{[i]}a)$, we use the same argument for g to obtain $mb_k = md_k$ and finally $b_k = d_k$.

We will now show that given the constant c_a of a root a, all of the other constants are known. Let $\{a_i\}_{i=1}^{r}$ be the sorted set of roots of f (and g). Fix a root a_i and assume c_{a_i} is known. Between two consecutive roots a_k and a_{k+1}, there is one essential monomial (otherwise there would be another root between them). Clearly, this essential monomial is quasi-essential at the roots also. Therefore there is a monomial of the form $c_{a_i}{}^{[b_k]}0\lambda^k a_i^{n-k}$ and also of the form $c_{a_{i+1}}{}^{[b_k]}0\lambda^k a_{i+1}^{n-k}$. We conclude that $c_{a_i} a_i^{n-k} = c_{a_{i+1}} a_{i+1}^{n-k}$ and therefore we can calculate $c_{a_{i+1}}$ from c_{a_i}, a_i and a_{i+1}. This argument works in the other direction as well, and thus all of $\{c_{a_i}\}_{i=1}^{r}$ are known.

Finally, since f and g are equal at some point, they have equal constants $c_a = e_a$ for some a. The rest of the constants are equal since they are calculated from this constant and the other roots (which are equal for f and g). We obtained the desired result: if f and g are root-equivalent and are equal at some point, then they have exactly the same monomials, and therefore are equal everywhere. If they are equal everywhere, then clearly they are equal at some point and root-equivalent. □

We will now prove this monomial equality for several variables using an induction whose base is the above lemma. In several variables, the corner roots are much more complex and interesting as we will see later in this thesis. First we prove a simple lemma which helps us show that any point of equality between two root-equivalent polynomials means equality at every point.

Lemma 9.6. *Let $f = \sum h_i$ be a polynomial in n variables with a non-empty set of corner roots, and let $c = (c_1, ..., c_n)$ be any point. Then there is a variable λ_i, and a corner root a such that $a = (x_1, ..., x_{i-1}, c_i, x_{i+1}, ...x_n)$.*

Proof. Having a corner root set is equivalent to f having at least two monomials, as we have seen when proving monomial equality. These two monomials must differ at the power of at least one variable, call it λ_j. Now fix any $\lambda_i = c_i$ such that $i \neq j$. Clearly we obtain a polynomial with $n-1$ variables and at least two different monomials. Therefore, this polynomial has a corner root $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and a above is a corner root of f as desired. □

Theorem 9.7. *Let f and g be two polynomials with non-empty corner root sets, then f and g are root-equivalent and are equal at one point $\iff$ f and g have the same monomials $\iff$ $f = g$ everywhere.*

Proof. Assume that f and g are root-equivalent and there exists a point c such that $f(c) = g(c) = x$. By the lemma above there is a variable, call it λ_n, and a corner root a such that $a = (x_1, ..., x_{n-1}, c_n)$.

At any given corner root the layers behave like classical polynomials. For example consider $f(\lambda) = {}^{[2]}\lambda + {}^{[1]}0$ then $f({}^{[x]}0) = {}^{[2x+1]}0$. Since f equals g at every layer of the corner root, their essential monomials at the corner root must be equal up to multiplication by a constant. In view of the point of equality they are exactly equal.

Similarly to the proof of the above lemma, by choosing different variables and direction we prove that f and g must consist of exactly the same monomials, and the rest of the proof follows. □

Corollary 9.8. *If f and g are two polynomials with non-empty corner root sets such that f and g are root-equivalent, then there is some constant c such that $f = cg$.*

9.2 Primary polynomials

Lemma 9.9. *Let f be a polynomial in one variable. Then there is a factorization of f into primary polynomials of its roots, and a power of λ.*

Proof. Take the highest $m \in \mathbb{N}$ such that λ^m divides each monomial in f. After we factor out λ^m, the constant monomial is essential. We call the new polynomial f_1, and order the root set $\{a_i\}_{i=1}^m$ where $a_1 < a_2 < ... < a_m$. The essential part at a_1 is primary in a_1. Let j be the highest power of λ in this primary polynomial, and let c_j be the coefficient of λ^j. We normalize by multiplying by c_j^{-1}, and call this primary polynomial g_1.

Now take the essential part at a_2. The highest power of λ of a monomial in g_1 is the smallest power here. Therefore by factoring out λ^j where $j = \deg(g_1)$ of the essentials above, we get a primary polynomial g_2 in a_2. Let us look at $g_1 g_2$ as a polynomial. First, it has the roots a_1 and a_2. Also, when $a < a_2$, $g_2(a) = b_n$. Therefore, $g_1 g_2$ and f_1 have the same layers at images of all layers of a_1. When $a > a_1$, the essential monomial of g_1 at a is λ^j and therefore $f(a_2) = g_2(a_2)g_1(a_2)$. Thus g_1g_2 has the same layers as f_1 for a_2 as well.

We normalize g_2g_1 and continue to get g_3 as before. $g_3(a)$ for $a < a_3$ is exactly the constant we normalized g_2g_1 with, and therefore $g_3g_2g_1$ has the same layers on a_1, a_2 and a_3. We continue this process and obtain that $f_1 = g_ng_{n-1}...g_1$ due to Theorem 9.7, and thus $f = \lambda^m g_ng_{n-1}...g_1$, as desired. $\square$

Example 9.10. We will now factor a polynomial into primary polynomials. Consider

$$f = 4\lambda^5 + 5\lambda^4 + 6\lambda^3 + 6\lambda^2.$$

First we factor λ^2 and are left with

$$f_1 = 4\lambda^3 + 5\lambda^2 + 6\lambda + 6.$$

The root set here is $\{a_1 = 0, a_2 = 1\}$. The essential monomials in a_1 are $6\lambda + 6$. We normalize by -6 and obtain $g_1 = \lambda + 0$. The essential monomials in a_2 are $4\lambda^3 + 5\lambda^2 + 6\lambda$. We factor by $\lambda^j = \lambda^1$ and obtain $g_2 = 4\lambda^2 + 5\lambda + 6$. $g_1(a)g_2(a) = g_1(a) \cdot 6 = 6a + 6 = f_1(a)$ for $a < a_2 = 1$, specifically $f_1(a_1) = g_1(a_1)g_2(a_1)$. When $a > a_1$ then $g_1(a) = a$; therefore $g_1(a)g_2(a) = a \cdot g_2(a) = f_1(a)$, specifically $f_1(a_2) = g_1(a_2)g_2(a_2)$. In this case, the factorization process ends here, and

$$f = \lambda^2 g_1g_2 = \lambda^2(\lambda + 0)(4\lambda^2 + 5\lambda + 6).$$

9.3 Unique factorization

Theorem 9.11. *Let f be a univariate polynomial. Then f factors uniquely into polynomials which are primary at the corner roots, possibly including λ^m for some $m \in \mathbb{N}$ (up to multiplication by constants).*

Proof. By the above lemma we know that such a factorization exists. Suppose that there are two different factorizations of f, called h_1 and h_2. Take the smallest corner root a. Let g be a primary polynomial in a root bigger than a. Then $c := g(a)$ is the constant of g. Now $g({}^{[i]}a) = c$ for all i, since a is smaller than the corner root of g. Thus, both h_1 and h_2 have the same layers at the primary in a up to a constant (since the other primaries yield constants in a). Thus the primaries in a of both h_1 and h_2 are equal up to multiplication by a constant. We factor out the primaries in a, and continue this process to see that all the primaries are equal up to a multiplication by constants. $\square$

Now the natural question is whether or not a primary polynomial factors uniquely into irreducibles. Define $\mathbb{P}_a$ to be the set of polynomials of the form $f = \sum_{k=0}^{n} {}^{[b_k]}0\lambda^k a^{n-k}$, and the function $\psi : \mathbb{P}_a \to \mathbb{Q}^+[\lambda]$ defined by $\psi(f) = \sum_{k=0}^{n} s(b_k)x^k$. $\mathbb{Q}^+$ is the set of positive rational numbers.

Lemma 9.12. *Let f and g be primary polynomials in a. Then $\psi(f+g) = \psi(f) + \psi(g)$ and $\psi(fg) = \psi(f)\psi(g)$. Moreover, ψ is an isomorphism up to multiplication by a tangible constant in $\mathbb{P}_a$.*

Proof. Let f be of the form $f = \sum_{i=0}^{n} a_i\lambda^i$ and $g = \sum_{j=0}^{m} a_j\lambda^j$. Assume $n \geq m$; then

$$f + g = \sum_{k=0}^{m}(a_k + b_k)\lambda^k + \sum_{k=m+1}^{n}(a_k)\lambda^k.$$

Therefore,

$$\psi(f+g) = \sum_{k=0}^{m}(s(a_k+b_k))x^k + \sum_{k=m+1}^{n}(a_k)\lambda^k = \sum_{k=0}^{m}(s(a_k)+s(b_k))x^k + \sum_{k=m+1}^{n}(s(a_k))\lambda^k =$$

$$= \sum_{k=0}^{n} s(a_k)x^k + \sum_{k=0}^{m} s(b_k)x^k = \psi(f) + \psi(g).$$

Now for multiplication,

$$fg = \sum_{i,j=1}^{i=n,j=m}(a_i + b_j)\lambda^{i+j}.$$

$$\psi(fg) = \sum_{i,j=1}^{i=n,j=m}(s(a_i+b_j))x^{i+j} = \sum_{i,j=1}^{i=n,j=m}(s(a_i)+s(b_j))x^{i+j} = \psi(f)\psi(g).$$

Next, ψ is clearly onto, and in order to prove it is also an isomorphism we must verify it is injective. Assume $\psi(f) = \psi(g)$, $\sum_{k=0}^{n} s(b_k)x^k = \sum_{k=0}^{n} s(a_k)x^k$. We see that $\forall k, s(a_k) = s(b_k)$. Since both f and g are primary in a, the tangible values of a_k, b_k are fixed up to a multiplication by a tangible constant c, and so we get $f = cg$. $\square$

Corollary 9.13. *We have unique factorization for primary polynomials iff there is unique factorization in $\mathbb{Q}^+[\lambda]$.*

Unfortunately, unique factorization fails in polynomials over the positive rational numbers (or even the positive real numbers). Take the following example:

$$x^6+2x^5+3x^4+2x^3+3x^2+2x+2 = (x^4+x^2+1)(x^2+2x+2) = (x^2+x+1)(x^4+x^3+x^2+2).$$

In order to verify that these polynomials are irreducible, we will look at the complete factorization over $\mathbb{R}$:

$$x^6+2x^5+3x^4+2x^3+3x^2+2x+2 = (x^2+x+1)(x^2-x+1)(x^2+2x+2).$$

One can easily see that the above factorizations are obtained by combining the left pair of polynomials $[(x^2+x+1)(x^2-x+1)]$ or the right one $[(x^2-x+1)(x^2+2x+2)]$. It is immediate that primary polynomials do not factor uniquely in this structure. (I thank Professor Uzi Vishne for his elegant counterexample.)

10 Full Rational Layer

We have seen that unique factorization fails for primary polynomials, due to the lack of negative layers. Unlike positive layers, negative layers first arise in light of the factorization problem. We will see that this expansion will be interesting in itself, and will solve our factorization problem.

From now on, we fix $L = \mathbb{Q}$.

10.1 Layer zero elements

One should introduce a new layer – zero, which changes the rules. Consider the primary polynomial $\lambda^2 + 2\lambda + 4$; here 2λ is quasi-essential. However, if we change the polynomial slightly into $\lambda^2 + {}^{[0]}2\lambda + 4$, we turn the middle monomial into an inessential one. This is true since this monomial does not change the size of the polynomial at any point, but it also does not change the layer because it contributes zero. These layer zero monomials will be the only exception to unique factorization in one variable.

For example, let us look at the following polynomial:

$$f = {}^{[0]}0(\lambda^2 + 1\lambda + 0).$$

f is equal to ${}^{[0]}0(\lambda + {}^{[y]}1)(\lambda + {}^{[x]}(-1))$ for any $x, y \in \mathbb{Q}$. Unfortunately, this is not the only type of counterexample to unique factorization. However, the only counterexamples to unique factorization involve layer zero monomials which do not factor polynomials without zero layer monomials, as we will see in the next subsections.

10.2 Monomial equality

First we must obtain the same monomial equality result that we have seen only for positive layers. We wish to prove that if two polynomials are equal as functions, they have the same essential and quasi-essential monomials.

We notice that root-equivalence is weaker than monomial equivalence in this structure. Consider the polynomials $f = 2\lambda^4 + {}^{[0]}3\lambda^3 + 0$, $g = \lambda^4 + {}^{[0]}2\lambda^2 + 0$. Both f and g have corner roots 1 and -1. Also $\forall i, f({}^{[i]}(-1)) = g({}^{[i]}(-1)) = 0$ and $s(f({}^{[i]}1)) = s(g({}^{[i]}1)) = i^4$. Therefore f and g are root-equivalent, but clearly not equal. We need to capture the monomials with layer zero coefficients which are lost on corner roots because they are inessential there. We capture these monomials by looking at the slopes of the polynomial as well.

Lemma 10.1. *Let p_1 and p_2 be two primary polynomials in a. Assume that $\forall \ell \in \mathbb{Q}, s(p_1({}^{[\ell]}a)) = s(p_2({}^{[\ell]}a))$, and assume that p_1 and p_2 have no monomials with layer zero coefficients. Then p_1=cp_2 for some constant c.*

Proof. Assume that $\deg(p_1) \geq \deg(p_2)$. Now choose a polynomial p with high layer coefficients so that $p_1 + p$ and $cp_2 + p$ are both primary polynomials in a with positive layer coefficients (c is a constant). Also we choose p so that it does not have any monomials in which both p_1 and cp_2 are missing. Due to our results in the positive layer section, we know that $p_1 + p = cp_2 + p$. Now since p_1 and p_2 do not have layer zero monomials, we can remove p from the equations by adding ${}^{[-1]}0p$: $p_1 + p + {}^{[-1]}0p$ is p_1 with the addition of layer zero monomials; assume h is such a monomial. Since p_1 does not have monomials of layer zero h cannot be a monomial of p_1, and therefore it must be a monomial of p. $cp_2 + p + {}^{[-1]}0p = p_1 + p + {}^{[-1]}0p$ so h must not be a monomial of either p_1 nor p_2 implying h is in p which is absurd. We conclude that $p_1 + p + {}^{[-1]}0p$ has no layer zero monomials and is equal to p_1, and $cp_2 + p + {}^{[-1]}0p = cp_2$ so $p_1 = cp_2$. □

Lemma 10.2. *Let x, y be two tangibles, and $r, t \in R$. Then there is no more than one monomial $a\lambda^k$ such that $ax^k = r$ and $ay^k = t$.*

Proof. This is a case of two linear equations in two variables. $a = (x^{-1})^k r$ since $ax^k = r$ and therefore $(x^{-1})^k r y^k = t$. Due to our assumptions, there is only one k such that $(x^{-1}y)^k = r^{-1}t$. Then $a = r(x^{-1})^k$. □

Since the layer zero is not our main interest and it turns out to be a counterexample to unique factorization, we will prove an easier theorem for monomial equality. Instead of building an exact description of the minimal amount of data we need in order to reconstruct the polynomial, we assume that we have all the data.

Theorem 10.3. *For any two polynomials f and g, f and g have the same monomials if and only if $f = g$ everywhere.*

Proof. Assume $f = g$ everywhere. We already know that each monomial with layer different from zero must be quasi-essential at some point. As we have seen, this monomial has to be in the polynomial (both in f and g).

Assume that h is a monomial with layer zero. Then h must be essential at some point $a = (a_1, ...a_n)$. As we have seen, when all variables are specialized to a except for λ_i, then h is a monomial in one variable which is essential at least between $x < a_i < y$ for some x, y. Due to our assumptions there are at least two points x', y' such that $x < x' < y' < y$ and the monomial is essential at x' and y'. Thus $h(x') = f(x')$ and $h(y') = f(y')$, and as a consequence of the above lemma, h is known. Therefore we know the exponent of λ_i. We can calculate the exponent of all the variables this way, and then the coefficient is given because we know $h(a) = f(a)$.

In conclusion, any monomial of f or g is uniquely determined by the polynomial as a function, as desired. The other direction is trivial. □

10.3 Regular polynomials

We will describe the multiplication process of polynomials, since it differs from classic multiplication and is essential to our paper. Even though multiplication is distributive, some monomials of the product are inessential and therefore deleted, since we view polynomial as functions. The multiplication is well defined due to theorem 10.3.

Example 10.4. Consider the two polynomials $f = (-2)\lambda^2 + \lambda + 1$, $g = (-5)\lambda^3 + (-1)\lambda + 0$. Both f and g have corner roots 1 and 2. Thus $f \cdot g =$

$$= ((-2)\lambda^2)((-5)\lambda^3 - 1\lambda) + \lambda((-5)\lambda^3 + (-1)\lambda + 0) + 1((-1)\lambda + 0) =$$

$$= (-7)\lambda^5 + (-5)\lambda^4 + (-3)\lambda^3 + (-1)\lambda^2 + {}^{[2]}0\lambda + 1.$$

Whereas $(-7)\lambda^5$ is essential at $\lambda > 2$, $(-7)\lambda^5 + (-5)\lambda^4 + (-3)\lambda^3 + (-1)\lambda^2$ are quasi-essential at $\lambda = 2$, $(-1)\lambda^2$ is essential at $1 < \lambda < 2$, $(-1)\lambda^2 + {}^{[2]}0\lambda + 1$ are quasi-essential at $\lambda = 1$, and 1 is essential at $\lambda < 1$.

Definition 10.5. A polynomial $f = \sum {}^{[\ell_I]}a_I\lambda^I \in R[\lambda_1, ..., \lambda_n]$ is called *regular* if $\ell_I \neq 0$ for all I.

We wish to prove that regular polynomials have unique factorization. In order to give this result more meaning, we would also like to prove that the product of two regular polynomials is also regular.

Lemma 10.6. *Let g and h be polynomials. A monomial of $f = gh$ is essential at a $\Longleftrightarrow$ it is a product of an essential monomial of g at a and an essential monomial of h at a.*

Proof. A monomial of f is the sum of products of monomials from g and monomials from h. For example, $g = \lambda + 0$, $h = \lambda + {}^{[2]}0$, $f = gh = \lambda^2 + \lambda^{(3)} + {}^{[2]}0$. The middle monomial $\lambda^{(3)}$ is the sum of $\lambda \cdot {}^{[2]}0 + 0 \cdot \lambda$.

Assume that u is an essential monomial of f at point a, then $u = g_1h_1 + ... + g_kh_k$. If for any i g_i of h_i are inessential then u is inessential, thus at a for all of the monomials g_i, h_i are quasi-essential. Therefore g_ih_j is also quasi-essential at a for every $1 \leq i, j \leq k$. Assume $k > 1$, the monomials g_1h_1 and g_1h_2 must have different powers of the variables otherwise h_1 and h_2 have the same powers which is absurd. We obtained two different monomials which are quasi-essential at a which contradicts the assumption that u is essential at a, and thus $k = 1$.

As a consequence of the contradiction, any essential monomial of f is a product of an essential monomial of g and an essential monomial of h as desired.

In the other direction, assume u is a monomial of f which is the product of monomials of g and h which are essential at a, $u = g_1h_1$. Clearly u is essential at a as any other product of two monomials g_ih_j is inessential at a due to g_i, h_j or both. $\square$

Corollary 10.7. *Let $f = gh$ be a factorization of a polynomial f. Then f is regular $\iff$ g and h are regular.*

Proof. Any monomial of layer zero which is not essential at a certain point must be inessential at that point. Thus in order for a polynomial to be regular, all of its essential monomials must be of layer different from zero.

A product of monomials $f_i = g_j h_k$ is an essential monomial of f at $a \iff g_j$ and h_k are essential at a due to the corollary above. The layer of f_i is the product of the layers of g_j and h_k, Thus the layer of f_i is zero $\iff$ the layer of either g_j or h_k is zero.

Therefore f is regular $\iff$ all monomials of f are of layer other than zero $\iff$ all of the monomials of g and h are with layer different than zero $\iff$ g and h are regular. $\square$

10.4 Factorization

Definition 10.8. Given an essential monomial h of a polynomial f, the largest corner root of f at which h is quasi-essential is called the *big root* of h and the smallest corner root is called the *small root* of h. Note that h may have one corner root which is both the big root and small root of h.

Example 10.9. Consider polynomial

$$f = \lambda^2 + \lambda + {}^{[1/4]}0.$$

The only corner root of f is${}^{[-1/2]}0$. Thus it is both the big root and small root of every monomial.

Lemma 10.10. *Let f be a polynomial with a constant term, and let h be an essential monomial of f so that $s(h) \neq 0$, and the big root and small root of h are distinct. Then f factors uniquely, up to multiplication by a constant into $f = g_1 g_2$, so that g_1 contains all of the corner roots greater or equal to the big root of h, and g_2 contains all of the corner roots less than or equal to its small root. Also any irreducible factor of f divides either g_1 or g_2.*

Proof. Let $f = \sum_{i=1}^{n} c_i \lambda^i$ and $h = c_k \lambda^k$ for some k. Define $g_2 = \sum_{i=1}^{k} c_i \lambda^i$, and $g_1 = c_k^{-1} \sum_{i=k}^{n} c_i \lambda^{i-k}$ (here we use the fact that $s(h) \neq 0$). Multiplication is very easy since the corner roots of g_1 are bigger than those of g_2. When a point has value smaller or equal to the small root of h, then in g_1 the essential monomial is 0. Therefore $0 \cdot g_2$ are essential and quasi-essential monomials of $g_1 g_2$. When a point has value greater than the small root of h, the only essential monomial of g_2 is $c_k \lambda^k$. Multiplying by g_1 we get $c_k \lambda^k g_1 = c_k \lambda^k c_k^{-1} \sum_{i=k}^{n} c_i \lambda^{i-k} = \sum_{i=k}^{n} c_i \lambda^i$. Together we get $g_1 g_2 = f$.

Now, assume that there is another factorization $f = u_1u_2$ with the same properties. Since f has a constant term, then so must u_1 and u_2. The constant of u_1 multiplied by u_2 must equal monomials of f which are essential and quasi-essential at the corner roots of u_2 and g_2, as before. So $g_2 = cu_2$ for some constant c. By a similar argument, $d\lambda^k u_1$ are the remaining monomials of f for some constant d, and so the factorization $f = g_1g_2$ is unique up to multiplication by constants.

Let $f = J_1J_2...J_l$ be a factorization of f into irreducible factors. We will show that each J_i has corner roots which are all larger than the big root of h or all smaller than the small one. Otherwise, there is a factor $J = J_i$ that has a corner root bigger than the big root of h and a corner root smaller than the small root of h. The polynomial J is irreducible, and so from the first part of the theorem we have already proved, the essential monomials which are not at the edges must have layer zero coefficient (or J would factor further). Therefore between the big and small roots of h, the essential monomial of J is of layer zero, and so the essential monomial at this point of f is of layer zero which is absurd. □

10.5 Primary polynomials

We have expanded the layer structure further in order to solve the problem of unique factorization of primary polynomials. Next we will verify that this property indeed holds.

Recall that the function $\psi : \mathbb{P}_a \to \mathbb{Q}[x]$, where $\mathbb{P}_a$ is the semiring of primary polynomials in a and

$$\psi(f) = \sum_{k=0}^{n} s(b_k)x^k$$

for $f = \sum_{k=0}^{n} b_k\lambda^k$. We will use ψ here to prove unique factorization for primary polynomials.

In the case of positive layers, ψ is an isomorphism. In this structure, we have for example $\psi({}^{[0]}\lambda + 0) = 1 = \psi({}^{[0]}\lambda^2 + 0)$, so ψ is not an isomorphism. Fortunately, it is still a homomorphism, since this part of the proof is not affected by the existence of layer zero.

Lemma 10.11. *Let f and g be primary univariate polynomials without monomials of layer zero. Then $\psi(f) = \psi(g) \iff f = cg$ for some tangible constant c .*

Proof. The proof is similar to the positive layer case. Knowing that f and g has no layer zero monomials leaves only one choice up to a tangible constant for $\psi^{-1}(\psi(f))$. □

It is important to note that primary polynomials cannot have layer zero monomials other than the leading monomial and the constant. The reason is that a quasi-essential monomial with layer zero is an inessential monomial and therefore is not part of the polynomial.

Theorem 10.12. *Let f be a primary univariate polynomial (with $L = \mathbb{Q}$). Then f factors uniquely into irreducible factors (up to multiplication by a constant).*

Proof. Assume that f is a regular primary polynomial, and that it factors in two different ways $f = g_1 \cdots g_n = h_1 \cdots h_m$. Since f is regular, so are $g_1, ..., g_n, h_1, ..., h_m$. As a consequence of the lemma, these factorizations are the same up to a multiplication by a constant.

Now we take f with a leading monomial of layer zero. f must be of the form $f = {}^{[0]}\lambda^n + g$ for g with $\deg(f) > \deg(g)$ and g with lead monomial of layer different than zero. First we will show that f factors into

$$({}^{[0]}\lambda + a)^{n-\deg(g)}(-a)^{n-\deg(g)}g = ({}^{[0]}\lambda^{n-\deg(g)} + a^{n-\deg(g)})(-a)^{n-\deg(g)}g = {}^{[0]}\lambda^n + g = f.$$

Note that g has a leading monomial coefficient of size $a^{n-\deg(g)}$, and note that we strike out any middle monomial with layer zero.

Take a factorization $f = g_1 \cdots g_k$. At least one factor must have a layer zero leading monomial. We rearrange and rename the factors to obtain $f = gh$, where g has leading monomial of layer zero and h does not. The polynomial f factors further to $({}^{[0]}\lambda + a)^l(-a)^l uh$ where u has no leading monomial of layer zero, and neither does uh. Since the sum of the lowest degree monomials of f equals to uh and also to g we have $uh = g$, and $l = n - \deg(g)$. Thus any factorization of f is a sub-factorization of $({}^{[0]}\lambda + a))^{n-\deg(g)}$.

A similar argument applies to f with layer zero constant. $f = g\lambda^k + {}^{[0]}a^{\deg(g)+k}$ where g has a constant term $a^{\deg(g)-k}$. We factor f; $f = g(\lambda + {}^{[0]}a)^k$. Like the leading monomial of layer zero argument, f factors uniquely into $g\lambda^k + {}^{[0]}a^{\deg(g)+k}$.

In conclusion, a general primary polynomial f factors into $f = ({}^{[0]}\lambda + a)^m(\lambda + {}^{[0]}a)^k g$ where g has no layer zero monomials, and not a layer zero constant. It is easy to see that the middle monomials of this product are $a^m\lambda^k g$, therefore g, k and m are determined uniquely according to the middle monomials of f. We already know that such g factors uniquely, and so every primary polynomial f factors uniquely into irreducible factors. □

10.6 Main result

We are now ready to prove our main theorem.

Theorem 10.13. *Any regular polynomial f factors uniquely into irreducible factors.*

Proof. From previous sections we know that f factors uniquely into primary polynomials around each essential monomial with layer different from zero. Thus a regular polynomial f factors uniquely into primary polynomials at each of its corner roots. Moreover, we proved that primary polynomials factor uniquely into irreducible factors. Thus f factors uniquely into irreducible factors. □

10.7 Non-regular polynomial factorization

Next we will discuss non-regular polynomials. We will first describe the basic irreducible factors, then examine some counterexamples to unique factorization, and thereafter we will sum up the factorization process for a general polynomial.

10.7.1 Basic irreducible factors

Unlike the case of positive layers, here we have an irreducible polynomial that is not primary. Fortunately, this is the only form of an irreducible polynomial other than the primary polynomials

$$\lambda^m + {}^{[0]}0b\lambda^k + c. \tag{1}$$

We list three properties of this form:

1. c must be of layer different from zero, or this polynomial will be reducible.
2. Not all polynomials of this form are irreducible.
3. We choose this polynomial not to be primary; thus it has two corner roots.

Lemma 10.14. *Non primary polynomials of the form*

$$\lambda^m + {}^{[0]}b\lambda + c$$

and

$$\lambda^{k+1} + {}^{[0]}b\lambda^k + c,$$

are irreducible.

Proof. Since a polynomial of this form has two corner roots, there are three types of polynomials that can factor it: Primaries in the first corner root, primaries in the second corner root, and polynomials with both of these roots. We will try all of these options to determine when these polynomials are irreducible.

First, we will try to multiply two primary polynomials, one for each corner root. Since the leading monomial and the constant do not have layer zero, the leading monomial and constant of their factors also must not be of layer zero. Therefore any such primaries are regular (since quasi-essential monomials of ghost layer zero are deleted). Clearly, this product cannot yield a polynomial of the form 1 (since the product is regular as well).

Next, we try to multiply a primary polynomial in the small root, with a polynomial having two corner roots. It is easier to look first at the essential monomials only, so we will assume g and h have no monomials which are never essential. Define $g = r+w$, $h = t+u+v$ where r, w, t, u, v are monomials. Then $gh = tr + ur + uw + rv + vw$ where uw and rv are never essential. In order for the result to be of the desired form, $s(uw)$ must be zero, and $s(r), s(w), s(t), s(v)$ must not be zero; therefore $s(u) = 0$. We remain with the monomial rv which is never essential, and therefore we have failed to achieve the desired form.

We get a similar result by multiplying a primary in the big root together with a polynomial with both corner roots. $g = r+w$, $h = t+u+v$, $gh = tr+tw+ru+uw+vw$ with tw and

ru never essential. We must have $s(uw) = 0, s(w) \neq 0 \Rightarrow s(u) = 0$, and $s(r), s(t), s(v) \neq 0$. We again get a polynomial with a monomial which is never essential.

Now we multiply two polynomials both having two corner roots. $g = r + y + t$, $h = u + v + w$, $gh = ru + rv + uy + yv + bw + tv + tw$ with the essentials ru, yv, tw. We do not want any monomials which are never essential, so we have $s(rv), s(uy), s(yw), s(tv) = 0$. Clearly $s(r), s(u) \neq 0$, and so $s(y), s(v) = 0$. Finally we obtain $gh = ru + yv + tw$. This is the only way a polynomial of the above form will factor, and when this factorization is impossible then the polynomial is irreducible.

Since our g and h each have three monomials, then y and v must have degree at least 1 in λ. Thus $f = gh = ru + yv + tw$ must have $\deg(yv) \geq 2$, and any polynomial of the form

$$\lambda^m + {}^{[0]}b\lambda + c$$

which is not primary is irreducible. Moreover, since $\deg(r) > \deg(y)$ and $\deg(u) > \deg(v)$, then $\deg(ru) \geq \deg(yv) + 2$. Thus the second type of irreducible non-regular polynomials is

$$\lambda^{k+1} + {}^{[0]}b\lambda^k + c.$$

□

We will later see that these are the only examples of irreducible polynomials which are not primary.

10.7.2 Counterexamples

We provide a few counterexamples to unique factorization for non-regular polynomials. We will start with the trivial case, and finish with more complex cases.

We have already seen that a product of irreducible non-regular polynomials is a product of pairs of matching monomials. Changing the layers of matching monomials, we can produce different factorizations:

$$(\lambda^2 + {}^{[0]}1\lambda + {}^{[1/3]}0)(\lambda^2 + {}^{[0]}1\lambda + {}^{[3]}0) = (\lambda^4 + {}^{[0]}2\lambda^2 + 0) = (\lambda^2 + {}^{[0]}1\lambda + 0)^2.$$

Next, we notice that layer zero coefficients eliminate quasi-essential monomials. We will build an example with different quasi-essential monomials which disappear. We will multiply an irreducible polynomial, with a primary polynomial at a corner root which is between the big and small root of the irreducible one. We will first study the general case, and then bring a concrete example.

Take $g = a + {}^{[0]}b + c$ with big root a_1 and small root a_2, and take $h = d + e + f$ primary at a_3 so that $a_1 > a_3 > a_2$. Think of e as any number of quasi-essential monomials (including none). Then $f = gh = ad + {}^{[0]}0(bd + be + bf) + cf$ where be is quasi-essential of layer zero and so is inessential.

A concrete example: $g = (\lambda^2 + {}^{[0]}1\lambda + 0)$ with corner roots $-1, 1$, and a primary at 0: $h = \lambda^2 + \lambda + 0$. Then $f = gh = \lambda^4 + {}^{[0]}1\lambda^3 + {}^{[0]}01\lambda + 0$. The monomial $1\lambda^2$ would be quasi-essential at 0, but the layer zero makes it inessential. If instead $h = \lambda^2 + 0$, the product $f = gh$ remains the same.

We have seen how to factor a non-regular polynomial with two corner roots. We will now give an example of different ways to factor the same polynomial.

Example 10.15. Take $f = (-2)\lambda^6 + {}^{[0]}\lambda^4 + 0$ with corner roots 1 and 0. Then

$$f = ((-1)\lambda^3 + {}^{[0]}\lambda^2 + 0)^2$$

but also

$$f = ((-1)\lambda^2 + {}^{[0]}\lambda + 0)((-1)\lambda^4 + {}^{[0]}\lambda^3 + 0).$$

So far we have seen various examples of non-unique factorizations of polynomials with the same corner roots. Now we will show an example of factorization into polynomials with different corner roots.

Example 10.16.

$$f = 1\lambda^8 + {}^{[0]}4\lambda^5 + {}^{[0]}4\lambda^4 + 0.$$

This polynomial has corner roots $\{-1, 0, 1\}$. We factor it into two factors, the first with corner roots $\{-1, 1\}$ and the second with corner roots either $\{-1, 0\}$ or $\{0, 1\}$.

$$f = (\lambda^6 + {}^{[0]}3\lambda^3 + 0)(1\lambda^2 + {}^{[0]}1\lambda + 0)$$

$$f = (2\lambda^6 + {}^{[0]}4\lambda^4 + 0)((-1)\lambda^2 + {}^{[0]}\lambda + 0).$$

These polynomial are not all irreducible, but they factor into polynomials having the same corner roots, which differ in the two cases.

10.8 Irreducible polynomials

We have introduced the basic irreducible polynomial, but we have yet to prove that it is the only non-regular irreducible polynomial. We will answer this question now.

Theorem 10.17. *The only irreducible non-primary polynomials are the basic irreducible polynomials.*

Proof. As we have seen before, any polynomial with a middle essential monomial with layer different from zero is reducible. Therefore we are only interested in polynomials which are not primary, having middle essential monomials of layer zero.

We start with the case of a polynomial with more than one corner root (since it is not primary) with quasi-essential monomials of layer different from zero at some corner root a. The polynomial is of the form $f = a_n\lambda^n + ... + {}^{[0]}a_k\lambda^k + a_m\lambda^m + ... + a_0$, where ${}^{[0]}0a_k\lambda^k + a_m\lambda^m$ have the same tangible value at a. We claim that

$$f = (a_m\lambda^m + ... + a_0)a_m^{-1}(a_n\lambda^{n-m} + ... + {}^{[0]}a_k\lambda^{k-m} + a_m).$$

It is easy to see that the corner roots of the left polynomial are the corner roots of f which are less than or equal to a, and the right polynomial has all of the corner roots of f which are larger or equal to a. When $\lambda > a$, the essential part is

$$(a_m\lambda^m)a_m^{-1}(a_n\lambda^{n-m} + ... + {}^{[0]}a_k\lambda^{k-m}) = a_n\lambda^n + ... + {}^{[0]}a_k\lambda^k.$$

When $\lambda \leq a$ the essential part is

$$(a_m\lambda^m + ... + a_0)a_m^{-1}(a_m) = a_m\lambda^m + ... + a_0.$$

(Note that the quasi-essential monomials at a multiplied by layer zero are inessential.) Thus we have proved that f factors into the above polynomials.

We are left with the case of polynomials with three or more corner roots and only essential monomials:

$$f = g + b_1\lambda^{m_1} + b_2\lambda^{m_2} + b_3\lambda^{m_3} + b_4,$$

where $g = \lambda^{m_1+1}g'$ for some polynomial g'. Let $a_n > a_{n-1} > ... > a_1$ be the corner roots of f, $n \geq 3$. We claim that

$$f = (b_2^{-1}b_3\lambda^{m_3-m_2}(g + b_1\lambda^{m_1}) + b_3\lambda^{m_3} + b_4)(b_2b_3^{-1}\lambda^{m_2-m_3} + 0).$$

The corner root of the second polynomial is a_2. Indeed, if $t(b_2x^{m_2}) = t(b_3x^{m_3})$ then $t(b_2b_3^{-1}x^{m_2-m_3}) = 0$. It is easy to verify that the corner roots of the right polynomial are $a_n, a_{n-1}, ..., a_3, a_1$. The only non-trivial case is of a_3:

$$b_2^{-1}b_3\lambda^{m_3-m_2}b_1\lambda^{m_1} + b_3\lambda^{m_3} = b_3\lambda^{m_3}(b_2^{-1}b_1\lambda^{m_1-m_2} + 0).$$

Therefore the product indeed equals f.

Note that this holds for polynomials with leading monomial and/or constant of layer zero. □

10.9 Summary of the factorization process

In this section we describe the factorization process for a general polynomial, and then give an example.

The first step is partitioning the polynomial around its essential middle monomials of layer different from zero. Afterwards, we factor the polynomial until we get to an irreducible factor, or to a primary polynomial. We factor the primary polynomials so they do not have layer zero leading monomials or constants. Finally, we factor the regular primary part in the same way polynomials factor over $\mathbb{Q}$.

10.9.1 Example

Consider the polynomial:

$$f = {}^{[0]}(-10)\lambda^{10} + (-4)\lambda^8 + (-1)\lambda^7 + {}^{[0]}3\lambda^5 + 5\lambda^3 + {}^{[-1]}5.$$

First we see that $\lambda = 0$ is the smallest corner root since it is the solution of the equation $5\lambda^3 + {}^{[-1]}5$. $5\lambda^3$ is the essential monomial with two distinct big and small roots, and thus f factors into

$$f = ({}^{[0]}(-15)\lambda^7 + (-9)\lambda^5 + (-6)\lambda^4 + {}^{[0]}(-2)\lambda^2 + 0)(5\lambda^3 + {}^{[-1]}5) =$$

$$= (-10)({}^{[0]}\lambda^7 + 6\lambda^5 + 9\lambda^4 + {}^{[0]}13\lambda^2 + 15)(\lambda^3 + {}^{[-1]}0).$$

Now the smallest corner root of ${}^{[0]}\lambda^7 + 6\lambda^5 + 9\lambda^4 + {}^{[0]}13\lambda^2 + 15$ is $\lambda = 1$. The next essential monomial is ${}^{[0]}13\lambda^2$, which has layer zero and therefore we will factor it later to basic irreducible factors. The next corner root is $\lambda = 2$, and the essential monomial is $9\lambda^4$. Therefore f factors further into

$$f = (-10)({}^{[-1]}(-9)\lambda^3 + (-3)\lambda + 0)(9\lambda^4 + {}^{[0]}13\lambda^2 + 15)(\lambda^3 + {}^{[-1]}0) =$$

$$= (-10)({}^{[0]}\lambda^3 + 6\lambda + 9)(\lambda^4 + {}^{[0]}4\lambda^2 + 6)(\lambda^3 + {}^{[-1]}0).$$

We obtained three factors – primary at 3, the basic irreducible factor at 2 and 1, and a primary at 0. We further factor each of these polynomials:

$${}^{[0]}\lambda^3 + 6\lambda + 9 = ({}^{[0]}\lambda + 3)^2(\lambda + 3)$$

$$\lambda^4 + {}^{[0]}4\lambda^2 + 6 = (\lambda^2 + {}^{[0]}2\lambda + 3)^2$$

$$\lambda^3 + {}^{[-1]}0 = (\lambda + {}^{[-1]}0)(\lambda^2 + \lambda + 0).$$

Thus:

$$f = (-10)({}^{[0]}\lambda + 3)^2(\lambda + 3)(\lambda^2 + {}^{[0]}2\lambda + 3)^2(\lambda + {}^{[-1]}0)(\lambda^2 + \lambda + 0).$$

11 Several Variables

11.1 Unique factorization of primary polynomials

Since changing the layer of the primary point a does not change its being a primary point, we will assume from now on that the primary point is tangible.

Theorem 11.1. *Let f be a regular primary polynomial at a. Then f factors uniquely into irreducible factors.*

Proof. The key to this proof is to build a homomorphism between primary polynomials at a given point and polynomials over $\mathbb{Q}$. First we need to verify that a primary polynomial factors into primary polynomials. Let g and h be polynomials so that $f = gh$. Assume g has a monomial u which is inessential at a, and let v be any monomial of h. Then uv is inessential at a. However, u must not be inessential so us must be at least quasi-essential for some s in h. Thus us is a monomial of f which is not quasi-essential at a, which contradicts the definition of a primary polynomial.

We define $\mathbb{P}_a$ to be the set of regular polynomials f which are primary in a and for which the tangible value of $f(a)$ is 0. The latter means that the constant c in the form $f = c\sum {}^{[b_I]}0 a_I \lambda^I$ is 0. We will show that $\mathbb{P}_a$ is closed under multiplication, and also under factorization.

Assume $g, h \in \mathbb{P}_a$. Clearly, any monomial u of $f = gh$ is the sum of products $u = g_1h_1 + ... + g_kh_k$ where g_i are monomials of g, and h_i are monomials of h. Since g and h are primary in a and their tangible value is 0, the tangible value of $u(a)$ is also 0. Due to Corollary 10.7, f is regular. Thus f is primary in a and the tangible value of $f(a)$ is 0, and in other words $f \in \mathbb{P}_a$.

Assume $f \in \mathbb{P}_a$, and assume f factor into $f = gh$. Due to Corollary 10.7, g and h are regular. Assume g has a monomial u which is inessential at a. u must not be completely inessential, so u must be at least quasi-essential for some point b. Let s be a monomial which is not inessential in h at b. Thus us is a monomial of f which is not inessential at b. However, since u in inessential at a then us is inessential at a; thus us is a monomial of f which is not quasi-essential at a, and that is absurd. Therefore both g and h are primary at a. The tangible value of $f(a)$ is 0, so the product of the tangible values of $g(a)h(a)$ must be 0 as well. Assume that the tangible value of $g(a)$ is c; then $f = gh = c^{-1}cgh = c^{-1}gch$. The tangible value of $c^{-1}g(a)$ is clearly 0, and since the tangible value of f is 0 then so is the tangible value of $ch(a)$. Thus $c^{-1}g, ch \in \mathbb{P}_a$.

Define $\psi : \mathbb{P}_a \to \mathbb{Q}[x_1, ..., x_n]$, by sending $f = \sum {}^{[b_I]}0 a_I \lambda^I$ to $\psi(f) = \sum b_I x^I$.

We wish to prove that $\psi(fg) = \psi(f)\psi(g)$. For all $f, g \in \mathbb{P}_a$ we know we can write the polynomials in the form $f = \sum {}^{[b_I]}0 a_I \lambda^I$ and $g = \sum {}^{[d_I]}0 a_I \lambda^I$. We also know that the product fg is also in $\mathbb{P}_a$, and thus $fg = \sum {}^{[e_I]}0 a_I \lambda^I$. Define e_I to be the sum of products of the form $b_J d_K$ such that $J + K = I$. If $e_I = 0$ then we already proved that the monomial ${}^{[0]}0 a_I \lambda^I$ is inessential and should be deleted. $\psi(fg) = \sum e_I x^I$. Now, $\psi(f)\psi(g) = (\sum b_I x^I)(\sum d_I x^I) = \sum e_I x^I$, as desired.

Having seen that ψ is an homomorphism, we now prove it is an isomorphism. Clearly ψ is onto, and we will prove it is also injective. Assume $\psi(f) = \psi(g) = \sum b_I x^I$. Then f and g both must equal $\sum {}^{[b_I]}0 a_I \lambda^I$. Note that this is mainly due to the fact that f and g are regular.

Since ψ is a multiplicative isomorphism, and since $\mathbb{Q}[x_1, ..., x_n]$ has unique factorization, $\mathbb{P}_a$ has unique factorization as well. Now, let f be any primary polynomial at a. If c is the tangible value of $f(a)$, then $c^{-1}f \in \mathbb{P}_a$; $c^{-1}f$ factors uniquely up to multiplication by a constant and clearly so does f as desired. $\square$

Example 11.2. $\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2$ is irreducible, like $x^2 + xy + y^2$. However,

$$\lambda_1^2 + {}^{[2]}\lambda_1\lambda_2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2,$$

as $x^2 + 2xy + y^2 = (x + y)^2$.

11.2 Non-primary polynomials

Primary polynomials play an important role in our theory. Recall that the essential part at a certain point is a primary polynomial. Thus, at any corner root, the essential part factors uniquely as a polynomial. One might think this will lead to a proof of unique factorization of general polynomials. However this idea fails, as we will now show.

Let us focus on polynomials in two variables. The corner root set consists of line segments, and the points where these segments intersect. As we have seen, the monomials which are quasi-essential on the line segment are quasi-essential at the intersection.

Assume that the quasi-essential monomials at point a factor into h_1h_2, at point b the quasi-essential monomials factor into g_1g_2 and at the line segment between a, b the quasi-essential monomials factor into r_1r_2. Clearly, r_1r_2 are obtained by deleting inessential monomials from h_1h_2 and g_1g_2. Considering r_1 as a part of h_1 and g_1, one can reconstruct part of the factors of the polynomial by identifying g_1 with h_1. However, consider the case that $r_1 = r_2$. One may identify g_1 with h_2 and g_2 with h_1 or g_i with h_i, for $i = 1, 2$. As we will see in the next example, this provides a counterexample to unique factorization.

Example 11.3.

$$f = f_1f_2 = (\lambda_2 + \lambda_1 + \lambda_1^2 + (-1)\lambda_1^3)(\lambda_2 + 0 + \lambda_1^2 + (-2)\lambda_1^4)$$

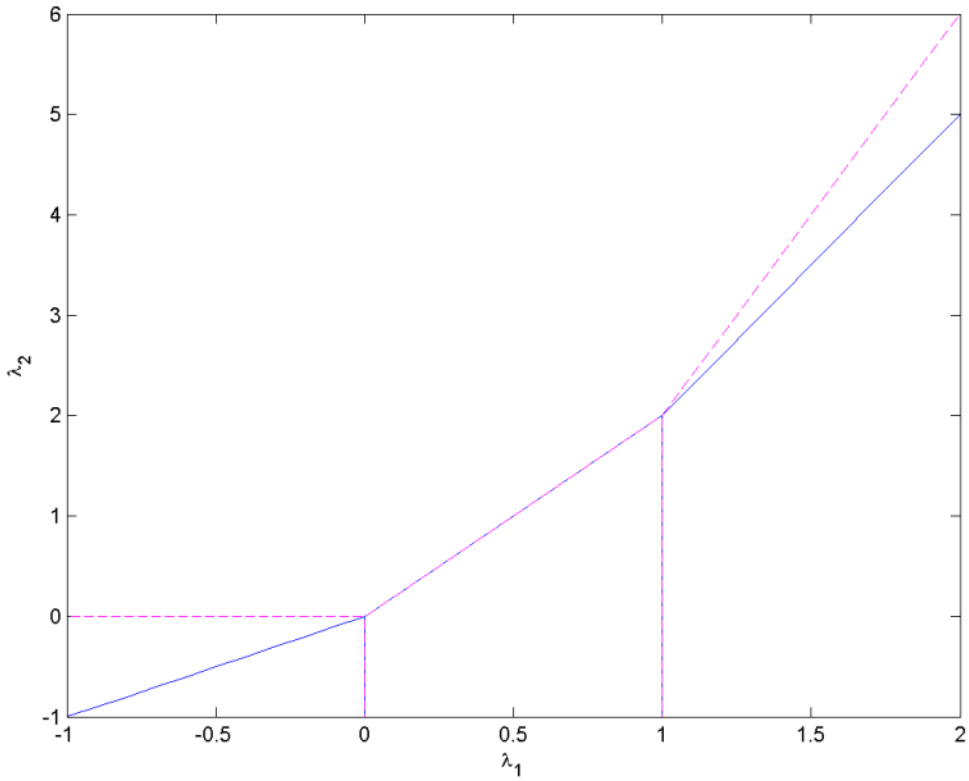
$$g = g_1g_2 = (\lambda_2 + \lambda_1 + \lambda_1^2 + (-2)\lambda_1^4)(\lambda_2 + 0 + \lambda_1^2 + (-1)\lambda_1^3).$$

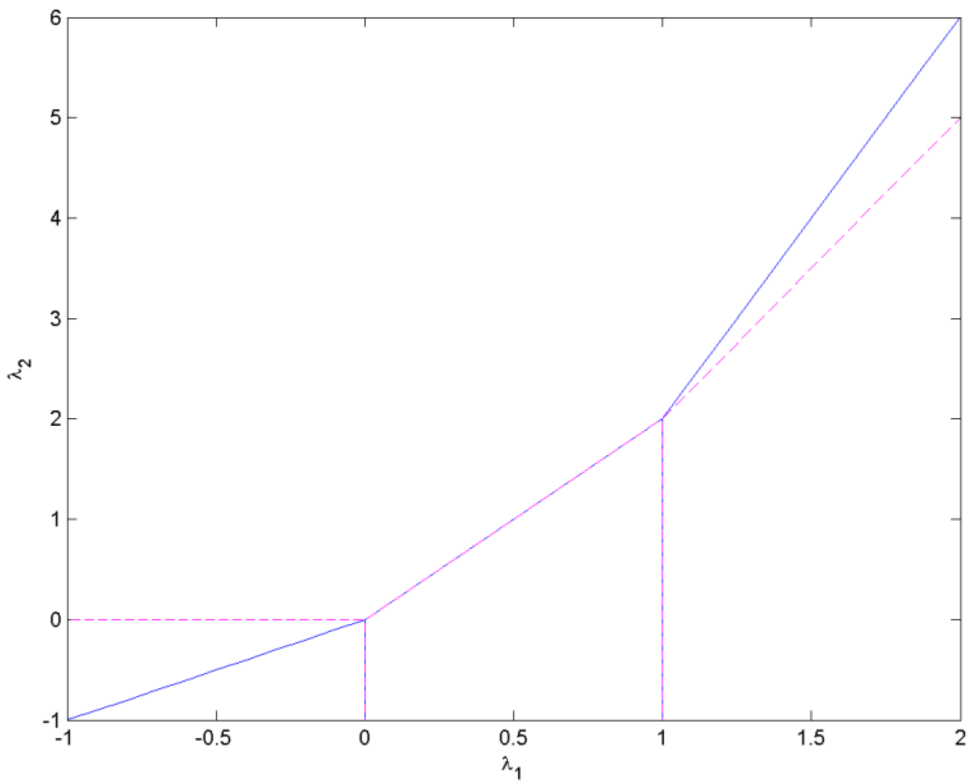
We will prove that f_1, f_2, g_1, g_2 are irreducible and that $f = g$, and thus unique factorization fails.

In the above notation, the points a, b are $a = (0, 0), b = (1, 2)$. For both g and f at a, the quasi-essential monomials are $h_1h_2 = (\lambda_2 + \lambda_1 + \lambda_1^2)(\lambda_2 + 0 + \lambda_1^2)$. At b, $g_1g_2 = (\lambda_1 + \lambda_1^2 + (-1)\lambda_1^3)(0 + \lambda_1^2 + (-2)\lambda_1^4)$. At the line between a and b, the quasi-essential monomials are $r_1r_2 = (\lambda_2 + \lambda_1^2)^2$. The reconstruction of the irreducible factors of the polynomial can yield either f or g. Next we will prove that this is a good example (i.e., $f = g$).

Recall that two regular polynomials are identical if they are root-equivalent. Therefore, it is enough to show that $f = g$ on the corner roots of f and g in order to prove that $f = g$ everywhere. The corner root set of f_1f_2 is the union of the corner root set of f_1 and

f_2. In the following figures, we can see the corner roots of f_1, g_1 (solid) and of f_2, g_2 (dashed).





As we can see in the following table in detail, both f and g have the same quasi-essential monomials at the corner roots set, as desired.

corner root set	quasi-essential monomials
$\{\lambda_1 = \lambda_2 < 0\}$	$\lambda_2 + \lambda_1$
$\{\lambda_2 = 0, \lambda_1 < 0\}$	$\lambda_2(\lambda_2 + 0)$
$\{\lambda_1 = \lambda_2 = 0\}$	$(\lambda_2 + \lambda_1 + \lambda_1^2)(\lambda_2 + 0 + \lambda_1^2)$
$\{\lambda_1 = 0, \lambda_2 < 0\}$	$(\lambda_1 + \lambda_1^2)(0 + \lambda_1)$
$\{\lambda_1^2 = \lambda_2, 0 < \lambda_1 < 1\}$	$(\lambda_2 + \lambda_1^2)(\lambda_2 + \lambda_1^2)$
$\{\lambda_1 = 1, \lambda_2 = 2\}$	$(\lambda_2 + \lambda_1^2 + (-1)\lambda_1^3)(\lambda_2 + \lambda_1^2 + (-2)\lambda_1^4)$
$\{\lambda_1 = 1, \lambda_2 < 2\}$	$(\lambda_1^2 + (-1)\lambda_1^3)(\lambda_1^2 + (-2)\lambda_1^4)$
$\{(-1)\lambda_1^3 = \lambda_2, 1 < \lambda_1\}$	$(\lambda_2 + (-1)\lambda_1^3)((-2)\lambda_1^4)$
$\{(-2)\lambda_1^4 = \lambda_2, 1 < \lambda_1\}$	$(\lambda_2 + (-2)\lambda_1^4)(\lambda_2)$

Next we will prove that f_1, f_2, g_1, g_2 are irreducible.

Lemma 11.4. *Let f be a polynomial of the form $f = \lambda_2 + g(\lambda_1)$ where g is a polynomial in one variable, then f is irreducible.*

Proof. Assume that $f = f_1 f_2$. Since λ_2 is a monomial of f, without loss of generality, f_1 must have $c\lambda_2$ as a monomial, and f_2 must have c^{-1} as a monomial for some constant c. Clearly f_1 and f_2 cannot have any monomials of the type $x\lambda_1^n\lambda_2^m$ where $m > 1$, since f does not have such monomials. Thus $f_1 = h_1(\lambda_1)\lambda_2 + r_1(\lambda_1)$, $f_2 = h_2(\lambda_1)\lambda_2 + r_2(\lambda_1)$.

Fix $\lambda_1 = a$. For any $\lambda_2 > r_i(a)h_i^{-1}(a)$, there is a monomial of the form $c\lambda_1^n\lambda_2$ of f_i which is not inessential. Take $\lambda_2 > r_1(a)h_1^{-1}(a) + r_2(a)h_2^{-1}(a)$ to obtain a monomial of the form $c\lambda_1^n\lambda_2^2$ of f which is not inessential, in contradiction to the form of f.

Assume h has a monomial of the form $c\lambda_1^n$ with $n > 0$, and assume that this monomial is essential or quasi-essential at a point b. Take $\lambda_2 > g(b)$, to obtain an essential or quasi-essential monomial of f of the form $c\lambda_1^n\lambda_2$, which is absurd. Thus $f_2 = c$ where c is a constant.

Therefore $f = cf_1$ is the only factorization of f and thus f is irreducible, as desired. $\square$

To conclude, f_1, f_2, g_1, g_2 are irreducible due to the lemma above and so f_1f_2 and g_1g_2 are two different factorizations of the same polynomial $f = g$.

Part III
The tropical determinant and linear dependence

12 Introduction

The determinant of a matrix is the signed sum of products of the form $a_{1\sigma(1)}\cdots a_{n\sigma(n)}$ for $\sigma \in Sym(n)$, which we call tracks. Since in tropical algebra addition is actually the maximum, the sign of a permutation and matrix singularity are not naturally defined. We will present an appropriate definition for the determinant and show that it is null (i.e., of layer zero) if and only if its rows are tropically linearly dependent.

We are interested in tracks having the property that its elements dominate all entries of their columns. We will use combinatorial methods to produce such tracks, and prove the main theorem. In part, we use a version of the well known Hungarian algorithm (ref. [13]).

I would like to thank Prof. Uzi Vishne for his help in formalizing this part.

13 Critical Matrixes

Definition 13.1. An entry a_{ij} of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *column-critical* if it is maximal within its column, i.e., if $\forall k : a_{ij} \geq a_{kj}$.
A matrix A is called *critical* if there exists a permutation σ such that $a_{1\sigma(1)}, ..., a_{n\sigma(n)}$ are column-critical.

In this section we present a version of the Hungarian algorithm which solves the assignment problem (having n tasks and n workers and each task pays differently to the various workers, how do you assign the tasks to the workers so that all tasks are done with minimal cost?, ref. [13]). We use classical algebraic operations over the real numbers, and use the result in the tropical section.

Theorem 13.2. *For any matrix A there exists a matrix*

$$B(a_1, ..., a_n) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix}$$

such that $A + B(a_1, ..., a_n)$ is critical.

Proof. It is enough to achieve criticality of the main diagonal after a permutation of the rows and columns.

We will use induction on the size of the matrix, considering that the case $n = 1$ is obvious. Therefore we assume that given an $(n+1) \times (n+1)$ matrix A, for any value of a_{n+1} there exists a matrix $B(a_1, ..., a_n, a_{n+1})$ such that the minor of $D = A + B(a_1, ..., a_n, a_{n+1})$ obtained by striking the last row and column, is critical. Moreover, by permuting the rows and columns and choosing a_{n+1} appropriately, we may assume that $c_{11}, ..., c_{nn}$ are column-critical and there is some column-critical element in the last row. We assume that the last element on the diagonal is not critical, for otherwise we are done.

Given a column-critical element in the last row at column i, one can switch the i_{th} row with the $n+1$ row while keeping a column-critical element at the i_{th} entry of the diagonal. We will search for such a permutation of the rows that will place a column-critical element at the last entry of the diagonal. If such a permutation does not exist, we will expand the number of rows we can use by increasing the constants of B.

It is fairly obvious that each column has a column-critical entry. Define

$$C_1 = \{i : c_{i,n+1} \text{ is column-critical}\},$$

and define C_k to be the set of numbers i such that there is a column-critical entry c_{im} with m in C_{k-1}. We claim that $C_{k-1} \subseteq C_k$. Indeed the c_{ii} are column-critical for all $i < k$. Thus $C_{r-1} = C_r$ for some r. We define the *critical set* of D to be $CS(D) := C_r \subseteq \{1, 2, ..., n+1\}$

If $n+1 \in CS(D)$, then by the construction of the critical set we can permute the rows to obtain only column-critical entries in the diagonal. Indeed, $n+1$ was added due to some column-critical entry $c_{n+1,m}$ such that $m \in CS(D)$. Thus we can switch rows $n+1$ and m keeping the m_{th} entry of the diagonal column-critical. There is an entry $c_{m,p}$ such that $p \in CS(D)$. Since we started with rows which have a column-critical entry in the $n+1_{th}$

column, this process will end with all entries of the diagonal being column-critical.

Assume $n+1 \notin CS(D)$ for every possible D. Pick D such that $CS(D)$ has maximal size. We will reorder the rows and columns of D so that the elements of $CS(D)$ will be $1, 2, ..., k$. This is permitted since we permute rows and columns of numbers in $CS(D)$ with previous numbers which are not in $CS(D)$, updating $CS(D)$ accordingly. Now we have a matrix in which none of the entries in the left-lower $n+1-k$ by k block are column-critical. If they were, they would be contained in $CS(D)$ since there is a column-critical entry above them in the diagonal.

Define d to be the minimal difference between any entry of the block and the column-critical entry in its column;

$$d = min\{a_{ii} - a_{mi} | m > i, 1 \leq i \leq k\}$$

Adding d to the last $n-k$ constants $B(a_1, ..., a_k, a_{k+1}+d, ..., a_{n+1}+d)$ will add a column-critical entry and enlarge $CS(D)$, contradiction.

Indeed, $A + B(a_1, ..., a_k, a_{k+1}+d, ..., a_{n+1}+d)$ still has the first n entries of the diagonal column-critical: since d is minimal and positive, it will not remove column-critical entries in any of the first k columns. Thus, the column-critical entries of the diagonal remain critical.

□

14 Exploded-Layered Tropical Determinant

In this section we introduce some ELT linear algebra definitions, where R is an ELT algebra over the reals.

Definition 14.1. For any two elements ${}^{[\ell_1]}\lambda_1, {}^{[\ell_2]}\lambda_2$ we define the *tangible distance* to be $|\lambda_1 -_{\mathbb{R}} \lambda_2|$.

Definition 14.2. The vectors $v_1, ..., v_m \in R^n$ are called *linearly dependent* if there exist

$$a_1, ..., a_m \in R^*$$

such that

$$s\Big(\sum_{i=1}^{m} a_i v_i\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

Definition 14.3. Consider a matrix $A = (a_{ij}) \in R^{n \times n}$. The *ELT determinant* of A is

$$|A| = \sum_{\sigma \in S^n} {}^{[sign(\sigma)]}0_{\mathbb{R}} \cdot a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

We are now ready to present a key definition.

Definition 14.4. Consider a matrix $A = (a_{ij}) \in R^{n\times n}$. We define the *critical layers matrix*, $S_A \in \mathbb{F}^{n\times n}$, of A in the following way:

$$(S_A)_{ij} := \begin{cases} s(a_{ij}) & a_{ij} \text{ is column-critical} \\ 0_{\mathbb{F}} & \text{otherwise} \end{cases}$$

The critical layers matrix enables us to use classical linear algebra, when possible.

We write $R^* = \{x \in R | s(x) \neq 0\}$.

14.1 Statement of the main theorem

Theorem 14.5. *Consider $A \in R^{n\times n}$. Then the rows of A are linearly dependent, iff the columns of A are linearly dependent, iff $s\Big(|A|\Big) = 0_{\mathbb{F}}$.*

14.2 Proof of the main theorem

We prove several lemmas which together prove the theorem.

Lemma 14.6. *If $A \in R^{n\times n}$ is a critical matrix such that $|S_A| = 0$, then the rows of A are dependent.*

Proof. Since $|S_A| = 0$, after some permutation of the rows we know that there exist nonzero scalars such that

$$\ell_1 Row_1(S_A) + ... + \ell_k Row_k(S_A) = 0.$$

If $k = n$ we are done since

$$s\Big({}^{[\ell_1]}0 \cdot Row_1(A) + ... + {}^{[\ell_n]}0 \cdot Row_n(A)\Big) = (0, ..., 0).$$

Assume $k < n$. Let C be the set of all columns with no column-critical element at any of the rows $1, ..., k$. If $C = \phi$ then

$$s\Big({}^{[\ell_1]}0 \cdot Row_1(A) + ... + {}^{[\ell_k]}0 \cdot Row_k(A)\Big) = (0, ..., 0).$$

Tropically dividing (i.e. classically subtracting) rows $k + 1, ..., n$ by some ${}^{[1]}y$, it is easy to see for large enough y that

$$s\Big({}^{[\ell_1]}0\cdot Row_1(A)+...+{}^{[\ell_k]}0\cdot Row_k(A)+{}^{[1]}(-y)\cdot Row_{k+1}(A)+...+{}^{[1]}(-y)\cdot Row_n(A)\Big) = (0, ..., 0).$$

Indeed, the last rows are too small to contribute. Therefore the rows of A are linearly dependent.

Assume that C is not empty. Take the least tangible distance between column-critical elements in the columns of C and elements in the same columns in rows $1, ..., k$, and call this tangible number $d \in \mathbb{R}$. By choice of C we know that $d > 0$. We tropically divide rows $k+1, ..., n$ by ${}^{[1]}d$, and call the new matrix A_1.

By our choice of d, in every column of C there exists a column-critical element in at least one of the rows $k+1, ..., n$. Moreover there is no column critical element in these rows in the columns which are not in C. Therefore in S_B we have $n-k$ rows with $|C|$ columns, where $n-k \geq |C|$ since A is critical. We can choose a linear combination that will annihilate the C columns in the sum of the first k rows of A_1. If that sum is already null it is enough to use the above technique in order to lower rows $k+1, ..., n$.

If we have scalars with value $0_{\mathbb{F}}$ in the above linear combination, we continue in the same way until we obtain A_p with linearly dependent rows. Since there must be some scalar different than $0_{\mathbb{F}}$, this process terminate. The change of A does not change the linear dependence of its rows, and therefore the rows of A are linearly dependent.

□

Lemma 14.7. *If A is a critical matrix such that $s\Big(|A|\Big) = 0_{\mathbb{F}}$, then the rows of A are linearly dependent.*

Proof. Since A is critical, the ELT determinant of A is the sum of tracks which contain only column-critical elements. Indeed such a track exists and dominates any other track.

Also the sum of the rows of A is the sum of the column-critical elements in each column. Therefore $|S_A| = s\Big(|A|\Big) = 0$, and we are done by the previous lemma.

□

Lemma 14.8. *If A is a matrix such that $s\Big(|A|\Big) = 0$, then the rows of A are dependent.*

Proof. By theorem 13.2, there exists constants $a_1, ..., a_n$ such that $\forall i : s(a_i) = 1_{\mathbb{F}}$ and the matrix B obtained by multiplying each row i of A by a_i is critical.

It it easy to see that $s\Big(|B|\Big) = 0$ and so its rows are linearly dependent. Therefore the rows of A are also linearly dependent.

□

Lemma 14.9. *Let $B = \{v_1, ..., v_n\}$ be a linearly dependent set of vectors. If every strict subset of B is linearly independent, then the matrix A with rows B is critical.*

Proof. Consider the matrix A with rows $v_1, ..., v_n$, and assume A is not critical. Also assume that

$$s\Big(v_1 + ... + v_n\Big) = 0.$$

By permuting the rows and columns we may assume that A satisfies the following conditions:

- The first k elements of the diagonal are column-critical.
- The last $n-k$ element of the diagonal are not column-critical.
- There is no such permutation of the matrix for larger k.

We assume $k<n$, for otherwise A is critical. Next we build a set S in the following way:

1. Adjoin to S the indices of rows with column-critical elements at columns $k+1,...,n$
2. Adjoin to S any index of a row with a column-critical element at a column with an index already in S
3. Repeat step 2 until no new index is added.

We notice that $S \subseteq \{1,...,k\}$, for otherwise there exists a permutation that increases k.

For any set I of indices, we denote the set of rows with indices in I by R_I. We denote by C_I the set of columns with indices in I.

Any element in a row from $k+1,...,n$ and in a column from $k+1,...,n$ could not be column-critical since k is maximal. Also by our choice any row from $k+1,...,n$ cannot have a column-critical element in a column with an index in S.

Now we denote the number of indices in S by $r=|S|$. If $r=k$ then the last rows do not have any column-critical elements and so rows $1,...,k$ are linearly dependent. Therefore we may assume that $r<k$.

We claim that

$$\Big[s\big(\sum_{j\in S} v_j\big)\Big]_i = 0$$

for all $i \in S \cup \{k+1,...,n\}$. Indeed there are no column-critical elements in these columns in any of the rows in $R_{\overline{S}}$, where $\overline{S}=\{1,2,...,k\}-S$.

We will show that there exists scalars $a_i \in R$ such that

$$s\Big(\sum_{j\in S} v_j + \sum_{i\in\overline{S}} a_i v_i\Big) = (0,0,...,0).$$

Then $R_{\{1,2,...,k\}}$ is a linearly dependent proper subset of $\{v_1,...,v_n\}$, contradiction.

Let d be the least tangible distance between column-critical elements of columns in $C_{\overline{S}}$ with the next maximal element in the same column in rows in R_S. We then tropically divide the rows in $R_{\overline{S}}$ by ${}^{[1]}d$. We now have at least one column in $C_{\overline{S}}$ with a column-critical element in a row from R_S and in a row from $R_{\overline{S}}$. Denote the matrix after this change by D.

In S_D there must exist scalars such that

$$a\sum_{i\in S} R_i(D) + \sum_{j\in\overline{S}} a_j R_j(D) = 0.$$

Assume that $a_j = 0$ for all $j \in \overline{S}$. Then for each column p with a column-critical element in $\sum_{i\in S} v_i$ it is true that

$$s\Big(\sum_{i\in S} v_i\Big)_p = 0.$$

In that case we can add the indices of the columns with column-critical elements in $\sum_{i\in S} v_i$ to S, and continue the process.

Otherwise, for all $j \in \overline{S}$ such that $a_j \neq 0$ add j to S and continue the process. Indeed the element in the j column of the j row is column-critical.

Since S strictly increases, we obtain our result after a finite number of steps.

□

Lemma 14.10. *Assume that A is a critical matrix such that $s\Big(\sum R_i(A)\Big) = 0$. Then $s\Big(|A|\Big) = 0$.*

Proof. Since A is critical, $|A|$ is the sum of tracks containing only column-critical elements, and therefore $s\Big(|A|\Big) = |S_A|$. Moreover,

$$(0_{\mathbb{F}}, ..., 0_{\mathbb{F}}) = s\Big(\sum R_i(A)\Big) = \sum R_i(S_A),$$

and therefore $|S_A| = 0_{\mathbb{F}}$. □

Lemma 14.11. *If $A \in R^{n\times n}$ is a matrix with linearly dependent rows, then $s\Big(|A|\Big) = 0_{\mathbb{F}}$.*

Proof. If A is critical, then we are done by the previous lemma. Otherwise, we may assume by lemma 14.9 and a permutation of the rows that the first $n-1$ rows of A are linearly dependent.

By induction on the size of the matrix, we obtain

$$s\Big(|A|\Big) = s\Big(\sum_{i=1}^{n} {}^{[(-1)^{n+i}]}0 \cdot a_{ni}|A_{ni}|\Big) = \sum_{j\in M_A} (-1)^{n+j} s\Big(a_{nj}\Big)\cdot s\Big(|A_{nj}|\Big) = 0,$$

where M_A is the set of cofactors $a_{nj}|A_{nj}|$ with maximal tangible value.

□

Together we have proved for any matrix $A \in R^{n\times n}$ that the rows of A are linearly dependent if and only if $s\Big(|A|\Big) = 0_{\mathbb{F}}$. Since $|A| = |A^t|$, the analogous statement about the columns follows.

14.3 Adjoining the element $-\infty$

In $\overline{R}$ we define linear dependence a bit differently and prove an analogous theorem.

Definition 14.12. The vectors $v_1, ..., v_m \in \overline{R}^n$ are called *linearly dependent* if there exist

$$a_1, ..., a_m \in \overline{R}^*$$

such that

$$s\Big(\sum_{i=1}^{m} a_i v_i\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}),$$

with some $a_i \neq -\infty$.

Example 14.13. The rows of the matrix

$$\begin{pmatrix} {}^{[1]}1 & -\infty \\ -\infty & {}^{[0]}1 \end{pmatrix}$$

are dependent. Indeed,

$$(-\infty) \cdot ({}^{[1]}1, -\infty) + (-\infty, {}^{[0]}1) = (-\infty, {}^{[0]}1),$$

and $s\Big(-\infty, {}^{[0]}1\Big) = (0, 0)$.

Lemma 14.14. *If $A \in \overline{R}^{n\times n}$ is a critical matrix such that $|S_A| = 0_{\mathbb{F}}$, then the rows of A are dependent.*

Proof. We repeat the proof of 14.6 with a minor change. In the case the minimal distance $d \in \mathbb{R}$ is $-\infty$ we may choose the next minimal distance. If no such minimal distance exists, then we already have a linear combination with layer zero.

□

Lemma 14.15. *Let $A \in \overline{R}^{n\times n}$ be a matrix. Either $|A| = -\infty$ or there exist scalars $a_i \in R^*$ such that the matrix with rows $a_i R_i(A)$ is critical.*
Moreover, if $|A| = -\infty$ then there exist a number $r < n$ and $n - r$ rows with $r + 1$ columns having only $-\infty$ elements.

Proof. Assume $|A| \neq -\infty$. Assume that B is a matrix obtained by permutation and multiplication by scalars $a_i \in R^*$ of the rows of A. Also assume that the b_{ii} are critical and different from $-\infty$ for $1 \leq i \leq k$, such that k is maximal with respect to a choice of such a matrix B.

If $k = n$ then B is critical and we are done. Therefore we assume that $k < n$.

Next we build the set S as in the previous proof:

1. Adjoin to S the indices of rows with column-critical elements (different than $-\infty$) at columns $k+1, ..., n$.
2. Adjoin to S any index of a row with a column-critical element (different than $-\infty$) at a column with an index already in S.
3. Repeat step 2 until no new index is added.

We notice that $S \subseteq \{1, ..., k\}$, for otherwise there exist a permutation that enlarge k.

We assume that B is such that S is maximal possible.

If $i \notin S$ and $j \in S$ or $(j > k)$ then $b_{ij} = -\infty$. Otherwise we can multiply the rows with index which is not in S by a scalar, and enlarge S.

Write $|S| = r$. We obtained $n-r$ rows with $r+n-k \geq r+1$ columns of $-\infty$. Therefore any track of the determinant must contain a choice of $n-r$ columns out of the $n-r$ rows, one of which must be $-\infty$. Therefore $|B| = -\infty$ in contradiction.

□

Lemma 14.16. *Let $A \in \overline{R}^{n\times n}$ be a matrix. If $|A| = -\infty$ then the rows of A are linearly dependent.*

Proof. Due to lemma 14.15 there are $n-r$ rows for which there is a submatrix having $r+1$ columns containing only $-\infty$ elements. Therefore we wish to prove that $m = n-r$ rows with $m-1 = n-r-1$ columns are necessarily linearly dependent. It is trivial that two rows with one column different than $-\infty$ are linearly dependent. we will use an induction on the number of rows.

Take the first $m-1$ rows with the $m-1$ columns that do not necessarily contain only $-\infty$ element. Multiply this submatrix with scalars to obtain a critical $(m-1)\times(m-1)$ matrix. If this is not possible then we have $m-1-r'$ rows with $m-1-r'+1$ $-\infty$ columns and we are done by induction.

We only need to prove that if A is a critical matrix then for any vector v, the rows of A together with v are linearly dependent. Add v as the last row to A and add last column with $-\infty$ elements, and call the obtained matrix B. B remain critical and $|S_B| = 0$ therefore its rows are linearly dependent.

□

Lemma 14.17. *If $A \in \overline{R}^{n\times n}$ is a matrix such that $|A| \neq -\infty$ and $s\Big(|A|\Big) = 0_{\mathbb{F}}$, then the rows of A are linearly dependent.*

Proof. Consider B a critical matrix with rows $a_i R_i(A)$, with $a_i \in R^*$. Is is true that $s\Big(|B|\Big) = 0_{\mathbb{F}}$.

Since B is critical, $s\Big(|B|\Big) = |S_B| = 0_{\mathbb{F}}$ and therefore the rows of B are linearly dependent. Thus the rows of A are linearly dependent.

□

Lemma 14.18. *Let $v_1, ..., v_n \in \overline{R}^n$ be linearly dependent. Then $s\Big(|A|\Big) = 0_{\mathbb{F}}$ for the matrix A with rows $v_1, ..., v_n$.*

Proof. If $|A| = -\infty$ the result is trivial. Also, if any strict subset of $\{v_1, ..., v_n\}$ is linearly dependent we are done by induction (similarly to the case with out $-\infty$).

Therefore there exist scalars $a_i \in R^*$ such that $s\Big(w_1 + ... + w_n\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}})$ with $w_i := a_i v_i$. Consider B to be the matrix with rows w_i.

Since $|B| \neq -\infty$, there is at least one maximal track that does not contain $-\infty$. Replacing any $-\infty$ element in B with a small enough scalar from R we obtain a matrix $B' \in R^{n\times n}$ such that $|B| = |B'|$ and $s\Big(\sum R_i(B')\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}})$.

Therefore $0_{\mathbb{F}} = s\Big(|B'|\Big) = s\Big(|B|\Big)$, and $s\Big(|A|\Big) = 0_{\mathbb{F}}$.

□

In conclusion:

Theorem 14.19. *If $A \in \overline{R}^{n\times n}$ is a matrix, then the rows of A are linearly dependent $\iff s\Big(|A|\Big) = 0_{\mathbb{F}} \iff$ the columns of A are linearly dependent.*

14.4 Calculation of the ELT determinant

Next we review the process of calculating a determinant, and its complexity.

1. Apply the algorithm above to the matrix A in order to obtain a critical matrix B, with complexity $o(n^4)$. (Best implementation is $o(n^3)$ [23].)
2. If $|A| = -\infty$ we are done.
3. The tangible value of $|A|$ is $t\Big(c_1^{-1} \cdots c_n^{-1} \cdot |B|\Big)$, where c_i's are scalars multiplying the rows of A in order to become rows of B.
4. Calculate the determinant of S_B with complexity $o(n^3)$. This is the layer of $|A|$.

14.5 Rank of a matrix

In this section we generalize Theorem 14.19 and prove that the row rank of an ELT matrix is equal to the rank of its columns.

Definition 14.20. Let $A \in (\overline{R}^*)^{m\times n}$ be an ELT matrix. The maximal number of linearly independent rows from A, is called the *row rank* of A.

Similarly, the *column rank* of A is the maximal number of linearly independent columns of A.

Definition 14.21. Let $A \in (\overline{R}^*)^{m\times n}$ be an ELT matrix. The *submatrix rank* of A is the maximal size of a square nonsingular submatrix of A. If no such matrix exists, then the rank of A is defined to be zero.

Theorem 14.22. *Let $A \in (\overline{R}^*)^{m\times n}$ be an ELT matrix. Then the row rank of A is equal to the column rank of A and to the submatrix rank of A.*

14.5.1 Proof of the rank theorem

Definition 14.23. We define the unit vectors $e_i \in \overline{R}^n$ for all $1 \le i \le n$ by

$$[e_i]_j = \begin{cases} -\infty & i \neq j \\ {}^{[1]}0_{\mathbb{R}} & i = j \end{cases}$$

Lemma 14.24. *Let $A \in (\overline{R}^*)^{m\times n}$ be an ELT matrix, with $n > m$. If all $m \times m$ submatrices of A are singular, then the rows of A are linearly dependent.*

Proof. Assume the rows of A, $v_i = R_i(A)$, are linearly independent. Add unit vectors to the set $\{v_1, ..., v_m\}$ such that it remains independent. If we obtain a set with n row vectors, the matrix with these rows must be non-singular, in contradiction to the assumption that all submatrices of A are singular.

Therefore we may assume that $A \in (\overline{R}^*)^{m\times n}$ is a matrix such that $n > m$, and for all $1 \le i \le n$ the set $\{v_1, ..., v_m, e_i\}$ is linearly dependent. Since the rows of A are independent, for all $1 \le i \le n$ there exist scalars $\alpha_{ij} \in \overline{R}^*$ such that

$$s\Big(\alpha_{i1}v_1 + ... + \alpha_{im}v_m + e_i\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}})$$

For all $1 \le i \le n$, define

$$u_i = \alpha_{i1}v_1 + ... + \alpha_{im}v_m.$$

Let $x_i = s\Big([u_i]_i\Big)$, then

$$s\Big([u_i]_j\Big) = \begin{cases} 0_{\mathbb{F}} & i \neq j \\ x_i & i = j \end{cases}.$$

If $x_i = 0_{\mathbb{F}}$ for some i then $s\Big([u_i]_j\Big) = 0_{\mathbb{F}}$ for every $1 \le j \le n$, implying that $v_1, ..., v_m$ are linearly dependent. Therefore we may assume that $x_i \neq 0_{\mathbb{F}}$, for all $1 \le i \le n$.

Let U be the matrix with rows $u_1, ..., u_n$. By theorem 13.2 we may assume that U is critical.

If U is non-singular, then there must be a dominant track all of whose entries have nonzero layers. Therefore, we may assume that the diagonal of U is a dominant track.

Now, if

$$t\Big(\big[\alpha_{ij}v_j\big]_i\Big) = t\Big([u_i]_i\Big)$$

and

$$t\Big(\big[\alpha_{kj}v_j\big]_k\Big) = t\Big([u_k]_k\Big)$$

then

$$t(\alpha_{ij}) = t(\alpha_{kj}).$$

Indeed, otherwise

$$[U]_{kk} < [U]_{ik},$$

which is absurd.

Put more simply, any vector v_j may contribute to the dominant entries of U at only one tangible value of its coefficient. Therefore, any vector v_j may contribute no more than one vector of layers, as following.

Next for all $1 \leq j \leq m$ we write

$$w_j = \begin{cases} s\Big([v_j]_i\Big) & \exists i : t\Big(\big[\alpha_{ij}v_j\big]_i\Big) = t\Big([u_i]_i\Big) \\ 0_{\mathbb{F}} & otherwise \end{cases},$$

with $w_j \in \mathbb{F}^n$.

Together,

$$Rowspan\Big(S_U\Big) \subseteq span\{w_1, ..., w_m\}.$$

Since $Rowspan\Big(S_U\Big) = \mathbb{F}^n$ and $n > m$, we have a contradiction.

Therefore U must be singular. Therefore there exists scalars $\beta_i \in \overline{R}^*$, not all $-\infty$ such that

$$s\Big(\beta_1 u_1 + ... + \beta_n u_n\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

For all $1 \leq i \leq n$,

$$[\beta_i u_i]_i < \big[\beta_1 u_1 + ... + \beta_n u_n\big]_i.$$

Therefore for all $1 \leq i \leq n$ one may choose [1]d_i from a range

$$0 < d_i < dist(\big[\beta_1 u_1 + ... + \beta_n u_n\big]_i, [\beta_i u_i]_i),$$

such that

$$s\Big(d_1\beta_1u_1 + ... + d_n\beta_nu_n\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

Furthermore, write

$$d_1\beta_1u_1 + ... + d_n\beta_nu_n = a_1v_1 + ... + a_mv_m.$$

Since there are infinitely many choices for each d_i, one may choose them so that $a_j \in \overline{R}^*$ for all $1 \leq j \leq m$.

Therefore $v_1, ..., v_m$ are linearly dependent.

□

Now we can conclude the proof of theorem 14.22. If a matrix A has row rank k it has k independent rows and by the lemma it must have a $k \times k$ nonsingular submatrix. Since any $k+1$ rows are linearly dependent, it is clear that any $(k+1) \times (k+1)$ submatrix must be singular. Therefore the submatrix rank of A is k, and is equal to the row rank of A.
Since the submatrix rank of A and A^t are equal, the column rank must be equal to the rank as well.

15 The Characteristic Polynomial and Eigenvalues

Definition 15.1. We define the *identity* matrix $I_n \in \overline{R}^{n\times n}$,

$$[I_n]_{ij} = \begin{cases} {}^{[1]}0 & i = j \\ -\infty & i \neq j \end{cases}.$$

It is easy to prove that $I_n \cdot A = A$ for all $A \in \overline{R}^{n\times k}$.

Definition 15.2. Let $A \in \overline{R}^{n\times n}$ be a matrix. The *characteristic polynomial* of A is defined to be

$$f_A(\lambda) := |A + {}^{[-1]}0 \cdot \lambda I_n|.$$

Definition 15.3. Let $A \in \overline{R}^{n\times n}$ be a matrix. A vector $v \in (\overline{R}^*)^n$ is called an *eigenvector* of A with an *eigenvalue* $x \in R^n$ if $v \neq (-\infty, ..., -\infty)$ and

$$Av = xv.$$

Theorem 15.4. *Let $A \in \overline{R}^{n\times n}$ be a matrix with eigenvalue x. Then $s\Big(f_A(x)\Big) = 0_{\mathbb{F}}$.*

Proof. Choose v to be an eigenvector of the eigenvalue v, then $Av = xv$. Therefore

$$Av + {}^{[-1]}0 \cdot xv = {}^{[0]}0 \cdot xv.$$

In other words,

$$s\Big(Av + {}^{[-1]}0 \cdot xv\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

Thus

$$s\Big((A + {}^{[-1]}0 \cdot xI_n)v\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

By theorem 14.19 we conclude that $s\Big(|A + {}^{[-1]}0 \cdot xI_n|\Big) = 0_{\mathbb{F}}$, i.e., $s\Big(f_A(x)\Big) = 0_{\mathbb{F}}$.

□

Note that the other direction is not necessarily true, i.e., there could be an ELT root of the characteristic polynomial which is not an eigenvalue. Indeed, if one might try to prove that direction he will encounter the following problem:

$$Av \neq Av + {}^{[-1]}0 \cdot xv + xv.$$

Example 15.5. Consider the matrix $A \in R^{2\times 2}$,

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},$$

(all layers equal 1).

The characteristic polynomial is

$$f_A(\lambda) = |A + {}^{[-1]}0 \cdot \lambda I_2| = \begin{vmatrix} 1 + {}^{[-1]}0 \cdot \lambda & 2 \\ 2 & 3 + {}^{[-1]}0 \cdot \lambda \end{vmatrix} =$$

$$= (1 + {}^{[-1]}0 \cdot \lambda)(3 + {}^{[-1]}0 \cdot \lambda) + {}^{[-1]}4 = \lambda^2 + {}^{[-1]}3 \cdot \lambda + {}^{[0]}4.$$

If $\lambda = {}^{[0]}1$, $\lambda = {}^{[\ell]}\alpha$ with $\alpha < 1$ or $\lambda = {}^{[1]}3$ then $s\Big(f_A(\lambda)\Big) = 0$.

The only eigenvalue of A is $\lambda = {}^{[1]}3$,

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = {}^{[1]}3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One may also define an ELT eigenvalue and eigenvector in the following way.

Definition 15.6. Let $A \in \overline{R}^{n\times n}$ be a matrix. A vector $v \in (\overline{R}^*)^n$ is called an *eigenvector* of A with an *eigenvalue* $x \in R^n$ if $v \neq (-\infty, ..., -\infty)$ and

$$s\Big(Av + {}^{[-1]}0xv\Big) = (0_{\mathbb{F}}, ..., 0_{\mathbb{F}}).$$

This definition is similar to the concept of 'ghost surpass' by Izhakian, Knebusch and Rowen ([12]).

Corollary 15.7. *Let $A \in \overline{R}^{n\times n}$ be a matrix. Then x is an ELT eigenvalue of A if and only if $s\Big(f_A(x)\Big) = 0_{\mathbb{F}}$.*

16 Inner Products And Orthogonality

In this section we introduce a definition for an inner product and orthogonality. Although we prove that an orthogonal set of vectors is linearly independent, if we add an orthogonal vector to a linearly independent set, we may obtain a linearly dependent set.

16.1 Inner product

Definition 16.1. Let R be an ELT algebra over the reals with $L = \mathbb{C}$. An *ELT inner product* is a function

$$<,>: \overline{R}^n \times \overline{R}^n \to \overline{R},$$

that satisfy the following three axioms for all vectors $v, u, w \in \overline{R}^n$ and all scalars $a, b \in R$.

1. $< av + bu, w >= a < v, w > +b < u, w >$.
2. $t\Big(< v, u > \Big) = t\Big(< u, v > \Big)$, and $s\Big(< v, u > \Big) = \overline{s\Big(< u, v > \Big)}$.
3. $s\Big(< v, v > \Big) \geq 0$ and if $v \in \overline{R^*}^n$ then $s\Big(< v, v > \Big) = 0 \iff v = (-\infty, ..., -\infty)$.

For short notation we will write uv instead of $< u, v >$ for the remainder of this section.

Example 16.2. For any two vectors $v_1, v_2 \in \overline{R}^n$

$$v_1 = ({}^{[z_1]}\lambda_1, ..., {}^{[z_n]}\lambda_n)$$

$$v_2 = ({}^{[w_1]}\alpha_1, ..., {}^{[w_n]}\alpha_n),$$

we define the *standard* inner product

$$v_1 v_2 := {}^{[z_1]}\lambda_1 {}^{[\overline{w_1}]}\alpha_1 + ... + {}^{[z_n]}\lambda_n {}^{[\overline{w_n}]}\alpha_n.$$

The first two axioms are trivial to prove, we will prove the third.

If $v = (-\infty, ..., -\infty)$ then $v^2 = -\infty$ and thus $s\Big(v^2\Big) = 0_{\mathbb{C}}$.

Otherwise write

$$v = ({}^{[z_1]}\lambda_1, ..., {}^{[z_n]}\lambda_n),$$

$$S = \{i | \lambda_i = \max_{1 \leq j \leq n} \lambda_j\}.$$

Then

$$v^2 = \sum_{i \in S} {}^{[|z_i|^2]}2\lambda_i,$$

and

$$s\Big(v^2\Big) = \sum_{i \in S} |z_i|^2 \geq 0_{\mathbb{R}}.$$

since $v \in \overline{R^*}^n$ and $v \neq (-\infty, ..., -\infty)$ then $z_i \neq 0_{\mathbb{C}}$ for all $i \in S$, thus $s\Big(v^2\Big) \neq 0_{\mathbb{C}}$.

Next we prove an ELT Cauchy-Schwartz lemma.

Lemma 16.3. *For every two vectors $v_1, v_2 \in \overline{R^*}^n$,*

$$(v_1 v_2)^2 \leq v_1^2 \cdot v_2^2.$$

In other words, either

$$v_1^2 \geq v_1 v_2,$$

or

$$v_2^2 \geq v_1 v_2.$$

Proof. If

$$v_1 = (-\infty, ..., -\infty)$$

then

$$v_2^2 \geq v_1 v_2 = -\infty,$$

therefore we assume that

$$v_1 \neq (-\infty, ..., -\infty).$$

For any scalar ${}^{[z]}0 \in R$,

$$(v_1 + {}^{[z]}0 v_2)^2 = v_1^2 + {}^{[z]}0 v_2 v_1 + v_1({}^{[z]}0 v_2) + ({}^{[z]}0 v_2)^2.$$

Now

$$t\Big({}^{[z]}0 v_2 v_1 + v_1({}^{[z]}0 v_2)\Big) = t(v_1 v_2),$$

and

$$s\Big({}^{[z]}0 v_2 v_1 + v_1({}^{[z]}0 v_2)\Big) = 2Re\Big(z \cdot s(v_1 v_2)\Big).$$

Since $v_1, v_2 \in \overline{R^*}^n$ we may choose z such that both

$$v_1 + {}^{[z]}0 v_2 \in \overline{R^*}^n,$$

and

$$2Re\Big(z \cdot s(v_1 v_2)\Big) \leq 0_{\mathbb{R}}.$$

Assume that $v_1^2 + v_2^2 < v_1v_2$, then

$$(v_1 + {}^{[z]}0v_2)^2 = {}^{[z]}0v_2v_1 + v_1({}^{[z]}0v_2),$$

and

$$s\Big((v_1 + {}^{[z]}0v_2)^2\Big) = 2Re\Big(z \cdot s(v_1v_2)\Big) \leq 0_{\mathbb{R}}.$$

Therefore

$$v_1 + av_2 = (-\infty, ..., -\infty),$$

which contradicts our assumption that $v_1 \neq (-\infty, ..., -\infty)$.

□

Now we extend this lemma to several vectors.

Lemma 16.4. *Let R be a max-plus algebra over the reals. If $v_1, ..., v_k \in (\overline{R})^n$ then there exists some p for which*

$$v_p^2 \geq \sum_{1 \leq j \neq p \leq k} v_jv_p.$$

Proof. Assume

$$\forall p : \max_{1 \leq i,j \leq k} \{v_iv_j\} > v_p^2,$$

and choose specific $i \neq j$ such that $\forall p : v_iv_j > v_p^2$. It follows that $v_iv_j > v_i^2$. Thus by lemma 16.3 $v_iv_j \leq v_j^2$, which contradicts our assumption.

Therefore there exists some p for which

$$v_p^2 \geq \max_{1 \leq i,j \leq k} \{v_iv_j\},$$

and specifically

$$v_p^2 \geq \sum_{1 \leq j \neq p \leq k} v_jv_p.$$

□

16.2 Orthogonality

Definition 16.5. Consider $v_1, v_2 \in \overline{R}^n$. We say v_1, v_2 are *orthogonal* and write $v_1 \perp v_2$ if

$$s\Big(v_1v_2\Big) = 0_{\mathbb{C}}.$$

Theorem 16.6. *If $v_1, ..., v_k \in (\overline{R}^*)^n$ are vectors such that*

$$\forall i : v_i \neq (-\infty, ..., -\infty)$$

and

$$\forall i \neq j : v_i \perp v_j,$$

then $v_1, ..., v_k$ are linearly independent.

Proof. Assume that $v_1, ..., vk$ are linearly dependent. Then there exists $\alpha_1, ..., \alpha_k \in R^*$ such that

$$s\Big(\alpha_1 v_1 + ... + \alpha_k v_k\Big) = (0_{\mathbb{C}}, ..., 0_{\mathbb{C}}).$$

If $u_i = \alpha_i v_i$, then by lemma 16.4 there exists p such that

$$u_p^2 \geq \sum_{1 \leq j \neq p \leq k} u_j u_p.$$

Multiplying by u_p we obtain

$$s\Big(u_1 u_p + ... + u_k u_p\Big) = s\Big((0_{\mathbb{C}}, ..., 0_{\mathbb{C}}) u_p\Big) = 0_{\mathbb{C}}.$$

Therefore $\forall i \neq p : s\Big(u_i u_p\Big) = 0_{\mathbb{C}}$ and u_p^2 dominates all other term. It follows that $s\Big(u_p^2\Big) = 0_{\mathbb{C}}$, which is absurd.

□

Lemma 16.7. *Let $v_1, ..., v_k \in (\overline{R}^*)^n$ such that $k < n$, $v_i \perp v_j$ for all $i \neq j$ and $v_i \neq (-\infty, ..., -\infty)$ for all i. Then there exists $v_{k+1}, ..., v_n$ such that the set $\{v_1, ..., v_n\}$ is orthogonal.*

Proof. Assume $V_1, ..., V_k \in K^n$ are lifts of $v_1, ..., v_k$, i.e. $ELTrop(V_i) = v_i$ for all i. Then there exists vectors $U_{k+1}, ..., U_n \in K^n$ such that $U_j \perp V_i$ for all $1 \leq i \leq k$ and $k+1 \leq j$, and $U_j \perp U_p$ for all $j \neq p$

It is easy to see that if $u_j = ELTrop(U_j)$ then $\{v_1, ..., v_k, u_{k+1}, ..., u_n\}$ is an orthogonal set.

□

Example 16.8. In this example we consider the linearly independent set $S = \{v_1, v_2\}$ which is not orthogonal, and a vector v_3 which is orthogonal to S. Unfortunately, the set $\{v_1, v_2, v_3\}$ is linearly dependent.

$$v_1 = ({}^{[1]}2, {}^{[-1]}2, {}^{[-1]}1),$$
$$v_2 = ({}^{[-1]}2, {}^{[1]}2, {}^{[-1]}1),$$
$$v_3 = ({}^{[1]}1, {}^{[1]}1, {}^{[2]}1).$$

These vectors are linearly dependent since

$$v_1 + v_2 + v_3 = ({}^{[0]}2, {}^{[0]}2, {}^{[0]}1).$$

However, it is easy to see that v_1, v_2 are linearly independent, and that v_3 is orthogonal to both v_1 and v_2.

$$v_2 v_3 = v_1 v_3 = {}^{[0]}3.$$

Part IV

An exploded-layered version of Payne's generalization of Kapranov's theorem

17 Introduction

Kapranov's theorem (ref. [4]) states that the tropicalization of a verity of a polynomial f is equal to the tropical verity of the tropicalization of f. Since the inclusion

$$Trop(X) \subseteq V(Trop[f])$$

is trivial, Kapranov's theorem states, in other words, that for each element $a \in V(Trop[f])$ there exists a lift $x \in X$ such that $Trop(x) = a$.

Payne further claimed that not only does there exists a lift, but also there exists specific lifts whose leading monomials are canceled by f. This regard to the coefficient is similar to Parker's exploded structure. The exploded approach contributes algebraic operations on these coefficients.

In this part we formalize Payne's generalization in terms of the ELT algebra, give a constructive algebraic proof for the hypersurface case which gives some hope for an algebraic proof of the general case, and at the end present an elegant proof using an idea of Tabera [22].

I would like to thank Prof. Steve Shnider for his collaboration on this subject.

18 Main Theorem

Theorem 18.1. *Consider a polynomial $f \in K[x_1, ..., x_n]$ and a point $a \in R^n$, such that $a \in V(f)$ where $V(f)$ is the ELT variety of f. Then there exists a lift $x \in (K^*)^n$ such that $ELTrop(x) = a$ and $f(x) = 0$.*

Corollary 18.2. *Consider a polynomial* $f \in K[x_1, ..., x_n]$*, and the variety of its roots*

$$X(f) := \{x \in (K^*)^n : f(x) = 0\}.$$

Then

$$V(f) = ELTrop(X(f)).$$

18.1 Proof of the case of one variable

Lemma 18.3. *If* $f \in K[x]$ *factors into*

$$f(x) = (x - a_1) \cdots (x - a_n),$$

then

$$ELTrop[f](\lambda) = (\lambda + ELTrop(-a_1)) \cdots (\lambda + ELTrop(-a_n)).$$

Example 18.4. Consider the polynomial

$$f(x) = (x - 1)(x + 1) = x^2 - 1.$$

Then

$$ELTrop[f] = \lambda^2 + {}^{[-1]}0.$$

We get the factorization

$$(\lambda + {}^{[1]}0)(\lambda + {}^{[-1]}0) = \lambda^2 + {}^{[0]}0\lambda + {}^{[-1]}0 = \lambda^2 + {}^{[-1]}0,$$

since the middle monomial is inessential.

Example 18.5. Consider

$$f(x) = (x - a_1)(x - a_2)(x + a_3)(x + a_4) = (x - 1)(x - t)(x + t)(x + t^2).$$

We will explore the coefficient of x^2 in f:

$$t - t - t^2 - t^2 - t^3 + t^3.$$

We notice that cancellation of the leading monomials $a_1a_2 + a_1a_3 = t - t$ occurs since a_2, a_3 has the same tangible value $(t(ELTrop(a_2)) = t(ELTrop(a_3)) = -1)$.

Next we prove the lemma.

Proof. We wish to prove that $ELTrop[f] = g$, where

$$g(\lambda) = (\lambda + ELTrop(-a_1)) \cdots (\lambda + ELTrop(-a_n)).$$

We notice that

$$f(x) = \sum_{k=0}^{n} \Big[\sum_{i_{1k}, \ldots, i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \Big] x^{n-k},$$

$$g(\lambda) = \sum_{k=0}^{n} \Big[\sum_{i_{1k},...,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}}) \Big] \lambda^{n-k},$$

and

$$ELTrop[f] = \sum_{k \in J} \Big[ELTrop\Big(\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \Big) \Big] \lambda^{n-k}$$

where $k \in J$ if and only if $\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \neq 0$.

In order to prove $g = ELTrop[f]$ we will prove that for each $0 \leq k \leq n$ either

1. $k \notin J$ and $\Big[\sum_{i_{1k},...,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}}) \Big] \lambda^{n-k}$ is inessential,
2. $k \in J$ and $\sum_{i_{1k},...,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}}) =$ $ELTrop\Big(\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \Big)$,
3. $k \in J$,$\Big[\sum_{i_{1k},...,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}}) \Big] \lambda^{n-k}$ is inessential in g and $\Big[ELTrop\Big(\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \Big) \Big] \lambda^{n-k}$ is inessential in $ELTrop[f]$.

Assume that $k \notin J$, i.e. $\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) = 0$.

In order to simplify the notation we will assume that the a_i's are ordered from the smallest power of t in the leading monomial, to the largest

$$t(ELTrop(a_1)) \geq t(ELTrop(a_2)) \geq ... \geq t(ELTrop(a_n))$$

Since cancellation occurs in the leading monomial, we conclude that

$$t(ELTrop(a_{k+1})) = t(ELTrop(a_k)).$$

Assume that

$$t(ELTrop(a_j)) = t(ELTrop(a_{j+1})) = ... = t(ELTrop(a_k)),$$

and if $j > 1$ then

$$ELTrop(a_{j-1}) > ELTrop(a_j).$$

Write $C = t\Big(ELTrop(a_1 \cdots a_{j-1}) \Big)$ and $d = t(ELTrop(a_j))$. We know that

$$t\Big[ELTrop\Big(\sum_{i_{1k},...,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}}) \Big) \Big] = Cd^{k-j+1},$$

since $a_1, ..., a_n$ are ordered.

Thus

$$g(\lambda) = ... + {}^{[b_0]}C\lambda^{n-j+1} + ... + {}^{[0]}Cd^{k-j+1}\lambda^{n-k} + {}^{[b_{k+1}]}Cd^{k-j+2}\lambda^{n-k+1} + ...$$

Clearly all the monomials listed have the same tangible value at d, and therefore ${}^{[0]}Cd^{k-j+1}x^{n-k}$ is inessential, yielding (1).

Next we assume that

$$\sum_{i_{1k},\dots,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}}) \neq ELTrop\Big(\sum_{i_{1k},\dots,i_{kk}} (-a_{i_{1k}}) \cdots (-a_{i_{kk}})\Big).$$

Therefore cancellation occurs at the leading monomials, and

$$\Big[\sum_{i_{1k},\dots,i_{kk}} ELTrop(-a_{i_{1k}}) \cdots ELTrop(-a_{i_{kk}})\Big]\lambda^{n-k}$$

is inessential in g, yielding (3).

Finally, to prove (3), assume that

$$t(ELTrop(a_k)) = ... = t(ELTrop(a_{k+m})).$$

The polynomial f contains the monomials ${}^{[b_0]}C\lambda^{n-j+1}$ and ${}^{[b_{k+m}]}C\lambda^{k+m-j+1}$ since there is no cancellation at the leading monomials for these powers. Indeed $ELTrop(a_{j-1}) > ELTrop(a_j)$ and $ELTrop(a_{k+m}) > ELTrop(a_{k+m+1})$ (if $k+m+1 \leq n$).

The monomial ${}^{[b_k]}B\lambda^{n-k}$ is indeed inessential in f since $B < Cd^{k-j+1}$.

□

Corollary 18.6. *Let $f \in K[x]$ be a univariate polynomial. If $a \in R^*$ is such that*

$$s\Big(ELTrop[f](a)\Big) = 0,$$

then there exists a lift $y \in K$ such that $f(y) = 0$ and $ELTrop(y) = a$.

18.2 Proof of the multivariate case

First we notice that is it enough to prove the theorem for the point $a = ({}^{[1]}0, ..., {}^{[1]}0)$. Indeed, suppose we are assuming the theorem is true for $({}^{[1]}0, ..., {}^{[1]}0)$, given any point

$$a = ({}^{[c_1]}\lambda_1, ..., {}^{[c_n]}\lambda_n),$$

the polynomial

$$g(x_1, ..., x_n) := f(c_1^{-1}t^{\lambda_1}x_1, ..., c_n^{-1}t^{\lambda_n}x_n)$$

has a lift. The result follows.

We use induction on the number of variables. Since K is algebraically closed, Lemma 18.3 holds and Corollary 18.6 proves the one variable case.

If we find a specialization $x_1 = y$ such that the tropicalization of the polynomial

$$g(x_2, ..., x_n) := f(y, x_2, ..., x_n)$$

in $n-1$ variables has an ELT root then by induction we have a lift as needed. Furthermore, if $g \equiv 0$ we have found a lift as well.

Example 18.7. Consider the polynomial $f(x,y) = x(y-1)+t$. The point $({}^{[p]}0, {}^{[1]}0)$ is an ELT root of $ELTrop[f]$ for each p.
We specialize $y = 1$ and obtain $g_y(x) = t$. The polynomial $ELTrop[g_y]$ has no ELT roots and therefore we need a better choice of y.

We define a partial order relation on K: $\forall a, b \in K$ we say $a \prec b$ if the power series b contains all of the monomials of a and $-Trop(b-a)$ is bigger than every power of t in a. In particular, if $a \prec b$ then the powers of t in monomials of a are bounded. We will use this order to build a root or a specialization by finding the next monomial at each step.

Consider a series $a \in K$ and a polynomial $f \in K^*[x_1, ..., x_n]$

$$f(x_1, ..., x_n) = f_0(x_2, ..., x_n) + f_1(x_2, ..., x_n)(x_1 - a) + ... + f_n(x_2, ..., x_n)(x_1 - a)^n.$$

By change of the first variable, $z_a = x_1 - a$, we obtain the polynomial

$$f.a := f_0 + f_1 z_a + ... + f_n z_a^n \in K^*[z_a, x_2, ..., x_n]$$

with $f_i \in K[x_2, ..., x_n]$.

Definition 18.8. Consider a polynomial $f \in K[x_1, ..., x_n]$. A series $a \in K^*$ with leading monomial 1 is called *constructive* if for every monomial ct^α in a, the series $b \prec a$ consisting of all monomials of a with powers smaller than α it is true that

$$s\Big(ELTrop[f.b]({}^{[c]}(-\alpha), {}^{[1]}0, {}^{[1]}0, ..., {}^{[1]}0)\Big) = 0_k.$$

Lemma 18.9. *If $f, g, h \in K[x_1, ..., x_n]$ are polynomials such that $f + g = h$ then either*

$$ELTrop[h]({}^{[1]}0, ..., {}^{[1]}0) = ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0) + ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0)$$

or

$$ELTrop[h]({}^{[1]}0, ..., {}^{[1]}0) < \min\{ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0), ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0)\}$$

Proof. We write $f = \sum a_I x^I$, $g = \sum b_I x^I$ and $h = \sum c_I x^I$, and notice that

$$ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0) = \sum ELTrop(a_I).$$

Next, for

$$m = \min\{t(a_I), t(b_J)\}$$

we define

$$M_a = \{I | t(a_I) = m\}, M_b = \{I | t(b_I) = m\}.$$

Now, if

$$\sum_{I \in M_a} a_I x^I + \sum_{J \in M_b} b_J x^J = 0$$

then

$$ELTrop[h]({}^{[1]}0, ..., {}^{[1]}0) < \min\{ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0), ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0)\}.$$

Otherwise

$$ELTrop[h]({}^{[1]}0, ..., {}^{[1]}0) = ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0) + ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0),$$

since ${}^{[\ell_1]}\lambda + {}^{[\ell_2]}\lambda = {}^{[\ell_1 + \ell_2]}\lambda$.

$\square$

Lemma 18.10. *Let $f \in K[x_1, ..., x_n]$ be a polynomial such that*

$$s\Big(ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0,$$

let $a \in K$ be a series with leading monomial 1, and write

$$f = f_0 + f_1(x_1 - a) + ... + f_n(x_1 - a)^n,$$

with $f_i \in K[x_2, ..., x_n]$. Now, if

$$ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0) \geq ELTrop[f - f_0]({}^{[1]}0, ..., {}^{[1]}0),$$

then for $g_a := f(a, x_2, ..., x_n)$ it is true that

$$s\Big(ELTrop[g_a]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0$$

Proof. First we look at $f - f_0$ and notice that

$$ELTrop[f - f_0] = ELTrop[x_1 - a] \cdot ELTrop[f_1 + f_2(x_1 - a) + ... + f_n(x_1 - a)^{n-1}].$$

Since $ELTrop[x_1 - a] = \lambda_1 - {}^{[1]}0$ then $s\Big(ELTrop[x_1 - a]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0$ and therefore

$$s\Big(ELTrop[f - f_0]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0.$$

Now $f_0 = (-f) + (f - f_0)$ thus together with the previous lemma we obtain

$$ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0) = ELTrop[-f]({}^{[1]}0, ..., {}^{[1]}0) + ELTrop[f - f_0]({}^{[1]}0, ..., {}^{[1]}0),$$

and so

$$s\Big(ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0.$$

Since $f_0 \in K[x_2, ..., x_n]$ it is true that $g_a = f_0$ and the result follows.

□

Lemma 18.11. *Let a be a constructive element of f with a bounded set of powers. Write*

$$f.a = f_0 + f_1 z_a + ... + f_n z_a^n.$$

Assume $f_0 \neq 0$ and that

$$ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0) < ELTrop[f.a - f_0]({}^{[1]}0, ..., {}^{[1]}0),$$

then a is not maximal.

Proof. For any λ,

$$f(a + ct^{\lambda}) = f_0 + ct^{\lambda} f_1 + ... + c^n t^{n\lambda} f_n$$

We can choose λ to be a corner root with f_0 dominating, and obtain a layer polynomial in c with a constant from f_0 and thus a non-trivial solution.

In order to contradict maximality we must prove that $a \prec a + ct^{\lambda}$, i.e. we must prove that λ is larger or equal to the supremum of the powers of t in monomials of a, call it M. Indeed, assume that λ is strictly smaller than M. Then there exists a series b such that $a = a' + b$ and $-Trop(b) > \lambda$ (i.e. the power of the leading monomial of b is larger than λ).

Next we present f_i using g_j and yield a contradiction.

$$f = f_0 + f_1(x - a) + ... + f_n(x - a)^n = f_0 + f_1(x - a' - b) + ... + f_n(x - a' - b)^n,$$

$$f = g_0 + g_1(x - a') + ... + g_n(x - a')^n.$$

Is it true that:

$$g_n = f_n$$

$$g_{n-1} = f_{n-1} + \binom{n}{1}(-b) f_n$$

$$g_{n-2} = f_{n-2} + \binom{n-1}{1}(-b) f_{n-1} + \binom{n}{2} b^2 f_n$$

and so forth. Therefore:

$$f_n = g_n,$$

$$\forall i : f_j = g_j + \sum_{i=j+1}^{n} a_{ij} b^{i-j} g_i,$$

and $b^j f_j = b^j g_j + \sum_{i=j+1}^{n} a_{ij} b^i g_i$.

Assume k is such that

$$ELTrop[b^k g_k]({}^{[1]}0, ..., {}^{[1]}0) \geq ELTrop[f.a']({}^{[1]}0, ..., {}^{[1]}0)$$

and for all $i > k$,

$$ELTrop[b^k g_k]({}^{[1]}0, ..., {}^{[1]}0) > ELTrop[b^i g_i]({}^{[1]}0, ..., {}^{[1]}0).$$

Then

$$ELTrop[b^k g_k] = ELTrop[b^k f_k] < ELTrop[t^{k\lambda} f_k].$$

Now

$$ELTrop[f_0] \leq ELTrop[b^k g_k] < ELTrop[t^{k\lambda} f_k],$$

which contradicts the choice of λ such that f_0 is dominating.

□

Lemma 18.12. *Let a be a constructive element of f with an unbounded set of powers. Then*

$$f(a, x_2, ..., x_n) \equiv 0$$

Proof. By scalar multiplication we may assume that the minimal power of t in any monomial of f is not negative. Assume that $f(a, x_2, ..., x_n) \neq 0$, then there must be a monomial $dx_2 \cdots x_n$ in f. We will consider two series $a' + b = a$ such that the minimal power in b is higher than any power in a'.

If the minimal power of b is large enough, the leading part of the monomial $dx_2 \cdots x_n$ must be in f_0 where

$$f.a' = f_0 + f_1 z_{a'} + ... + f_k z_{a'}^k.$$

Indeed, $f.a'(b, x_2, ..., x_n) = f(a, x_2, ..., x_n)$. If b is large enough, any monomial $f_i z_{a'}^i(b, x_2, ..., x_n)$ will be strictly larger than $dx_2, ..., x_n$.

However, if we choose b with minimal power larger than the minimal power of d,

$$ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0)$$

will strictly dominate at $ELTrop[b]$, which contradicts the assumption that a is constructive.

□

Due to Zorn's lemma, there exist a maximal constructive element y. We specialize $x_1 = y$ and obtain the result by induction.

18.3 The element $-\infty$

Corollary 18.13. *Consider a polynomial $f \in K[x_1, ..., x_n]$, and the variety of its roots*

$$X(f) := \{x \in K^n : f(x) = 0\}.$$

Then

$$V(f) = ELTrop(X(f)).$$

Proof. Specialization of a variable to 0_K is equivalent to deletion of all monomials containing this variable. The same is true for $-\infty$ in ELT algebra.

Therefore after deleting the relevant monomials in f and $ELTrop[f]$ we use theorem 18.1 on a polynomial with less variables to obtain the result.

□

19 Tabera's Proof

Here we will present Tabera's ([22]) proof of the hypersurface case of Payne's theorem. We will prove the multivariate case using an induction on the number of variables, and not re-prove the one variable case.

Similar to the proof above, Tabera also seeks a specialization of a variable such that the remaining polynomial will have a lift. The main difference between Tabera's proof and the proof above is that we constructed a specialization of the first variable, while Tabera finds the best variable to specialize.

Consider a polynomial $f \in K[x_1, ..., x_n]$ such that $s\Big(ELTrop[f]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0_k$. Write

$$f(x_1, ..., x_n) = t^\alpha \Big(f_0(x_1, ..., x_n) + \sum_{\beta \in I} t^\beta f_\beta(x_1, ..., x_n)\Big),$$

where $\beta > 0$ for all $\beta \in I$, and $f_0, f_\beta \in k[x_1, ..., x_n]$.

We have

$$ELTrop[f] = {}^{[1]}(-\alpha) ELTrop[f_0],$$

and therefore $s\Big(ELTrop[f_0]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0_k$. Moreover, $f_0(1, ..., 1) = 0$.

Example 19.1. Consider the polynomial

$$f(x, y) = x - 2y + 1 + tx + t^2 y.$$

In this case $t^\alpha = t^0, f_0 = x - 2y + 1, f_1 = x, f_2 = 2y$. Thus

$$ELTrop[f] = {}^{[1]}0 ELTrop[f_0] = {}^{[1]}0\lambda_1 + {}^{[-2]}0\lambda_2 + {}^{[1]}0,$$

and $f_0(1, 1) = 0$.

Assume that a specialization $x_i = 1$ yields

$$g := f(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n) = t^{\alpha}\Big(g_0 + \sum_{\beta \in I} t^{\beta} g_{\beta}\Big),$$

such that $g_0 \not\equiv 0$. Then $s\Big(ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0_k$, and therefore g has a lift, and so does f by induction.

If no such specialization exists then f_0 must be of the form

$$f_0 = (x_1 - 1) \cdots (x_n - 1) h(x_1, ..., x_n).$$

We specialize $x_1 = 1 + t^{\gamma}$, to obtain

$$f_0(1 + t^{\gamma}, x_2, ..., x_n) = t^{\gamma}(x_2 - 1) \cdots (x_n - 1) h(1 + t^{\gamma}, x_2, ..., x_n).$$

If we choose $\gamma > 0$ small enough we have

$$g(x_2, ..., x_n) := f(1 + t^{\gamma}, x_2, ..., x_n) = t^{\alpha'}\Big(g_0 + \sum_{\beta \in I'} t^{\beta} g_{\beta}\Big),$$

where

$$g_0 = (x_2 - 1) \cdots (x_n - 1) h'(x_2, ..., x_n).$$

Therefore $g_0(1, ..., 1) = 0$ and

$$s\Big(ELTrop[g]({}^{[1]}0, ..., {}^{[1]}0)\Big) = 0_k.$$

20 Applications To Linear Algebra

20.1 Resultant

In classical algebra the resultant of two univariate polynomials $f, g \in K[x]$ is zero if and only if they have a common root. Considering all polynomials $f, g \in K[x]$ with fixed degrees, the resultant is a polynomial in the coefficients of f and g.

Furthermore, it is well known that the resultant $Res(f, g)$ is equal to the determinant of the Sylvester matrix, where given

$$f = a_n x^n + ... + a_0$$

and

$$g = b_m x^m + ... + b_0,$$

the Sylvester matrix $Syl(f, g) \in \mathbb{R}^{(m+n)\times(m+n)}$, is defined by

$$Syl(f,g) := \begin{pmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & b_m & b_{m-1} & \cdots & b_0 \end{pmatrix}.$$

In this notation, $|Syl(f,g)| = Res(f,g) \in K[a_n,...,a_0,b_m,...,b_0]$.

Furthermore, the first m rows of $Syl(f,g)$ each have at most $n+1$ entries different than zero and the last n each have $m+1$. Therefore we may Consider $|Syl|$ to be a polynomial in $2mn+m+n$ variables, a_i's and b_j's.

20.1.1 ELT resultant

We will prove an analogous theorem for the ELT case. Assume R is an ELT algebra.

Theorem 20.1. *Let $f,g \in R[x]$ be two univariate polynomials. Then*

$$s\Big(|Syl(f,g)|\Big) = 0$$

if and only if f,g have a common ELT root.

Proof. Consider $|Syl| \in K[x_1,...,x_{2mn+m+n}]$ and $ELTrop[|Syl|] \in R[\lambda_1,...,\lambda_{2mn+m+n}]$. It is easy to see that $ELTrop[|Syl|]$ is the ELT polynomial of the ELT Sylvester matrix.

Consider two ELT univariate polynomials $f,g \in R[x]$. Then by theorem 18.1

$$s\Big(|Syl(f,g)|\Big) = 0$$

if and only if there are polynomials $F,G \in K[x]$ such that $|Syl(F,G)| = 0$ and $ELTrop[F] = f, ELTrop[G] = g$.

If $|Syl(F,G)| = 0$ then there exists $x \in \mathbb{R}$ such that $F(x) = G(x) = 0$. We conclude that $ELTrop[x]$ is a common ELT root of f and g.

On the other hand, assume f,g have a common ELT root $a \in R$. Take an arbitrary lift $x \in K$ such that $ELTrop(x) = a$. Due to lemma 18.3, we may choose two polynomials F,G such that $F(x) = G(x) = 0$ and $ELTrop[F] = f, ELTrop[G] = g$. Therefore $|Syl(F,G)| = 0$ and $s\Big(|Syl(f,g)|\Big) = 0$.

□

20.2 Cayley Hamilton theorem

Theorem 20.2. *Let $A \in \overline{R}^{n\times n}$ be a matrix. Then*

$$s\Big(f_A(A)\Big) = 0_{k^{n\times n}}$$

Proof. We may consider the entry in the i-th row and j-th column of $f_C(B)$ to be a polynomial in $2n^2$ variables (the entries of C and B are the variables). The EL tropicalization of this polynomial is the equivalent entry in $f_C(B)$ over the ELT algebra.

$$f_C(B) = B^n \pm (\sum c_{ii})B^{n-1} \pm (\sum c_{i_1 i_1} \cdots c_{i_{n-1} i_{n-1})})B \pm |C|I.$$

Indeed, since there are no cancelations,

$$ELTrop\Big[f_C(B)\Big] = f_{C'}(B'),$$

where C', B' are variables matrices in $\overline{R}^{n\times n}$.

For any matrix $A \in K^{n\times n}$ it is true that $\Big[f_A(A)\Big]_{ij} = 0$. Then by theorem 18.1

$$s\Big[f_{A'}(A')\Big]_{ij} = 0_k,$$

where $A' = ELTrop(A)$.

Since any ELT matrix has a lift, we conclude the theorem.

$\square$

Example 20.3. Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

$$f_A(B) = (a-x)(d-x) - bc = x^2 - (a+d)x + ad - bc = B^2 - (a+d)B + (ad-bc)I,$$

$$B^2 = \begin{pmatrix} x^2+yz & xy+yw \\ zx+wz & zy+w^2 \end{pmatrix}.$$

Therefore,

$$f_A(B) = \begin{pmatrix} x^2+yz-(a+d)x+ad-bc & xy+yw-(a+d)y \\ zx+wz-(a+d)z & zy+w^2-(a+d)w+ad-bc \end{pmatrix}.$$

Now,

$$f_A(A) = \begin{pmatrix} a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b \\ ca+dc-(a+d)c & cb+d^2-(a+d)d+ad-bc \end{pmatrix},$$

it is easy to see that every monomial cancels by a monomial of an equal tangible size.

References

[1] R. Bieri and J. Groves, *The geometry of the set of characters induced by valuations.* J. Reine Angew. Math. 347:163-195, (1984).

[2] P. Butkovic, *Max-algebra: the linear algebra of combinatorics?*, Linear Algebra Appl. 367, 313335, (2003).

[3] M. Develin, F. Santos, B. Sturmfels, *On the rank of a tropical matrix*, Combinatorial and computational geometry 52, 213-242, (2005).

[4] M. Einsiedler, M. Kapranov, D. Lind, *Non-archimedean amoebas and tropical varieties*, Journal fr die reine und angewandte Mathematik (Crelles Journal) 2006.601, 139-157, (2006).

[5] A. Gathmann, *Tropical algebraic geometry*, Jahresbericht der DMV 108 (2006) no. 1, 3-32.

[6] I. Itenberg, G. Mikhalkin, E. Shustin, *Tropical Algebraic Geometry*, Oberwolfach Seminars vol. 35, Birkhauser, (2007).

[7] Z. Izhakian, *Tropical arithmetic and algebra of tropical matrices*, Communications in Algebra 37.4, 1445-1468, (2009).

[8] Z. Izhakian, L. Rowen, *Supertropical algebra*, Advances in Mathematics 225.4, 2222-2286, (2010).

[9] Z. Izhakian, L. Rowen, *Ghost-graded supertropical algebra and its differential calculus*, Preprint, (2008).

[10] Z. Izhakian, L. Rowen, *Supertropical matrix algebra*, Israel Journal of Mathematics 182.1, 383-424, (2011).

[11] Z. Izhakian, L. Rowen, *The tropical rank of a tropical matrix*, Communications in Algebra 37.11, 3912-3927, (2009).

[12] Z. Izhakian, M. Knebusch, L. Rowen, *Dual spaces and bilinear forms in supertropical linear algebra*, Linear and Multilinear Algebra 60.7, 865-883, (2012).

[13] H. Kuhn, *The Hungarian method for the assignment problem*, Naval research logistics quarterly 2.1-2, 83-97, (1955).

[14] G. Mikhalkin, *Tropical geometry and its applications*, arXiv preprint math/0601041 (2006).

[15] G. Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, Journal of the American Mathematical Society 18.2, 313-377, (2005).

[16] B. Parker, *Exploded manifolds*, Adv. Math, 229, (2012).

[17] S. Payne, *Fibers of tropicalization*, Math. Z., 262(2):301311, (2009).

[18] J. Richer-Gebert, B. Sturmfels, T. Theobald, *First Steps In Tropical Geometry*, Idempotent Mathematics and Mathematical Physics: International Workshop, February 3-10, 2003, Erwin Schrdinger International Institute for Mathematical Physics, Vienna, Austria. Vol. 377. AMS Bookstore, 200, (2003).

[19] S. Sergeev, *Max-plus definite matrix closures and their eigenspaces*, Linear algebra and its applications 421.2, 182-201, (2007).

[20] E. Shustin, *A tropical approach to enumerative geometry*, Algebra Anal. 17, no. 2, 170214, (2005).

[21] D. Speyer, B. Sturmfels, *The tropical grassmannian*, Advances in Geometry 4.3, 389-411, (2004).

[22] L. Tabera, *Constructive proof of extended kapranov theorem*, Actas de X Encuentro en A'lgebra Computacional y Aplicaciones EACA2006, ISBN: 84-611-2311-5, pp 178-181, (2008).

[23] J. Wong, *A new implementation of an algorithm for the optimal assignment problem: An improved version of Munkres' algorithm*, BIT Numerical Mathematics, volume 19, issue 3, pp 418-424, (1979).

Department of mathematics, Bar-Ilan university, Ramat-Gan 52900, Israel
E-mail address: erez@math.biu.ac.il