LEBESGUE INTEGRAL

1. Compute

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin^2 nx}{(1+x^2)^n}\, dx$$

Hint: use the Dominated Convergence theorem.

2. Compute

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\, dx.$$

Hint: you can use that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$, and that $e^{x/n} = 1+\frac{x}{n}+\frac{1}{2}(\frac{x}{n})^2+\cdots \geq 1+\frac{x}{n}$.

3. Let $0 \leq f(x) \leq 1$ be a measurable function on $[b,+\infty)$ and m is the Lebesgue measure on $\mathbb{R}$.

(a) Prove that

$$\lim_{\alpha\to 1+} \int_{[b,+\infty)} f^\alpha(x)\, dm = \int_{[b,+\infty)} f(x)\, dm,$$

where the meaning of $\alpha \to 1+$ is that α decreases to 1; the limit is from the right.

(b)* (bonus question). Show that a similar statement for $\alpha \to 1-$ is false, in general. Example:

$$f(x) = \frac{1}{x\ln^2(x)}, \qquad x \in [2,+\infty).$$

4. Let (X,μ) be a measure space, and let $f \in L^p(X)$, $\|f\|_p = 1$. For $t > 0$ consider the set

$$A = \{x \in X : \ |f(x)| > t\}.$$

Show that

$$\mu(A) \leq \frac{1}{t^p}.$$